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Nonconstant Continuous Functions whose Tangential Derivative Vanishes along a Smooth Curve

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Abstract. We provide a simple example showing that the tangential derivative of a continuous function ϕ can vanish everywhere along a curve while the variation of ϕ along this curve is nonzero. We give additional regularity conditions on the curve and/or the function that prevent this from happening.

1 Introduction

In [5], H. Whitney shows that for $n \geqslant 1$ there exist a function $\phi \in C^{n-1}(\mathbb{R}^n)$ and a continuous (parametrized) arc $f \in C([0,1],\mathbb{R}^n)$ so that ϕ is not constant on im f, yet $\nabla\phi[f(t)] = 0$ for each $0 \leqslant t \leqslant 1$. Of course such an arc f must not be rectifiable.

In this paper, we explore the possibility of constructing a continuous function $\phi \in C(\mathbb{R}^n)$ and a smooth arc $f \in C^k([0,1],\mathbb{R}^n)$ so that ϕ is not constant on im f, yet the directional derivative of ϕ at $f(t)$ along the tangent line to f at t (called the *tangential* derivative of ϕ at $f(t)$ along f) vanishes for each $0 \leqslant t \leqslant 1$.

In Section 2, we provide for $n \geqslant 3$ an example of a pair (ϕ, f) satisfying these conditions with $k = \infty$ and such that $\phi \circ f$ is monotone. This situation is not possible in the plane for $k \geqslant 2$, as we show in Section 4.

In Section 3 we give some sufficient conditions on the pair (f, ϕ) so that the vanishing of the tangential derivative of ϕ at $f(t)$ along f for each $0 \leqslant t \leqslant 1$ implies that ϕ has zero variation on im f.

2 The Tangential Derivative

Let $f\colon [0,1] \to \mathbb{R}^n$ be of class C^1, injective, regular (*i.e.*, $f'(t) \neq 0$ whenever $0 \leqslant t \leqslant 1$) and let U be an open set containing im f.

Definition 2.1 For a given $\phi\colon U \to \mathbb{R}$, the *upper tangential derivative* of ϕ at $f(t)$ $(0 \leqslant t \leqslant 1)$ along f is the extended real number

$$\overline{D}_f\phi[f(t)] := \frac{1}{|f'(t)|}\overline{\lim_{h\to 0}}\,\frac{\phi[f(t)+hf'(t)] - \phi[f(t)]}{h},$$

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whereas the *lower tangential derivative* of ϕ at $f(t)$ $(0 \leqslant t \leqslant 1)$ along f is defined as

$$\underline{D}_f \phi[f(t)] := \frac{1}{|f'(t)|} \varliminf_{h \to 0} \frac{\phi[f(t) + hf'(t)] - \phi[f(t)]}{h}.$$

In the case where $\overline{D}_f \phi(x) = \underline{D}_f \phi(x)$ is a real number, we call its value the *tangential derivative* of ϕ at x along f.

Example 2.2 Let $f\colon [0,1] \to \mathbb{R}^3$ be the map of class C^∞, injective and regular, defined by the formula $f(t) = (\cos t, \sin t, t)$. For $t \in [0,1]$ we denote the tangent line to f at t by $\Lambda(t) = \{f(t) + \lambda f'(t) : \lambda \in \mathbb{R}\}$.

Claim 2.3 For $s \neq t$ in $[0,1]$, we have $\Lambda(s) \cap \Lambda(t) = \varnothing$.

To see this, assume that $f(s) + \lambda f'(s) = f(t) + \mu f'(t)$. This yields

$$\cos s - \lambda \sin s = \cos t - \mu \sin t,$$
$$\sin s + \lambda \cos s = \sin t + \mu \cos t,$$
$$s - t = \mu - \lambda.$$

From the two first equations, we infer

$$(\mu - \lambda) \sin(s - t) = 2 - 2\cos(s - t).$$

Replacing $\mu - \lambda$ by $s - t$ in the previous equality, using trigonometric identities, and defining $u = (s - t)/2$, we get $\sin^2 u = u \sin 2u$. It follows that either $u = 0$ or $\tan u = u$. As 0 is the only fixed point of tan in $[-1/2, 1/2]$, the latter condition also implies $u = 0$.

Let $\mathscr{F} := \bigcup_{t \in [0,1]} \Lambda(t)$, and let $p\colon \mathscr{F} \to [0,1]$ be the map that associates with $x \in \mathscr{F}$ the unique $t \in [0,1]$ for which one has $x \in \Lambda(t)$.

Claim 2.4 The function p is continuous on the closed set $\mathscr{F}$.

To prove that p is continuous on $\mathscr{F}$, assume that $(x_k)_{k \geqslant 1} \subseteq \mathscr{F}$ converges to $x_0 \in \mathscr{F}$. For each $k \geqslant 0$, let $t_k = p(x_k)$ and choose $\lambda_k \in \mathbb{R}$ for which $x_k = f(t_k) + \lambda_k f'(t_k)$. Next we show that any convergent subsequence of $(t_k)_{k \geqslant 1}$ converges to t_0. Assuming that the subsequence $(t_{k_l}) \subseteq (t_k)$ converges to $0 \leqslant t \leqslant 1$ (and extracting again a subsequence if necessary so that (λ_{k_l}) converges to some $\lambda \in \mathbb{R}$), observe that the continuity of f and f' yields $x_0 = f(t) + \lambda f'(t)$ and hence $x_0 \in \Lambda(t)$. According to Claim 2.3 we obtain $t = t_0$.

We easily show that $\mathscr{F}$ is a closed set using similar arguments.

For $x \in \mathscr{F}$ define $\phi(x) = p(x)$ and extend ϕ to $\mathbb{R}^3$ in a continuous way using the Tietze extension theorem. As ϕ is constant on $\Lambda(t)$ for each $0 \leqslant t \leqslant 1$, one has $D_f \phi[f(t)] = 0$ for each $0 \leqslant t \leqslant 1$. Yet

$$\phi[f(1)] - \phi[f(0)] = 1 - 0 = 1 > 0.$$

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Remark 2.5 Using a Whitney-like extension theorem instead of Tietze's theorem (see [1]), we can extend ϕ to a continuous function in $\mathbb{R}^3$ that is of class C^1 outside the helix $H = \operatorname{im} f$.

Let us summarize the preceding information.

Proposition 2.6 *There exists a function $\phi \in C(\mathbb{R}^3)$ together with a map $f \in C^\infty([0,1],\mathbb{R}^3)$, injective and regular, for which the following conditions are fulfilled:*

(i) *ϕ is constant on $f(t) + \operatorname{span}\langle f'(t)\rangle$ for all $0 \leqslant t \leqslant 1$;*

(ii) *$\phi[f(t)] = t$ for each $0 \leqslant t \leqslant 1$ (so $F := \phi \circ f$ is continuous and increasing).*

In particular, we have $D_f\phi[f(t)] = 0$ for each $0 \leqslant t \leqslant 1$, yet ϕ is not *constant on* $\operatorname{im} f$.

Remark 2.7 The oscillation $\operatorname{osc}(\phi, r)$ of $\phi\colon U \to \mathbb{R}$ at scale $r > 0$ is the extended real number

$$\operatorname{osc}(\phi, r) := \sup\{|\phi(x) - \phi(y)| : x, y \in U, |x - y| \leqslant r\}.$$

The oscillation of the function ϕ constructed in Example 2.2 satisfies

$$\varlimsup_{r\to 0} r^{-1/2} \operatorname{osc}(\phi, r) > 0.$$

This follows from the next proposition.

In the sequel, the notation $[0,1] - t$ stands for the set $\{h \in \mathbb{R} : t + h \in [0,1]\}$.

Proposition 2.8 *Assume that U is open and fix $\phi \in C(U)$. If*

(i) *$f\colon [0,1] \to U$ is of class $C^{1,1}$, injective and regular,*

(ii) *for every $0 \leqslant t \leqslant 1$, $D_f\phi[f(t)] = 0$, and*

(iii) *$\phi[f(t)] = t$ for $0 \leqslant t \leqslant 1$,*

then $\varlimsup_{r\to 0} r^{-1/2} \operatorname{osc}(\phi, r) > 0$.

Proof Let $L = \operatorname{Lip}(f') > 0$ (where $\operatorname{Lip}(f')$ denotes the Lipschitz constant of f'). For $0 \leqslant t \leqslant 1$ we compute

$$|h|^{-2}|f(t+h) - f(t) - hf'(t)| = |h|^{-2}\Big|\int_t^{t+h} [f'(s) - = f'(t)]\, ds\Big| \leqslant L$$

for each $h \in [0,1] - t$. Moreover, for $t \in [0,1]$ choose $\delta(t) > 0$ so that

$$|\phi[f(t) + hf'(t)] - \phi[f(t)]| \leqslant \tfrac{1}{2}|h|$$

whenever $h \in [0,1] - t$ satisfies $f(t) + hf'(t) \in U$ and $|h| \leqslant \delta(t)$. So for $h \in [0,1] - t$ satisfying $|h| \leqslant \delta(t)$, we have

$$\frac{|\phi[f(t+h)] - \phi[f(t) + hf'(t)]|}{\sqrt{|f(t+h) - f(t) - hf'(t)|}} \geqslant \frac{|h| - |\phi[f(t) + hf'(t)] - \phi[f(t)]|}{\sqrt{|f(t+h) - f(t) - hf'(t)|}} \geqslant \frac{1}{2L}.$$

The result follows as h can be chosen as small as we want. ∎

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3 Some Sufficient Conditions for a Mean Value Formula

We can avoid the situation that appears in Example 2.2 by imposing a Lipschitz condition on ϕ.

Theorem 3.1 *Assume that U is open, and let $\phi \in C(U)$. If*

(i) *$f\colon [0,1] \to U$ is of class C^1, injective and regular,*
(ii) *for every $0 \leqslant t \leqslant 1$, $\underline{D}_f \phi[f(t)] \geqslant 0$, and*
(iii) *ϕ is Lipschitz,*

then $\phi[f(1)] - \phi[f(0)] \geqslant 0$.

To prove Theorem 3.1, we will make use of Cousin's lemma, which essentially expresses the compactness property of closed intervals.

Lemma 3.2 (Cousin) *Given real numbers $a < b$ and a positive function $\delta\colon [a,b] \to (0,\infty)$, there exist real numbers $a = \alpha^0 < \alpha^1 < \cdots < \alpha^m = b$ and points $t^j \in [\alpha^{j-1}, \alpha^j]$, $1 \leqslant j \leqslant m$ such that $[\alpha^{j-1}, \alpha^j] \subseteq [t^j - \delta(t^j), t^j + \delta(t^j)]$ holds for each $1 \leqslant j \leqslant m$.*

This lemma is a crucial tool in the study of Riemann-type integrals; the interested reader will find a (simple) proof in [3], for example.

Proof Write $L = \operatorname{Lip}(f)$ and introduce $M = \max_{0\leqslant t\leqslant 1} |f'(t)| > 0$. Fix $\varepsilon > 0$. For each $t \in [0,1]$, choose $\delta_1(t) > 0$ such that one has

$$\phi[f(t) + hf'(t)] - \phi[f(t)] \geqslant -\frac{\varepsilon}{2M}|f'(t)|h, \tag{3.1}$$

$$\phi[f(t) - hf'(t)] - \phi[f(t)] \leqslant \frac{\varepsilon}{2M}|f'(t)|h \tag{3.2}$$

whenever $h \geqslant 0$ is such that $f(t) \pm hf'(t) \in U$ and satisfies $|h| \leqslant \delta_1(t)$. Also choose a real number $\delta_2(t) > 0$ for which one has

$$|f(t+\eta) - f(t) - \eta f'(t)| \leqslant \frac{\varepsilon}{2L+1}|\eta|,$$

whenever $\eta \in [0,1] - t$ satisfying $0 \leqslant |\eta| \leqslant \delta_2(t)$ is given. For $0 \leqslant t \leqslant 1$ define $\delta(t) = \min[\delta_1(t), \delta_2(t)]$. According to Cousin's lemma (Lemma 3.2) find $0 = \alpha^0 < \alpha^1 < \alpha^2 < \cdots < \alpha^m = 1$ together with points $t^j \in [\alpha^{j-1}, \alpha^j]$, $1 \leqslant j \leqslant m$ for which $[\alpha^{j-1}, \alpha^j] \subseteq [t^j - \delta(t^j), t^j + \delta(t^j)]$ holds for each $1 \leqslant j \leqslant m$. As ϕ is Lipschitz, we compute for $1 \leqslant j \leqslant m$

$$\phi[f(\alpha^j)] - \phi[f(t^j) + (\alpha^j - t^j)f'(t^j)] \geqslant -\frac{\varepsilon L}{2L+1}(\alpha^j - t^j) \geqslant -\frac{1}{2}\varepsilon(\alpha^j - t^j), \tag{3.3}$$

and similarly

$$\phi[f(t^j) - (t^j - \alpha^{j-1})f'(t^j)] - \phi[f(\alpha^{j-1})] \geqslant -\frac{1}{2}\varepsilon(t^j - \alpha^j). \tag{3.4}$$

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Moreover, (3.1) and (3.2) yield

$$\phi[f(t^j)+(\alpha^j-t^j)f'(t^j)]-\phi[f(t^j)]\geqslant-\frac{\varepsilon M}{2M}(\alpha^j-t^j)=-\frac{1}{2}\varepsilon(\alpha^j-t^j). \tag{3.5}$$

and similarly

$$\phi[f(t^j)]-\phi[f(t^j)-(t^j-\alpha^{j-1})f'(t^j)]\geqslant-\frac{1}{2}\varepsilon=(t^j-\alpha^{j-1}). \tag{3.6}$$

Fitting (3.3), (3.4), (3.5), and (3.6) together, we get

$$\phi[f(\alpha^j)]-\phi[f(\alpha^{j-1})]\geqslant-\varepsilon(\alpha^j-\alpha^{j-1});$$

we complete the proof by adding those inequalities obtained for $j=1,\dots,m$ and letting $\varepsilon\to 0$. ■

Whenever f' is Lipschitz, one can weaken the last condition in Theorem 3.1.

Theorem 3.3 *Assume that U is open and fix $\phi\in C(U)$. If*

(i) *$f\colon[0,1]\to U$ is of class $C^{1,1}$, injective and regular,*
(ii) *for every $x\in\operatorname{im} f$, $\underline{D}_f\phi(x)\geqslant 0$, and*
(iii) $\overline{\lim}_{r\to 0}\left[r^{-1/2}\operatorname{osc}(\phi,r)\right]=0,$

then $\phi[f(1)]-\phi[f(0)]\geqslant 0$.

Proof Let $L=\operatorname{Lip}(f')$ and assume $L>0$. As in the proof of Proposition 2.8, one observes

$$|h|^{-2}|f(t+h)-f(t)-hf'(t)|\leqslant L$$

holds for each $h\in[0,1]-t$. Let $M=\max_{0\leqslant t\leqslant 1}|f'(t)|>0$, fix $0<\varepsilon\leqslant 1$, and choose $r>0$ such that

$$|\phi(x)-\phi(y)|\leqslant\frac{\varepsilon}{2\sqrt{L}}|x-y|^{1/2}$$

holds whenever $x,y\in U$ satisfy $|x-y|\leqslant r$. For each $0\leqslant t\leqslant 1$, let $\delta_1(t)>0$ be such that (3.1) and (3.2) hold whenever $h\geqslant 0$ is such that $f(t)\pm hf'(t)\in U$ and satisfies $|h|\leqslant\delta_1(t)$. For $0\leqslant t\leqslant 1$ define $\delta(t)=\min[r^{1/2}M^{-1/2},\delta_1(t),\delta_2(t)]$, find real numbers α^j, $0\leqslant j\leqslant m$ and points t^j as in the proof of Theorem 3.1. Decomposing $\phi[f(\alpha^j)]-\phi[f(\alpha^{j-1})]$ as in the same proof, we get the estimates (3.5) and (3.6). Observe that

$$|f(\alpha^j)-f(t^j)-(\alpha^j-t^j)f'(t^j)|\leqslant L(\alpha^j-t^j)^2\leqslant r,$$

so that we have

$$\begin{aligned}\phi[f(\alpha^j)]-\phi[f(t^j)+(\alpha^j-t^j)f'(t^j)]&\geqslant-\frac{\varepsilon}{2\sqrt{L}}[L(\alpha^j-t^j)^2]^{1/2}\\&=-\frac{1}{2}\varepsilon(\alpha^j-t^j)\end{aligned}$$

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Similarly one shows

$$\phi[f(t^j)-(t^j-\alpha^{j-1})f'(t^j)]-\phi[f(\alpha^{j-1})]\geqslant -\frac{1}{2}\varepsilon(t^j-\alpha^j).$$

One finishes the argument as in the proof of Theorem 3.1. ∎

From the preceding statements we arrive at the following corollary.

Corollary 3.4 *Assume that U is open and that $f\in C^1([0,1],U)$ is injective and regular. Assume moreover that $D_f\phi[f(t)]=0$ for each $0\leqslant t\leqslant 1$. If one of the following conditions is fulfilled, then ϕ is constant on* im f*:*

(i) *ϕ is Lipschitz;*
(ii) *f' is Lipschitz and $\overline{\lim}_{r\to 0} r^{-1/2}\operatorname{osc}(\phi,r)=0$.*

Proof Fix $0\leqslant s<t\leqslant 1$ and define a map $\varphi\colon [0,1]\to[s,t]$ by $\varphi(\xi)=s+(t-s)\xi$. Apply Theorem 3.1 or 3.3 to $f\circ\varphi$ and ϕ (resp. $-\phi$) to get $\phi[f(t)]\geqslant\phi[f(s)]$ (resp. $\phi[f(t)]\leqslant\phi[f(s)]$). As $s<t$ are arbitrary we conclude that ϕ is constant on im f. ∎

In the plane, we cannot provide an analogue of Example 2.2.

4 The Planar Case

Set the following for this entire section: we have a curve $f\in C^2([0,1],\mathbb{R}^2)$ satisfying $f'(t)\neq 0$ for each $0\leqslant t\leqslant 1$ and a function $\phi\in C(\mathbb{R}^2)$. We also assume that f has nonzero normal curvature everywhere, *i.e.,* that $f''(t)$ is not collinear with $f'(t)$ for every $0\leqslant t\leqslant 1$.

4.1 About the Existence of Curvilinear Derivatives

Let us state and prove an easy geometrical fact.

Lemma 4.1 *Fix $0\leqslant t\leqslant 1$. For each $h\in[0,1]-t$ sufficiently small, the tangent lines to f at t and $t+h$ intersect at $x(t,h)$ and*

$$\max\{|f(t)-x(t,h)|,|f(t+h)-x(t,h)|\}\leqslant M|h|,$$

where $M:=\max_{0\leqslant t\leqslant 1}|f'(t)|$.

Proof Without loss of generality, assume

$$f(t)=(0,0),\quad f'(t)=(f_1'(t),0)\neq 0\quad\text{and}\quad f''(t)=(f_1''(t),f_2''(t))$$

with $f_2''(t)\neq 0$. Choose $\eta>0$ such that

$$\min_{\substack{ch\in[0,1]-t\\ |h|\leqslant\eta}}|f_2''(t+h)|>0,$$

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and assume $h \in [0,1]-t$ satisfies $|h| \leqslant \eta$. The coordinates of the point $x(t,h) = (x_1(t,h), x_2(t,h))$ are given by

$$x_1(t,h) = f_1(t+h) - f_1'(t+h)\frac{f_2(t+h)}{f_2'(t+h)} \quad \text{and} \quad x_2(t,h) = 0.$$

Also choose $\delta > 0$ such that for each $h \in [0,1]-t$ verifying $|h| \leqslant \delta$, one can find points θ_h and ϑ_h in $[0,1]$ for which

$$f_2'(t+h) = hf_2''(t+\theta_h h) \quad \text{and} \quad f_2(t+h) = \tfrac{1}{2}h^2 f_2''(t+\vartheta_h h).$$

For $h \in [0,1]-t$ satisfying $|h| \leqslant \min\{\eta,\delta\}$, compute

$$x_1(t,h) = f_1(t+h) - \frac{1}{2}hf_1'(t+h)\frac{f_2''(t+\vartheta_h h)}{f_2''(t+\theta_h h)}, \quad x_2(t,h) = 0$$

and

$$\frac{x_1(t,h)-f_1(t)}{hf_1'(t)} = \frac{1}{f_1'(t)}\frac{f_1(t+h)-f_1(t)}{h} = -\frac{1}{2}\frac{f_1'(t+h)}{f_1'(t)}\frac{f_2''(t+\vartheta_h h)}{f_2''(t+\theta_h h)}.$$

Thus we get

$$\lim_{h\to 0}\frac{|x(t,h)-f(t)|}{|hf'(t)|} = \left|\lim_{h\to 0}\frac{x_1(t,h)-f_1(t)}{hf_1'(t)}\right| = \frac{1}{2}.$$

On the other hand, compute for $h \in [0,1]-t$ satisfying $|h| \leqslant \min\{\eta,\delta\}$

$$\lim_{h\to 0}\frac{x_1(t,h)-f_1(t+h)}{h} = -\lim_{h\to 0}\frac{1}{2}f_1'(t+h)\frac{f_2''(t+\vartheta_h h)}{f_2''(t+\theta_h h)} = -\frac{1}{2}f_1'(t),$$

and from $f_2(t) = 0$ we get

$$\lim_{h\to 0}\frac{x_2(t,h)-f_2(t+h)}{h} = -\lim_{h\to 0}\frac{f_2(t+h)-f_2(t)}{h} = -f_2'(t) = 0.$$

Consequently we obtain

$$\lim_{h\to 0}\frac{|x(t,h)-f(t+h)|}{|hf'(t)|} = \frac{1}{2},$$

and the proof is complete. ■

Given a set $E \subseteq \mathbb{R}$, denote by $|E|$ the outer Lebesgue measure of E. The notation $\operatorname{dens}(E)$ stands for the set of all Lebesgue-density points of E. We know that

$$|(E \setminus \operatorname{dens} E) \cup (\operatorname{dens} E \setminus E)| = 0.$$

Proposition 4.2 *Assume that for almost every* $0 \leqslant t \leqslant 1$, $D_f\phi[f(t)] = 0$. *Then for almost every* $0 \leqslant t \leqslant 1$, *we have* $F'(t) = 0$ *where* $F := \phi \circ f$.

Proof To show this, call D the set of $0 \leqslant t \leqslant 1$ at which $D_f\phi[f(t)] = 0$ and observe that D is a set of full measure in $[0, 1]$. Define for each integer k and each $t \in D$

$$0 < \delta_k(t) := \sup\{\delta > 0 : |\phi[f(t) + hf'(t)] - \phi[f(t)]| \leqslant 2^{-k}|h| \text{ for each } |h| \leqslant \delta\},$$

and let $\delta_k(t) = 0$ for $t \notin D$. From the identity

$$\{0 \leqslant t \leqslant 1 : \delta_k(t) \leqslant \alpha\} = \bigcap_{\substack{\delta\in\mathbb{Q}\\ \delta>\alpha}} \bigcup_{\substack{h\in\mathbb{Q}\\ |h|\leqslant\delta}} \{0 \leqslant t \leqslant 1 : |\phi[f(t + h)] - \phi[f(t)]| > 2^{-k}|h|\}$$

valid for each $\alpha > 0$, we see that δ_k is a measurable function of t for each $k \in \mathbb{N}$. For $k \in \mathbb{N}$ and $l \in \mathbb{N}$, define

$$E_{k,l} = \{0 \leqslant t \leqslant 1 : \delta_k(t) \geqslant 2^{-l}\}.$$

Let $F_{k,l} := \operatorname{dens}(E_{k,l})$ for $k, l \in \mathbb{N}$, and observe that $F := \bigcap_{k\in\mathbb{N}} \bigcup_{l\in\mathbb{N}} F_{k,l}$ has full measure in $[0, 1]$.

Fix $t \in F$. By definition there exist increasing sequences (k_i) and (l_i) of integers such that $t \in F_{k_i,l_i}$ for each i. Observing that for h sufficiently small we have (see Lemma 4.1)

$$\frac{|F(t + h) - F(t)|}{|h|} \leqslant M\frac{|\phi[f(t + h)] - \phi[x(t, h)]|}{|f(t + h) - x(t, h)|} + M\frac{|\phi[f(t)] - \phi[x(t, h)]|}{|f(t) - x(t, h)|}.$$

For any i we thus get

$$\varlimsup_{\substack{ch\to 0\\ h\in F_{k_i,l_i}-t}} \frac{|F(t + h) - F(t)|}{|h|} \leqslant 2^{-k_i+1}M,$$

that is, $\operatorname{ap}\varlimsup_{h\to 0} \frac{|F(t+h)-F(t)|}{|h|} = 0$ (here and in the sequel, we will use the particle ap to indicate that we are dealing with an *approximate limit*; see [2, Section 2.9.12]).

We infer from [2, Theorem 3.1.8] that F has vanishing approximate derivative almost everywhere. It follows from [3, Theorem 6.6.8] that $F'(t) = 0$ for almost every $0 \leqslant t \leqslant 1$. ∎

4.2 Using de la Vallée Poussin's Theorem

In the case where F has bounded variation, de la Vallée Poussin's theorem [4, Chapter IV, Theorem 9.6] will allow us to prove a mean value formula for tangential derivatives.

Fix $F\colon [0,1] \to \mathbb{R}$. For real numbers $0 \leqslant a < b \leqslant 1$ define

$$\underline{V}_a^b F := \inf_{\mathcal{P}} \sum_{[c,d]\in\mathcal{P}} [F(d) - F(c)] \quad \text{and} \quad \overline{V}_a^b F := \sup_{\mathcal{P}} \sum_{[c,d]\in\mathcal{P}} [F(d) - F(c)],$$

where the infimum and the supremum are taken over all finite families $\mathcal{P}$ of nonoverlapping compact intervals contained in $[a,b]$.

Define the *classical variation* of F on $[a,b]$ by $V_a^b F = \overline{V}_a^b F + |\underline{V}_a^b F|$.

In the case where F has bounded variation (*i.e.,* satisfies $V_0^1 F < \infty$), define an outer measure V^*F on $[0,1]$ by

$$V^*F(E) = \inf\Big\{ \sum_{k=0}^{\infty} V_{a_k}^{b_k} F : 0 \leqslant a_k < b_k \leqslant 1, E \subseteq \bigcup_{k\in\mathbb{N}} \operatorname{int}_{[0,1]}[a_k,b_k] \Big\},$$

where $\operatorname{int}_{[0,1]}$ means "*relative interior in* $[0,1]$ of".

The following lemma is the first part of de la Vallée Poussin's theorem.

Lemma 4.3 (de la Vallée Poussin, Part I) *Assume that $F\colon [0,1] \to \mathbb{R}$ has bounded variation on $[0,1]$. Then the derivative (finite or infinite) $F'(t)$ exists for each t outside a V^*F-negligible set.*

The following proposition is an immediate corollary of Lemma 4.3.

Proposition 4.4 *Assume that ϕ is continuous and that for each $0 \leqslant t \leqslant 1$ we have $D_f\phi[f(t)] = 0$. Assume moreover that $F := \phi \circ f$ has bounded variation on $[0,1]$. Then $F'(t) = 0$, except on a set having zero V^*F measure.*

Proof As in the proof on Proposition 4.2, define for $k, l \in \mathbb{N}$ a positive function δ_k on $[0,1]$ and measurable subsets $E_{k,l}$ of $[0,1]$. For each $k \in \mathbb{N}$ observe $[0,1] = \bigcup_{l\in\mathbb{N}} E_{k,l}$ and call $D_{k,l}$ the countable set of isolated points in $E_{k,l}$. Aslo define $D = \bigcup_{k,l\in\mathbb{N}} D_{k,l}$.

Let us associate with F a set B according to Lemma 4.3. For each $t \in F := [0,1] \setminus (B \cup D)$ and for any k, l such that $t \in E_{k,l}$, observe that t is an accumulation point of $E_{k,l}$ and use Lemma 4.1 to infer

$$|F'(t)| = \lim_{\substack{h\to 0\\ h\in E_{k,l}-t}} \Big| \frac{F(t+h) - F(t)}{h} \Big| \leqslant 2^{-k+1} M.$$

As k can be chosen as large as we wish, we get $F'(t) = 0$.

It suffices now to observe that $B \cup D$ is V^*F negligible as V^*F cannot concentrate on points (F is continuous). ∎

Given any function F having bounded variation on $[0,1]$, define outer measures $\underline{F}^*$ and $\overline{F}^*$ on $[0,1]$ by

$$\underline{F}^*(E) = \inf\Big\{ \sum_{k=0}^{\infty} [-\underline{V}_{a_k}^{b_k} F] : 0 \leqslant a_k < b_k \leqslant 1, E \subseteq \bigcup_{k\in\mathbb{N}} \operatorname{int}_{[0,1]}[a_k,b_k] \Big\},$$

$$\overline{F}^*(E) = \inf\Big\{ \sum_{k=0}^{\infty} \overline{V}_{a_k}^{b_k} F : 0 \leqslant a_k < b_k \leqslant 1, E \subseteq \bigcup_{k\in\mathbb{N}} \operatorname{int}_{[0,1]}[a_k,b_k] \Big\};$$

and define for $E \subseteq [0,1]$, $F^*(E) = \overline{F}^*(E) - \underline{F}^*(E)$.

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Theorem 4.5 (de la Vallée Poussin, Part II) *Assume F has bounded variation on $[0,1]$ and let E_+ (resp. E_-) denote the set of points $0 \leqslant t \leqslant 1$ at which $F'(t) = +\infty$ (resp. $F'(t) = -\infty$). For any Borel set B*

$$V^*F(B) = F^*(B \cap E_+) + |F^*(B \cap E_-)| + \int_B |F'(t)|\, dt.$$

As a corollary, we get the following mean value formula.

Corollary 4.6 *Assume that ϕ is continuous and that $F := \phi \circ f$ has bounded variation. If, moreover, $D_f\phi[f(t)] = 0$ for each $0 \leqslant t \leqslant 1$, then ϕ is constant on* $\operatorname{im} f$.

Proof It suffices to observe that Proposition 4.4 and Theorem 4.5 yield

$$V^*F([0,1]) = \int_0^1 |F'(t)|\, dt = 0. \qquad \blacksquare$$

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