A UNIQUENESS THEOREM FOR HARMONIC FUNCTIONS ON HALF-SPACES

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An arbitrary point of the Euclidean space $\mathbf{R}^{n+1}$, where $n \geq 1$, is denoted by (X, y), where $X \in \mathbf{R}^n$ and $y \in \mathbf{R}$, and we denote the Euclidean norm on $\mathbf{R}^n$ by $\|\cdot\|$. If h is harmonic on the half-space $\Omega = \{(X, y) : y > 0\}$, then we define extended real-valued functions m and M as follows:

$$m(r) = \sup\{|h(X, y)| : (X, y) \in \Omega, \|(X, y)\| = r\} \qquad (r > 0)$$

and

$$M(y) = \sup\{|h(X, y)| : X \in \mathbf{R}^n\} \qquad (y > 0).$$

It is known [**1**], [**2**] that if

$$m(r) = O(e^{-\alpha r}) \qquad (r \to +\infty)$$

for some positive number α, then $h \equiv 0$ on Ω. Here we prove a similar result with M in place of m.

THEOREM . *If h is harmonic on Ω and*

$$M(y) = O(e^{-\beta y}) \qquad (y \to +\infty) \tag{1}$$

for every positive number β, then $h \equiv 0$ on Ω.

The example

$$h(X, y) = e^{-\beta y} \sin(x_1 \beta/\sqrt{n}) \ldots \sin(x_n \beta/\sqrt{n}),$$

where $X = (x_1, \ldots, x_n)$ shows that, in contrast with the $m(r)$ result, it is not enough to suppose only that (1) holds for some positive β.

We first establish the corresponding result for holomorphic functions on a half-plane. We write $\omega = \{z \in \mathbf{C} : \operatorname{Re} z > 0\}$.

LEMMA. *If f is holomorphic on ω and*

$$\sup\{|f(\xi + i\eta)| : \eta \in \mathbf{R}\} = O(e^{-\beta \xi}) \qquad (\xi \to +\infty) \tag{2}$$

for every positive number β, then $f \equiv 0$ on ω.

If $f \not\equiv 0$, then since f is bounded on some half-plane contained in ω, the three lines theorem (see, e.g., [**5**, p. 93]) implies that $\log \sup\{|f(\xi + i\eta)| : \eta \in \mathbf{R}\} = \lambda(\xi)$, say, is a convex function of ξ on some interval $(a, +\infty)$, so that $\xi^{-1}\lambda(\xi)$ is bounded below for large ξ and (2) fails for some positive β.

Now suppose that the hypotheses of the theorem are satisfied. By translating the origin if necessary, we may suppose that h is bounded on Ω. We associate to h a holomorphic function g on ω. This technique has been used elsewhere (see, e.g., [**3**], [**4**]), but here we can exploit the boundedness of h to express g simply.

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For each positive number b, let

$$\Omega_b = \{(X, y) \in \mathbf{R}^{n+1} : y > b\}$$

and

$$\omega_b = \{z \in \mathbf{C} : \operatorname{Re} z > b\},$$

and write

$$g_b(z) = c\int_{\mathbf{R}^n} (z-b)(V(z, X))^{-(n+1)/2} h(X, b)\, dX \qquad (z \in \omega_b), \tag{3}$$

where $c = \pi^{-(n+1)/2}\Gamma(\frac{1}{2}n + \frac{1}{2})$ and

$$V(z, X) = (z-b)^2 + \|X\|^2 \qquad (z \in \omega_b,\ X \in \mathbf{R}^n).$$

It is easy to see that V never takes values on the non-positive real axis. Hence for each fixed X in $\mathbf{R}^n$ the integrand in (3) is holomorphic on ω_b. Further, writing $z = \xi + i\eta$, we have

$$\begin{aligned} |V(z, X)|^2 &= (\|X\|^2 - \eta^2)^2 + 2(\|X\|^2 + \eta^2)(\xi - b)^2 + (\xi - b)^4 \\ &\geq (\|X\|^2 - \eta^2)^2 + (\xi - b)^4. \end{aligned} \tag{4}$$

Hence for all z belonging to a fixed compact subset of ω_b, the modulus of the integrand in (3) is dominated by a constant multiple of $(1 + \|X\|)^{-n-1}$, which is integrable on $\mathbf{R}^n$. It follows that g_b is holomorphic on ω_b. Also, by (4), if $z \in \omega_{b+1}$, then

$$\begin{aligned} |g_b(z)| &\leq c\,|z-b|\,M(b)\left(\int_{\|X\|<|\eta|\sqrt{2}} dX + \int_{\|X\|\geq|\eta|\sqrt{2}} (1 + \|X\|^4/4)^{-(n+1)/4}\, dX\right) \\ &\leq CM(b)(1+|z|)^{n+1}, \end{aligned} \tag{5}$$

where C is a positive constant depending only on n. Next note that, since h is bounded on Ω, it is equal on Ω_b to the half-space Poisson integral with boundary values h on $\mathbf{R}^n \times \{b\}$, that is

$$h(Z, y) = c\int_{\mathbf{R}^n} (y-b)(y^2 + \|Z - X\|^2)^{-(n+1)/2} h(X, b)\, dX$$

when $(Z, y) \in \Omega_b$. In particular, denoting the origin of $\mathbf{R}^n$ by O, we have $h(O, y) = g_b(y)$ when $y > b$. Hence if $y > b > b' > 0$, then $g_b(y) = g_{b'}(y)$ and it follows, since g_b and $g_{b'}$ are holomorphic on ω_b, that $g_b = g_{b'}$, on ω_b. Hence a holomorphic function g on ω is defined by writing $g(\xi + i\eta) = g_{\xi/2}(\xi + i\eta)$, and we have $g(y) = h(O, y)$ for all positive y.

Now define f on ω by $f(z) = z^{-n-1}g(z)$. Then f is holomorphic on ω and if $z = \xi + i\eta \in \omega_2$, it follows from (5) that

$$\begin{aligned} |f(z)| &= |z|^{-n-1}\,|g_{\xi/2}(z)| \\ &\leq C(3/2)^{n+1}M(\xi/2). \end{aligned}$$

Hence (2) holds for every positive number β, and it follows from the lemma that $f \equiv 0$ on

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ω and so $g \equiv 0$ on ω. In particular,

$$h(O, y) = g(y) = 0 \qquad (y > 0).$$

By translating the axes, we find that $h = 0$ on any semi-infinite line in ω parallel to the y-axis. Hence $h \equiv 0$ on ω.

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