Canad. Math. Bull. Vol. **56** (1), 2013 pp. 116–126
http://dx.doi.org/10.4153/CMB-2011-131-6

Central Extensions of Loop Groups and Obstruction to Pre-Quantization

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Abstract. An explicit construction of a pre-quantum line bundle for the moduli space of flat G-bundles over a Riemann surface is given, where G is any non-simply connected compact simple Lie group. This work helps to explain a curious coincidence previously observed between Toledano Laredo's work classifying central extensions of loop groups LG and the author's previous work on the obstruction to pre-quantization of the moduli space of flat G-bundles.

1 Introduction

The moduli space $\mathcal{M}(\Sigma)$ of flat G-bundles over a surface Σ with one boundary component is known to admit a pre-quantization at integer levels[1] when the structure group G is a simply connected compact simple Lie group. If the structure group is not simply connected, however, integrality of the level does not guarantee the existence of a pre-quantization. It was found in [5] that for non-simply connected G, $\mathcal{M}(\Sigma)$ admits a pre-quantization if and only if the underlying level is an integer multiple of $l_0(G)$ listed in Table 1.1 for all non-simply connected, compact, simple Lie groups G.

G	$SU(n)/\mathbb{Z}_k$ $n \geq 2$	$PSp(n)$ $n \geq 1$	$SO(n)$ $n \geq 7$	$PO(2n)$ $n \geq 4$	$Ss(4n)$ $n \geq 2$	PE_6	PE_7
$l_0(G)$	$\mathrm{ord}_k(\frac{n}{k})$	1, n even 2, n odd	1	2, n even 4, n odd	1, n even 2, n odd	3	2

Table 1.1: The integer $l_0(G)$. Here, $\mathrm{ord}_k(x)$ denotes the order of x mod k in $\mathbb{Z}_k = \mathbb{Z}/k\mathbb{Z}$.

A curiosity observed in [5] is that the integer $l_0(G)$ also appears in Toledano Laredo's work [9], which classifies positive energy projective representations of loop groups LG for non-simply connected, compact, simple Lie groups G. To be more specific, Toledano Laredo classifies central extensions

$$1 \to U(1) \to \widehat{LG} \to LG \to 1,$$

Received by the editors April 5, 2010; revised July 3, 2010.
Published electronically June 29, 2011.
The author was supported by a NSERC Postdoctoral Fellowship.
AMS subject classification: **53D**, 22E.
Keywords: loop group, central extension, prequantization.

[1]The level $l > 0$ encodes a choice of invariant inner product on the simple Lie algebra $\mathfrak{g}$ of G.

showing they can only exist at levels that are integer multiples of the so-called *basic level* $l_b(G)$, which is then computed for each non-simply connected G (see [9, Proposition 3.5]). By inspection, it is easy to see that $l_0(G) = l_b(G)$, and this paper aims to understand this coincidence.

The main result of this work, which helps to account for the observed coincidence, is an explicit construction of a pre-quantum line bundle over the moduli space $\mathcal{M}(\Sigma)$ of flat G-bundles in the case when the structure group G is non-simply connected. The construction is an extension of the well-known constructions in the case when G is simply connected (see [6,8]). It also appears in [1] for non-simply connected G, although using unnecessary assumptions on the underyling level. The necessary and sufficient condition for pre-quantization, found in [5], is that the underlying level must be an integer multiple of $l_0(G)$. Using the equality $l_0(G) = l_b(G)$, we show that the construction appearing in [1] applies at these levels.

The obstruction to applying this construction of the pre-quantum line bundle in the case of non-simply connected structure group G is related to a central extension

$$1 \to U(1) \to \widehat{\Gamma} \to \Gamma \to 1, \tag{1.1}$$

where $\Gamma \cong \pi_1(G) \times \pi_1(G)$ (see (4.4) in). The proof of Theorem 4.2 shows that this extension is trivial precisely when the underlying level is an integer multiple of the basic level $l_b(G)$. As a consequence, when the level is an integer multiple of the basic level, the well-known construction of the pre-quantum line bundle applies.

This paper is organized as follows. Section 2 reviews some of the relevant background material about loop groups and establishes some notation used throughout the paper. Section 3 reviews the construction of the moduli space, paying special attention to the fact that the underlying structure group is not simply connected. Finally, Section 4 contains the main results of this work, which include a careful study of the central extensions of the gauge groups and Theorem 4.2 whose proof shows that non-triviality of the central extension (1.1) is the obstruction to constructing the pre-quantum line bundle. This last section also contains the construction of the pre-quantum line bundle under the conditions when the above central extension is trivial.

2 Preliminaries and Notation

In this section, we establish notation that will be used in the rest of this paper and review some relevant background material.

Let G be a simply connected, compact, simple Lie group with Lie algebra $\mathfrak{g}$, and let $T \subset G$ be a maximal torus with Lie algebra $\mathfrak{t} \subset \mathfrak{g}$. For a non-trivial subgroup Z of the center $Z(G)$, let $G' = G/Z$ with maximal torus $T' = T/Z$, which identifies the quotient map $\pi\colon G \to G'$ as the universal covering homomorphism, and $Z \cong \pi_1(G')$. (Recall that all non-simply connected, compact, simple Lie groups G' are of this form.)

Let $\Lambda = \ker \exp_T$ be the integer lattice for G, and let $\Lambda' = \ker \exp_{T'}$ be the integer lattice for G', so that $\Lambda \subset \Lambda'$ and $Z \cong \Lambda'/\Lambda$.

Let $B(\cdot\,,\,\cdot)$ denote the *basic inner product*, the invariant inner product on $\mathfrak{g}$ normalized to make short co-roots have length $\sqrt{2}$.

Following [6], throughout this paper we fix a real number $s > 1$. For a given manifold X (possibly with boundary) and $p \leq \dim X$, let $\Omega^p(X;\mathfrak{g})$ be the space of $\mathfrak{g}$-valued p-forms on X of Sobolev class $s - p + \dim X/2$. For a compact Lie group K with Lie algebra $\mathfrak{k}$, the space $\Omega^0(X;\mathfrak{k}) = \operatorname{Map}(X,\mathfrak{k})$ is the Lie algebra of the group $\operatorname{Map}(X,K)$ of maps of Soboloev class $s + \dim X/2$.

Loop Groups and Central Extensions

For a compact Lie group K with Lie algebra $\mathfrak{k}$, let LK denote the (free) loop space $\operatorname{Map}(S^1,K)$, viewed as an infinite dimensional Lie group, with Lie algebra $L\mathfrak{k} = \operatorname{Map}(S^1,\mathfrak{k})$.

Given an invariant inner product $(\cdot\,,\,\cdot)$ on $\mathfrak{k}$, define the central extension $\widehat{L\mathfrak{k}} := L\mathfrak{k} \oplus \mathbb{R}$ with Lie bracket

$$\big[(\xi_1,t_1),(\xi_2,t_2)\big] := \Big([\xi_1,\xi_2], \int_{S^1}(\xi_1, d\xi_2)\Big).$$

If it exists, let $\widehat{LK}$ denote a $U(1)$-central extension of LK with Lie algebra $\widehat{L\mathfrak{k}}$.

For $K = G$, it is well known (see [7, Theorem 4.4.1]) that central extensions $\widehat{LG}$ are classified by their *level l*—the unique multiple of the basic inner product that coincides with the chosen inner product—which is required to be a positive integer. (Since G is simple, any invariant inner product on $\mathfrak{g}$ is necessarily of the form $lB(\cdot\,,\,\cdot)$ for some $l > 0$, called the *level*.)

For $K = G'$, however, central extensions $\widehat{LG'}$ are classified by their level l, which is required to be an integer multiple of $l_b(G')$, and a character $\chi\colon Z \to U(1)$ (see [9, Proposition 3.4]). The integer $l_b(G')$ is defined as follows.

Definition 2.1 Let G' be a compact simple Lie group with integer lattice Λ'. The *basic level* $l_b(G')$ is the smallest integer l such that the restriction of $lB(\cdot\,,\,\cdot)$ to Λ' is integral.

As mentioned in the introduction, $l_b(G') = l_0(G')$, which appears in Table 1.1 for each non-simply connected, compact, simple Lie group G'.

Let $L\mathfrak{g}^* = \Omega^1(S^1;\mathfrak{g})$, sometimes called the *smooth dual* of $L\mathfrak{g}$. The pairing $L\mathfrak{g} \times L\mathfrak{g}^* \to \mathbb{R}$ given by $(\xi, A) \mapsto \int_{S^1}(\xi, A)$ induces an inclusion $L\mathfrak{g}^* \subset (L\mathfrak{g})^*$. Additionally, define $\widehat{L\mathfrak{g}}^* := L\mathfrak{g}^* \oplus \mathbb{R}$ and consider the pairing $\widehat{L\mathfrak{g}} \times \widehat{L\mathfrak{g}}^* \to \mathbb{R}$ given by

$$\big((\xi,a),(A,t)\big) = \int_{S^1}(\xi,A) + at.$$

Since the central subgroup $U(1) \subset \widehat{LG}$ acts trivially on $\widehat{L\mathfrak{g}}$, the coadjoint representation of $\widehat{LG}$ factors through LG. The coadjoint action of LG on $\widehat{L\mathfrak{g}}^*$ is (see [7, Proposition 4.3.3]),

$$g \cdot (A,t) = \big(\operatorname{Ad}_g(A) - tg^*\theta^R, t\big),$$

where θ^R denotes the right-invariant Maurer–Cartan form on G.

Notice that for each real number λ, the hyperplanes $t = \lambda$ are fixed. Identifying $L\mathfrak{g}^*$ with $L\mathfrak{g}^* \times \{\lambda\} \subset \widehat{L\mathfrak{g}}^*$ yields an action of LG on $L\mathfrak{g}^*$, called the (affine) level λ action.

3 The Moduli Space of Flat Connections $\mathcal{M}'(\Sigma)$

In this section, we review the construction of the moduli space of flat connections following [1], with special attention to the case where G' is a non-simply connected, compact, simple Lie group. The reader may wish to consult [1,2,6] and the references therein for more details.

Let Σ denote a compact oriented surface of genus h with 1 boundary component. The affine space of connections $\mathcal{A}(\Sigma) = \Omega^1(\Sigma, \mathfrak{g})$ on the trivial G'-bundle over Σ admits an action of $\mathrm{Map}(\Sigma, G')$, the space of maps $g\colon \Sigma \to G'$, by gauge transformations $g \cdot A = \mathrm{Ad}_g A - g^*\theta^R$. The kernel of the restriction map

$$\mathrm{Map}(\Sigma, G') \to \mathrm{Map}(\partial\Sigma, G'), \qquad g \mapsto g|_{\partial\Sigma}$$

will be denoted $\mathrm{Map}_\partial(\Sigma, G')$. Define the moduli space of flat G'-connections up gauge transformations whose restriction to $\partial\Sigma$ is trivial by

$$\mathcal{M}'(\Sigma) := \mathcal{A}_{\mathrm{flat}}(\Sigma)/\mathrm{Map}_\partial(\Sigma, G').$$

The Atiyah–Bott [2] symplectic structure on $\mathcal{M}'(\Sigma)$ is obtained by symplectic reduction (as in [4, Chapter 23]), viewing the moduli space as a symplectic quotient of the affine space of connections $\mathcal{A}(\Sigma)$. Recall that the affine space $\mathcal{A}(\Sigma)$ carries a symplectic form $\omega_{\mathcal{A}}(a_1, a_2) = \int_\Sigma lB(a_1, a_2)$ and a Hamiltonian action of $\mathrm{Map}_\partial(\Sigma, G')$ with momentum map the curvature; therefore, the zero level set of the moment map is the space of flat connections $\mathcal{A}_{\mathrm{flat}}(\Sigma)$, and hence the resulting symplectic quotient is the moduli space $\mathcal{M}'(\Sigma)$.

The moduli space $\mathcal{M}'(\Sigma)$ carries an action by LG that can be described as follows. For $g \in \mathrm{Map}(\Sigma, G')$, the restriction $g|_{\partial\Sigma}$ is a contractible loop in G', since $\pi_1(G')$ is Abelian and $\partial\Sigma$ is homotopic to a product of commutators $\prod a_i b_i a_i^{-1} b_i^{-1}$ for loops a_i, b_i representing generators of $\pi_1(\Sigma)$. Thus the restriction map takes values in the identity component $\mathrm{Map}_0(\partial\Sigma, G')$, which, after choosing a parametrization $\partial\Sigma \cong S^1$, can be identified with the identity component L_0G' of the loop group LG'. The LG action on $\mathcal{M}'(\Sigma)$ is then defined using the natural projection $L\pi\colon LG \to LG'$ that takes values in L_0G', and the identification $\mathrm{Map}(\Sigma, G')/\mathrm{Map}_\partial(\Sigma, G') \cong L_0G'$. The LG action is Hamiltonian, with momentum map $\Phi'\colon \mathcal{M}'(\Sigma) \to L\mathfrak{g}^*$ given by pulling back the connection to the boundary.

The corresponding moduli space $\mathcal{M}(\Sigma) = \mathcal{A}_{\mathrm{flat}}/\mathrm{Map}_\partial(\Sigma, G)$ with simply connected structure group G is a finite covering of $\mathcal{M}'(\Sigma)$. This is a consequence of the following proposition found in [1].

Proposition 3.1 *The following sequences are exact:*

(3.1) $$1 \to Z \to \mathrm{Map}(\Sigma, G) \to \mathrm{Map}(\Sigma, G') \to Z^{2h} \to 1,$$

(3.2) $$1 \to \mathrm{Map}_\partial(\Sigma, G) \to \mathrm{Map}_\partial(\Sigma, G') \to Z^{2h} \to 1.$$

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In sequences (3.1) and (3.2), the maps into Z^{2h} are defined by sending $g \mapsto g_\sharp$ in $\mathrm{Hom}(\pi_1(\Sigma), \pi_1(G')) \cong Z^{2h}$. Since $A \in \mathcal{A}(\Sigma)$ may be viewed as either a G-connection or a G'-connection on the corresponding trivial bundle over Σ, the moduli space $\mathcal{M}(\Sigma)$ admits a residual $Z^{2h} \cong \mathrm{Map}_\partial(\Sigma, G')/\mathrm{Map}_\partial(\Sigma, G)$ action, identifying $\mathcal{M}'(\Sigma) = \mathcal{M}(\Sigma)/Z^{2h}$. Also, the momentum map $\Phi \colon \mathcal{M}(\Sigma) \to L\mathfrak{g}^*$ is clearly invariant under the Z^{2h}-action and descends to the momentum map $\Phi' \colon \mathcal{M}'(\Sigma) \to L\mathfrak{g}^*$ above. Viewed this way, Φ' sends an equivalence class of G'-connections to its restriction to the boundary, considered as a G-connection on $\partial\Sigma$.

For $\mu \in L\mathfrak{g}^*$, the symplectic quotient

$$\mathcal{M}(\Sigma)_\mu := \Phi^{-1}(LG \cdot \mu)/LG$$

represents the moduli space of flat connections on the trivial G bundle over Σ whose restriction to the boundary is gauge equivalent to μ. Equivalently, $\mathcal{M}(\Sigma)_\mu$ is the moduli space of flat connections on the trivial G-bundle whose holonomy along the boundary is conjugate to $\mathrm{Hol}(\mu)$. Similarly, the symplectic quotient $\mathcal{M}'(\Sigma)_\mu = (\Phi')^{-1}(LG \cdot \mu)/LG$ represents the moduli space of flat connections on the trivial G'-bundle over Σ whose holonomy along the boundary, when viewed as a G-connection on $\partial\Sigma$, is conjugate to $\mathrm{Hol}(\mu)$.

The connected components of the moduli space of flat G'-bundles over a *closed* surface may then be described in terms of the symplectic quotients $\mathcal{M}'(\Sigma)_\mu$ with $\mathrm{Hol}(\mu) \in Z$. To see this, let $\widehat{\Sigma}$ be the closed surface obtained by gluing a disc D to Σ by identifying boundaries. Recall that there is a bijective correspondence between isomorphism classes of principal G'-bundles $P \to \widehat{\Sigma}$ and $\pi_1(G') \cong Z$: every such bundle $P \to \widehat{\Sigma}$ is isomorphic to one that can be constructed by gluing together trivial bundles over both Σ and D with some transition function $f \colon S^1 = \Sigma \cap D \to G'$. By [3, Proposition 4.33], the holonomy around $\partial\Sigma$ of a flat connection on P coincides with $[f] \in \pi_1(G') \cong Z$. It follows that the moduli space $M_{G'}(\widehat{\Sigma})$ of flat G'-bundles over a closed surface $\widehat{\Sigma}$ up to gauge transformations may be written as the (disjoint)! union of the symplectic quotients $\mathcal{M}'(\Sigma)_\mu$, where $\mathrm{Hol}(\mu) \in Z$.

4 The Pre-Quantum Line Bundle $L'(\Sigma) \to \mathcal{M}'(\Sigma)$

In this section, we construct a pre-quantum line bundle $L'(\Sigma) \to \mathcal{M}'(\Sigma)$, which is an adaptation of a well-known construction in the case where the underlying structure group is simply connected (see [6, 8]). The construction appears in [1] (using unnecessary assumptions on the underlying level). The main contribution here is to verify that this construction applies under the necessary and sufficient conditions obtained in [5]. For simplicity, we consider the case of genus $h = 1$.

Central Extensions of the Gauge Group

An important part of the construction of the pre-quantum line bundle is a careful discussion of certain central extensions of various gauge groups.

Recall that the cocycle defined by the formula $c(g_1, g_2) = \exp i\pi \int_\Sigma lB(g_1^*\theta^L, g_2^*\theta^R)$

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defines central extensions

$$\begin{aligned} 1 \to U(1) \to \widehat{\mathrm{Map}}(\Sigma, G) \to \mathrm{Map}(\Sigma, G) \to 1, \\ 1 \to U(1) \to \widehat{\mathrm{Map}}(\Sigma, G') \to \mathrm{Map}(\Sigma, G') \to 1. \end{aligned} \tag{4.1}$$

It is known (see [6, p. 431]) that when l is an integer, the restriction of the central extension $\widehat{\mathrm{Map}}(\Sigma, G)$ to the subgroup $\mathrm{Map}_\partial(\Sigma, G)$ is trivial; that is, the exact sequence

$$1 \to U(1) \to \widehat{\mathrm{Map}}_\partial(\Sigma, G) \to \mathrm{Map}_\partial(\Sigma, G) \to 1 \tag{4.2}$$

splits, and we may view $\mathrm{Map}_\partial(\Sigma, G)$ as a subgroup of $\widehat{\mathrm{Map}}(\Sigma, G)$.

More precisely, the section $\sigma\colon \mathrm{Map}_\partial(\Sigma, G) \to \widehat{\mathrm{Map}}_\partial(\Sigma, G)$, $g \mapsto (g, \alpha(g))$ composed with the inclusion $\widehat{\mathrm{Map}}_\partial(\Sigma, G) \hookrightarrow \widehat{\mathrm{Map}}(\Sigma, G)$ embeds $\mathrm{Map}_\partial(\Sigma, G)$ as a normal subgroup in $\widehat{\mathrm{Map}}(\Sigma, G)$, where $\alpha\colon \mathrm{Map}_\partial(\Sigma, G) \to U(1)$ is defined as follows. For $g \in \mathrm{Map}_\partial(\Sigma, G)$, choose a homotopy $H\colon \Sigma \times [0, 1] \to G$ with $H_0 = g$, $H_1 = e$ and $H_t|_{\partial\Sigma} = e$ for $0 \le t \le 1$ and define

$$\alpha(g) = \exp \frac{-i\pi}{6} \cdot l \int_{\Sigma \times [0,1]} H^* \eta,$$

where $\eta = B(\theta^L, [\theta^L, \theta^L])$ denotes the canonical invariant 3-form on G. It is straightforward to check that α is well defined and satisfies the coboundary relation

$$\alpha(g_1 g_2) = \alpha(g_1)\alpha(g_2)c(g_1, g_2)$$

so that σ is indeed a section. That we may view $\mathrm{Map}_\partial(\Sigma, G)$ as a normal subgroup of $\widehat{\mathrm{Map}}(\Sigma, G)$ is also straightforward (*cf.* Lemma 4.1 and the proof of Corollary 4.3).

Therefore, one obtains the central extension

$$1 \to U(1) \to \widehat{\mathrm{Map}}(\Sigma, G)/\mathrm{Map}_\partial(\Sigma, G) \to LG \to 1$$

using the identification $LG \cong \mathrm{Map}(\Sigma, G)/\mathrm{Map}_\partial(\Sigma, G)$.

Assume that l is an integer. Under additional restrictions on l described in Theorem 4.2, the same holds for the central extension $\widehat{\mathrm{Map}}(\Sigma, G')$ in (4.1) and we obtain a central extension

$$1 \to U(1) \to \widehat{\mathrm{Map}}(\Sigma, G')/\mathrm{Map}_\partial(\Sigma, G') \to L_0 G' \to 1$$

using the identification $L_0 G' \cong \mathrm{Map}(\Sigma, G')/\mathrm{Map}_\partial(\Sigma, G')$.

Lemma 4.1 *Let* $\widehat{\mathrm{Map}}_\partial(\Sigma, G')$ *denote the restriction of the central extension* (4.1) *to* $\mathrm{Map}_\partial(\Sigma, G')$. *Using the section* $\sigma\colon \mathrm{Map}_\partial(\Sigma, G) \to \widehat{\mathrm{Map}}_\partial(\Sigma, G)$ *above and the inclusion* $\widehat{\mathrm{Map}}_\partial(\Sigma, G) \to \widehat{\mathrm{Map}}_\partial(\Sigma, G')$ *induced from the inclusion in* (3.2), *we may embed* $\mathrm{Map}_\partial(\Sigma, G)$ *as a normal subgroup in* $\widehat{\mathrm{Map}}_\partial(\Sigma, G')$.

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Proof The inclusion $\mathrm{Map}_\partial(\Sigma, G) \to \widehat{\mathrm{Map}}_\partial(\Sigma, G')$ is given by $g \mapsto (\pi g, \alpha(g))$, where $\pi\colon G \to G'$ is the universal covering homomorphism. To verify that this includes $\mathrm{Map}_\partial(\Sigma, G)$ as a normal subgroup, a direct calculation shows that it suffices to verify that for $g \in \mathrm{Map}_\partial(\Sigma, G)$ and $h \in \mathrm{Map}_\partial(\Sigma, G')$,

$$\alpha(h\pi g h^{-1}) = c(h, \pi g h^{-1}) c(\pi g, h^{-1}) \alpha(g). \tag{4.3}$$

(Note that $c(h, h^{-1}) = 1$, since $(h^*\theta^L, (h^{-1})^*\theta^R) = -h^*(\theta^L, \theta^L) = 0$.) Note that $h\pi g h^{-1}$ is clearly in $\mathrm{Map}_\partial(\Sigma, G)$ (using the inclusion of (3.2)) so that $\alpha(h\pi g h^{-1})$ is defined.

To compute $\alpha(h\pi g h^{-1})$, let $F\colon \Sigma \times [0,1] \to G$ be a homotopy for g such that $F_0 = g$, $F_1 = e$, and $F_t|_{\partial\Sigma} = e$ and let $H\colon \Sigma \times [0,1] \to G'$ be the homotopy $H(p,t) = h(p)\pi F(p,t) h(p)^{-1}$. Since $\pi\colon G \to G'$ is a covering projection, we may lift H to a homotopy $\tilde{H}\colon \Sigma \times [0,1] \to G$, and find that

$$\alpha(h\pi g h^{-1}) = \exp \frac{-i\pi}{6} \cdot l \int_{\Sigma \times [0,1]} \tilde{H}^* \eta = \exp \frac{-i\pi}{6} \cdot l \int_{\Sigma \times [0,1]} (h\pi F h^{-1})^* \eta.$$

A direct calculation now verifies that equation (4.3) holds. (See the proof of Corollary 4.3 for a sketch of a similar calculation.) ■

Theorem 4.2 *The restriction of the central extension* (4.1) *to* $\mathrm{Map}_\partial(\Sigma, G')$ *splits if the underlying level l is a multiple of the basic level $l_b(G')$.*

Proof It will be useful in what follows to choose representative loops in $T' \subset G'$ for elements of $Z \cong \pi_1(G')$. For each $z \in Z \cong \Lambda'/\Lambda$ let $\zeta_z \in \Lambda'$ be a (minimal dominant co-weight) representative for z. In particular, $\exp \zeta_z = z \in T \subset G$, and the loop $\zeta_z(t) = \exp(t\zeta_z)$ in $T' \subset G'$ represents z viewed as an element of $\pi_1(G')$.

For $\mathbf{z} = (z_1, z_2) \in Z \times Z$, construct a map $g_{\mathbf{z}}\colon \Sigma \to G'$ in $\mathrm{Map}_\partial(\Sigma, G')$ as follows. View the surface Σ as the quotient of the pentagon with oriented sides identified according to the word $aba^{-1}b^{-1}c$. Define $g\colon S^1 \to T'$ on the boundary of the pentagon so that $g|_a = \zeta_{z_1}$, $g|_b = \zeta_{z_2}$ and $g|_c = 1$. Since $\pi_1(T)$ is abelian, g is null homotopic and can be extended to the pentagon, defining $g_{\mathbf{z}}\colon \Sigma \to T' \to G'$. Note that the induced map $(g_{\mathbf{z}})_\sharp\colon \pi_1(\Sigma) \to \pi_1(G')$ satisfies $(g_{\mathbf{z}})_\sharp(a) = z_1$ and $(g_{\mathbf{z}})_\sharp(b) = z_2$, and hence $(g_{\mathbf{z}})_\sharp = \mathbf{z}$ in sequence (3.2).

Since sequence (4.2) splits, and by Lemma 4.1 we may view $\mathrm{Map}_\partial(\Sigma, G)$ as a normal subgroup of $\widehat{\mathrm{Map}}_\partial(\Sigma, G')$, the restriction of the central extension (4.1) to $\mathrm{Map}_\partial(\Sigma, G')$. Hence, by the exact sequence (3.2), we obtain a central extension

$$1 \to U(1) \to \widehat{\mathrm{Map}}_\partial(\Sigma, G')/\mathrm{Map}_\partial(\Sigma, G) \to Z \times Z \to 1. \tag{4.4}$$

Therefore, the central extension $\widehat{\mathrm{Map}}_\partial(\Sigma, G')$ fits in the following pullback diagram:

$$\begin{array}{ccc} \widehat{\mathrm{Map}}_\partial(\Sigma, G') & \longrightarrow & \widehat{\mathrm{Map}}_\partial(\Sigma, G')/\mathrm{Map}_\partial(\Sigma, G) \\ \downarrow & & \downarrow \\ \mathrm{Map}_\partial(\Sigma, G') & \longrightarrow & Z \times Z \end{array}$$

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where the map on the bottom of the square is the one appearing in (3.2). It follows that the central extension $\widehat{\mathrm{Map}}_\partial(\Sigma, G')$ splits if the central extension (4.4) is trivial.

Central $U(1)$-extensions over the abelian group $\Gamma = Z \times Z$ are determined by their commutator pairing $q\colon \Gamma \times \Gamma \to U(1)$. (In general, a trivial commutator pairing would only show that the given extension is abelian. However, abelian $U(1)$-extensions are necessarily trivial, since $U(1)$ is divisible.) For $\mathbf{z}$ and $\mathbf{w}$ in $Z \times Z$, recall that the commutator pairing is defined by

$$q(\mathbf{z}, \mathbf{w}) = \widehat{\mathbf{z}}\widehat{\mathbf{w}}\widehat{\mathbf{z}}^{-1}\widehat{\mathbf{w}}^{-1},$$

where $\widehat{\mathbf{z}}$ and $\widehat{\mathbf{w}}$ in $\widehat{\mathrm{Map}}_\partial(\Sigma, G')/\mathrm{Map}_\partial(\Sigma, G)$ are arbitrary lifts of $\mathbf{z}$ and $\mathbf{w}$ respectively.

Next, we compute the commutator pairing q and determine when it is trivial. To that end, let $g_{\mathbf{z}}$ and $g_{\mathbf{w}}$ be constructed as above. Then since $g_{\mathbf{z}}$ and $g_{\mathbf{w}}$ lie in T', $g_{\mathbf{z}} g_{\mathbf{w}} = g_{\mathbf{w}} g_{\mathbf{z}}$, and

$$(g_{\mathbf{z}}, 1)(g_{\mathbf{w}}, 1)(g_{\mathbf{z}}, 1)^{-1}(g_{\mathbf{w}}, 1)^{-1} = \big(1, c(g_{\mathbf{z}}, g_{\mathbf{w}})c(g_{\mathbf{w}}, g_{\mathbf{z}})^{-1}\big).$$

Therefore,

$$\begin{aligned} q(\mathbf{z}, \mathbf{w}) = c(g_{\mathbf{z}}, g_{\mathbf{w}})c(g_{\mathbf{w}}, g_{\mathbf{z}})^{-1} &= \exp \pi i \int_\Sigma \big(lB(g_{\mathbf{z}}^*\theta^L, g_{\mathbf{w}}^*\theta^R) - lB(g_{\mathbf{w}}^*\theta^L, g_{\mathbf{z}}^*\theta^R)\big) \\ &= \exp 2\pi i \int_\Sigma lB(g_{\mathbf{z}}^*\theta, g_{\mathbf{w}}^*\theta), \end{aligned}$$

where θ denotes the Maurer–Cartan form on the torus T'.

By collapsing the boundary of Σ to a point, we map view the maps $g_{\mathbf{z}}$ and $g_{\mathbf{w}}$ as maps from the 2-torus $T^2 \to T'$. If ω denotes the standard symplectic form on T^2 with unit symplectic volume, then $lB(g_{\mathbf{z}}^*\theta, g_{\mathbf{w}}^*\theta) = (lB(\zeta_{z_1}, \zeta_{w_2}) - lB(\zeta_{z_2}, \zeta_{w_1}))\omega$. Indeed,

$$\begin{aligned} &(g_{\mathbf{z}}^*\theta, g_{\mathbf{w}}^*\theta)((u_1, u_2), (v_1, v_2)) \\ &\quad = lB\Big(\theta\big(g_{\mathbf{z}*}(u_1, u_2)\big), \theta\big(g_{\mathbf{w}*}(v_1, v_2)\big)\Big) - lB\Big(\theta\big(g_{\mathbf{z}*}(v_1, v_2)\big), \theta\big(g_{\mathbf{w}*}(u_1, u_2)\big)\Big) \\ &\quad = lB(u_1\zeta_{z_1} + u_2\zeta_{z_2}, v_1\zeta_{w_1} + v_2\zeta_{w_2}) - lB(v_1\zeta_{z_1} + v_2\zeta_{z_2}, u_1\zeta_{w_1} + u_2\zeta_{w_2}) \\ &\quad = \big(lB(\zeta_{z_1}, \zeta_{w_2}) - lB(\zeta_{z_2}, \zeta_{w_1})\big)(u_1v_2 - v_1u_2). \end{aligned}$$

Therefore,

$$q(\mathbf{z}, \mathbf{w}) = \exp 2\pi i \big(lB(\zeta_{z_1}, \zeta_{w_2}) - lB(\zeta_{w_1}, \zeta_{z_2})\big),$$

and q is trivial if and only if l is a multiple of the basic level $l_b(G')$. ■

Corollary 4.3 *If the level is an integer multiple of the basic level, there is a central extension*

$$1 \to U(1) \to \widehat{\mathrm{Map}}(\Sigma, G')/\mathrm{Map}_\partial(\Sigma, G') \to L_0G'.$$

Proof As in the proof of Theorem 4.2, at any integer level, the central extension

$$1 \to U(1) \to \widehat{\mathrm{Map}}_\partial(\Sigma, G') \to \mathrm{Map}_\partial(\Sigma, G') \to 1$$

is the pullback of the central extension (4.4) over the abelian group $Z \times Z$. Moreover, if the underlying level is a multiple of the basic level, the proof of Theorem 4.2 shows that this extension is abelian and hence split.

Each choice of section $\delta \colon Z \times Z \to \widehat{\mathrm{Map}}_\partial(\Sigma, G')/\mathrm{Map}_\partial(\Sigma, G)$ of the central extension (4.4) induces a canonical section $s \colon \mathrm{Map}_\partial(\Sigma, G') \to \widehat{\mathrm{Map}}_\partial(\Sigma, G')$ as follows. For $g \in \mathrm{Map}_\partial(\Sigma, G')$, write $\delta(g_\sharp) = [(h, z)]$. Since $h_\sharp = g_\sharp$, by the exactness of (3.2), there is a unique $a \in \mathrm{Map}_\partial(\Sigma, G)$ with $h\pi a = g$. Define

$$s(g) = \big(g, c(h, \pi a) z \alpha(a)\big).$$

It is easy to check that s is well-defined and is indeed a section. It remains to verify that the induced inclusion

$$\mathrm{Map}_\partial(\Sigma, G') \xrightarrow{s} \widehat{\mathrm{Map}}_\partial(\Sigma, G') \hookrightarrow \widehat{\mathrm{Map}}(\Sigma, G')$$

includes $\mathrm{Map}_\partial(\Sigma, G')$ as a normal subgroup.

To that end, observe first that it suffices to check that $\mathrm{Map}_\partial(\Sigma, G')$ is closed under conjugation by elements of $\widehat{\mathrm{Map}}(\Sigma, G')$ in the image of $\widehat{\mathrm{Map}}(\Sigma, G) \to \widehat{\mathrm{Map}}(\Sigma, G')$ induced from (3.1). Indeed, sequences (3.1) and (3.2) show that each k in $\mathrm{Map}(\Sigma, G')$ can be expressed as $k = \pi x f$, where $f \in \mathrm{Map}_\partial(\Sigma, G')$ satisfies $k_\sharp = f_\sharp$ and $x \in \mathrm{Map}(\Sigma, G)$.

Let $g \in \mathrm{Map}_\partial(\Sigma, G')$ and choose $x \in \mathrm{Map}(\Sigma, G)$. Then

$$(\pi x, w) s(g) (\pi x, w)^{-1} = \big(\pi x g \pi x^{-1}, c(\pi x g, \pi x^{-1}) c(\pi x, g) c(h, \pi a) z \alpha(a)\big),$$

where $\delta(g_\sharp) = [(h, z)]$ and $h\pi a = g$ for $a \in \mathrm{Map}_\partial(\Sigma, G)$. Since $(\pi x g \pi x^{-1})_\sharp = g_\sharp$, $s(\pi x g \pi x^{-1}) = (\pi x g \pi x^{-1}, c(h, a') z \alpha(a'))$, where $\pi x g \pi x^{-1} = h a'$. Therefore we must verify that

$$c(\pi x g, \pi x^{-1}) c(\pi x, g) c(h, \pi a) \alpha(a) = c(h, a') \alpha(a'),$$

which, since $a' = a \cdot g^{-1} \pi x g \pi x^{-1}$, simplifies to

$$c(\pi x, g \pi x^{-1}) c(\pi x, g) = c(g, g^{-1} \pi x g \pi x^{-1}) \alpha(g^{-1} \pi x g \pi x^{-1}). \tag{4.5}$$

In order to compute $\alpha(g^{-1} \pi x g \pi x^{-1})$ in (4.5), let $F \colon \Sigma \times [0,1] \to G$ be a homotopy such that $F_0 = x$ and $F_1 = e$. (Such a homotopy exists, since G is 2-connected.) Let $H \colon \Sigma \times [0,1] \to G'$ be defined by $H(p,t) = g(p)^{-1} \pi F(p,t) g(p) \pi F(p,t)^{-1}$, and argue as in the proof of Lemma 4.1 that

$$\alpha(g^{-1} \pi x g \pi x^{-1}) = \exp \frac{-i\pi}{6} \int_{\Sigma \times [0,1]} (g \pi F g^{-1} \pi F^{-1})^* \eta.$$

A direct calculation verifies that equation (4.5) holds.

https://doi.org/10.4153/CMB-2011-131-6 Published online by Cambridge University Press

The main strategy to verify (4.5) is to recognize $\rho = (g\pi F g^{-1}\pi F^{-1})^*\eta$ as a coboundary $\rho = d\tau$ and use Stokes' Theorem, so that

$$\int_{\Sigma\times[0,1]} \rho = \int_{\partial\Sigma\times[0,1]} \tau + \int_{\Sigma\times 0} \tau + \int_{\Sigma\times 1} \tau,$$

where

$$\tfrac{1}{6}\tau = B\big((\pi F)^*\theta^L, (g\pi F^{-1})^*\theta^R\big) + B\big((\pi F)^*\theta^L, g^*\theta^R\big) - B\big(g^*\theta^L, (g^{-1}\pi F g\pi F^{-1})^*\theta^R\big).$$

The first term does not contribute because $g|_{\partial\Sigma} = e$, and the third term above does not contribute because $F_1 = e$. ■

The Pre-Quantum Line Bundle

As mentioned in the introduction, the construction of the pre-quantum line bundle over $\mathcal{M}'(\Sigma)$ appears in [1]. Nevertheless, the main steps in the construction are summarized next, focussing on the obstruction related to central extensions of the gauge group.

The pre-quantum line bundle $L'(\Sigma) \to \mathcal{M}'(\Sigma)$ is obtained through a reduction procedure. Recall that $\widehat{\mathrm{Map}}(\Sigma, G')$ acts on the trivial bundle $\mathcal{A}(\Sigma) \times \mathbb{C}$ by

$$(g, w)\cdot(A, a) = \Big(g\cdot A, \exp\big(-i\pi\int_\Sigma lB(g^*\theta^L, A)\big)\, wa\Big).$$

The 1-form $\alpha \mapsto \frac{1}{2}\int_\Sigma lB(A,\alpha)$ on $\mathcal{A}(\Sigma)$ defines an invariant connection, whose curvature can be verified to be $\omega_{\mathcal{A}}$.

By Corollary 4.3, when l is a multiple of $l_b(G')$ (see Definition 2.1), the central extension $\widehat{\mathrm{Map}}_\partial(\Sigma, G') \subset \widehat{\mathrm{Map}}(\Sigma, G')$ splits, and we may define the pre-quantum line bundle over $\mathcal{M}'(\Sigma)$ by

$$L'(\Sigma) = (\mathcal{A}_{\text{flat}}(\Sigma)\times\mathbb{C})/\mathrm{Map}_\partial(\Sigma, G').$$

As in the proof of Corollary 4.3, each choice of splitting of the central extension (4.4) induces a splitting of the central extension $\widehat{\mathrm{Map}}_\partial(\Sigma, G')$ over $\mathrm{Map}_\partial(\Sigma, G')$ used in the above construction. Since any two sections of the central extension (4.4) differ by a character $Z\times Z \to U(1)$, it is not hard to see that the set of pre-quantum line bundles are therefore in one-to-one correspondence with a group of characters $\mathrm{Hom}(Z\times Z, U(1))$ (*cf.* [1, Theorem 4.1(b)]).

Finally, note that since the symplectic quotients $\mathcal{M}'(\Sigma)_\mu$, where $\mathrm{Hol}(\mu)\in Z$, are the connected components of the moduli space $M_{G'}(\widehat{\Sigma})$ of flat G'-bundles over the closed surface $\widehat{\Sigma}$ (see the end of Section 3), the pre-quantum line bundle $L'(\Sigma)$ descends to a pre-quantization of $M_{G'}(\widehat{\Sigma})$.

Acknowledgements The author is grateful to E. Meinrenken for very useful conversations and to the referee for insightful comments.

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