Glasgow Math. J. **52** (2010) 227–240. © Glasgow Mathematical Journal Trust 2009.
doi:10.1017/S0017089509990292.

SOBOLEV INEQUALITIES FOR ORLICZ SPACES OF TWO VARIABLE EXPONENTS

PETER HÄSTÖ
Department of Mathematical Sciences, PO Box 3000, FI-90014, University of Oulu, Oulu, Finland
e-mail: peter.hasto@helsinki.fi

YOSHIHIRO MIZUTA
Department of Mathematics, Graduate School of Science, Hiroshima University, Higashi-Hiroshima 739-8521, Japan
e-mail: yomizuta@hiroshima-u.ac.jp

TAKAO OHNO
General Arts, Hiroshima National College of Maritime Technology, Higashino Oosakikamijima Toyotagun 725-0231, Japan
e-mail: ohno@hiroshima-cmt.ac.jp

and TETSU SHIMOMURA
Department of Mathematics, Graduate School of Education, Hiroshima University, Higashi-Hiroshima 739-8524, Japan
e-mail: tshimo@hiroshima-u.ac.jp

(Received 22 October 2008; accepted 1 June 2009; first published online 25 November 2009)

Abstract. Our aim in this paper is to deal with Sobolev's embeddings for Sobolev–Orlicz functions with $\nabla u \in L^{p(\cdot)} \log L^{q(\cdot)}(\Omega)$ for $\Omega \subset \mathbb{R}^n$. Here p and q are variable exponents satisfying natural continuity conditions. Also the case when p attains the value 1 in some parts of the domain is included in the results.

2000 *Mathematics Subject Classification.* 46E30

1. Introduction. Variable exponent spaces have been studied in many papers over the past decade; for a survey see [**6, 21**]. These investigations have dealt with the spaces themselves, with related differential equations and with applications. One typical feature is that the exponent has to be strictly bounded away from various critical values. More concretely, consider the example of the Sobolev embedding theorem. Such embeddings and embeddings of Riesz potentials have been studied, e.g., in [**1, 3, 5, 6, 9, 11, 14, 15, 18, 22**] in the variable exponent setting. Most proofs in the literature are based on the Riesz potential and maximal functions and thus lead to the additional, unnatural restriction $\inf p > 1$.

Early papers owing to Edmunds and Rákosník [**7, 8**] avoided this restriction by a use of ad hoc methods of proofs, but it turned out that these do not extend conveniently to other situations. Recently, Harjulehto and Hästö [**12**] introduced a method based on a weak-type estimate which covers the case $\inf p = 1$ and can be easily adopted also to other situations. Their result was extended to the case of unbounded domains in [**13**].

In this paper we consider more general variable exponents following Cruz-Uribe and Fiorenza [4]. To define these spaces let $p : \mathbb{R}^n \to [1,\infty)$ and $q : \mathbb{R}^n \to \mathbb{R}$ be continuous functions. We will be considering spaces of type $L^{p(\cdot)}\log L^{q(\cdot)}(\Omega)$. For simplicity we denote the function defining the space by Φ throughout the paper, i.e. $\Phi(x,t) = t^{p(x)}(\log(c_0+t))^{q(x)}$. By C we denote a generic constant whose value may change between appearances even within a single line.

We assume throughout the paper that our variable exponents p and q are continuous functions on $\mathbb{R}^n$, satisfying

(p1) $1 \leqslant p^- := \inf_{x\in\mathbb{R}^n} p(x) \leqslant \sup_{x\in\mathbb{R}^n} p(x) =: p^+ < \infty$;

(p2) $|p(x)-p(y)| \leqslant \dfrac{C}{\log(e+1/|x-y|)}$ whenever $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$;

(p3) $|p(x)-p(y)| \leqslant \dfrac{C}{\log(e+|x|)}$ whenever $|y| \geqslant |x|/2$;

(q1) $-\infty < q^- := \inf_{x\in\mathbb{R}^n} q(x) \leqslant \sup_{x\in\mathbb{R}^n} q(x) =: q^+ < \infty$;

(q2) $|q(x)-q(y)| \leqslant \dfrac{C}{\log(e+\log(e+1/|x-y|))}$ whenever $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$.

Moreover, we assume that

(Φ_1) there exists $c_0 \in [e,\infty)$ such that $\Phi(x,\cdot)$ is convex on $[0,\infty)$ for every $x \in \mathbb{R}^n$.

If there is a positive constant C_0 such that

$$C_0\,(p(x)-1)+q(x) \geqslant 0,$$

then condition (Φ_1) holds; this follows from a computation of the second derivative of $\Phi(x,\cdot)$. For example, this inequality holds if $p^- > 1$ or if $q^- \geqslant 0$. For later use it is convenient to note that (Φ_1) implies the following condition:

(Φ_2) $t \mapsto t^{-1}\Phi(x,t)$ is non-decreasing on $(0,\infty)$ for fixed $x \in \mathbb{R}^n$.

We define the space $L^\Phi(\Omega)$ to consist of all measurable functions f on the open set $\Omega \subset \mathbb{R}^n$ with

$$\int_\Omega \Phi\left(x, \frac{|f(x)|}{\lambda}\right) dx < \infty$$

for some $\lambda > 0$. We define the norm

$$\|f\|_{\Phi(\cdot,\cdot)(\Omega)} = \inf\left\{\lambda > 0 : \int_\Omega \Phi\left(x, \frac{|f(x)|}{\lambda}\right) dx \leqslant 1\right\}$$

for $f \in L^\Phi(\Omega)$. These spaces have been studied in [4, 18]. Note that $L^\Phi(\Omega)$ is a Musielak–Orlicz space [19]. In case $q \equiv 0$, $L^\Phi(\Omega)$ reduces to the variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$.

Our first aim in this paper is to prove a weak-type inequality of maximal functions in Theorem 2.5. Then we prove in Theorem 3.5 a weak-type estimate for the Riesz potential. These enable us to prove the main result of this paper, namely a Sobolev embedding for functions in $W^{1,\Phi}$. The Sobolev space $W^{1,\Phi}(\Omega)$ consists of those functions $u \in L^\Phi(\Omega)$ with a distributional gradient satisfying $|\nabla u| \in L^\Phi(\Omega)$. Further we denote by $W_0^{1,\Phi}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in the space $W^{1,\Phi}(\Omega)$ (cf. [10] for definitions of zero boundary value functions in the variable exponent context).

Let $p^\sharp(x)$ denote the Sobolev conjugate of $p(x)$; that is to say

$$1/p^\sharp(x) = 1/p(x) - \alpha/n.$$

https://doi.org/10.1017/S0017089509990292 Published online by Cambridge University Press

For the Sobolev emedding in $W^{1,\Phi}$ we need the conjugate exponent with $\alpha = 1$, which is denoted by p^*.

THEOREM 1.1. *Let p and q satisfy the above conditions. If $p^+ < n$, then*

$$\|u\|_{\Psi(\cdot,\cdot)(\Omega)} \leqslant c_1 \|\nabla u\|_{\Phi(\cdot,\cdot)(\Omega)}$$

for every $u \in W_0^{1,\Phi}(\Omega)$, where $\Phi(x, t) := (t \log(c_0 + t)^{q(x)/p(x)})^{p(x)}$ and $\Psi(x, t) := (t \log(c_0 + t)^{q(x)/p(x)})^{p^*(x)}$.

This extends [**11**, Proposition 4.2(1)] and [**13**, Theorem 3.4] which dealt with the case $q \equiv 0$.

2. Weak-type inequality of maximal functions.

In order to prove the main result of this section, namely a weak-type inequality for the maximal function, we start by presenting several preparatory results.

Let $B(x, r)$ denote the open ball centred at x with radius r. For a locally integrable function f on $\mathbb{R}^n$, we consider the maximal function Mf defined by

$$Mf(x) := \sup_B f_B = \sup_B \frac{1}{|B|} \int_B |f(y)|dy,$$

where the supremum is taken over all balls $B = B(x, r)$ and $|B|$ denotes the volume of B.

The following lemma is an improvement of [**18**, Lemma 2.6].

LEMMA 2.1. *Let f be a non-negative measurable function on $\mathbb{R}^n$ with $\|f\|_{\Phi(\cdot,\cdot)(\mathbb{R}^n)} \leqslant 1$. Set*

$$I := \frac{1}{|B(x, r)|} \int_{B(x,r)} f(y)\, dy$$

and

$$J := \frac{1}{|B(x, r)|} \int_{B(x,r)} \Phi(y, f(y))\, dy.$$

Then

$$I \leqslant C\{J^{1/p(x)}(\log(c_0 + J))^{-q(x)/p(x)} + 1\}.$$

Proof. By condition (Φ_2), for $K > 0$ we have

$$I \leqslant K + \frac{C}{|B(x, r)|} \int_{B(x,r)} f(y) \left(\frac{f(y)}{K}\right)^{p(y)-1} \left(\frac{\log(c_0 + f(y))}{\log(c_0 + K)}\right)^{q(y)} dy,$$

where the first term, K, represents the contribution to the integral of points where $f(y) < K$. If $J \leqslant 1$, then we take $K = 1$ and obtain

$$I \leqslant 1 + CJ \leqslant C.$$

https://doi.org/10.1017/S0017089509990292 Published online by Cambridge University Press

Now suppose that $J \geqslant 1$ and set

$$K := CJ^{1/p(x)}(\log(c_0 + J))^{-q(x)/p(x)}.$$

Note that $J^{C/\log(CJ^{1/n})} \leqslant C$ and $(\log(c_0 + J))^{C/\log(\log(e+CJ^{1/n}))} \leqslant C$. Since we assumed that $\|f\|_{\Phi(\cdot,\cdot)(\mathbb{R}^n)} \leqslant 1$, we conclude that

$$J \leqslant \frac{1}{|B(x,r)|}\int_{\mathbb{R}^n} \Phi(y, f(y))\, dy \leqslant \frac{1}{|B(x,r)|}.$$

Hence, by Conditions (p2) and (q2), we obtain, for $y \in B(x, r)$, that

$$\begin{aligned}
K^{-p(y)} &\leqslant \left\{CJ^{1/p(x)}(\log(c_0 + J))^{-q(x)/p(x)}\right\}^{-p(x)+C/\log(1/r)} \\
&\leqslant \left\{CJ^{1/p(x)}(\log(c_0 + J))^{-q(x)/p(x)}\right\}^{-p(x)+C/\log(CJ^{1/n})} \\
&\leqslant CJ^{-1}(\log(c_0 + J))^{q(x)}
\end{aligned}$$

and

$$\begin{aligned}
(\log(c_0 + K))^{-q(y)} &\leqslant \{C\log(c_0 + J)\}^{-q(x)+C/\log(\log(e+1/r))} \\
&\leqslant \{C\log(c_0 + J)\}^{-q(x)+C/\log(\log(e+CJ^{1/n}))} \\
&\leqslant C(\log(c_0 + J))^{-q(x)}.
\end{aligned}$$

Consequently it follows that

$$I \leqslant CJ^{1/p(x)}(\log(c_0 + J))^{-q(x)/p(x)}.$$

Combining this with the estimate $I \leqslant C$ from the previous case yields the claim. □

In view of Lemma 2.1, for each bounded open set G in $\mathbb{R}^n$ we can find a positive constant C such that

$$\{Mf(x)\}^{p(x)} \leqslant C\{Mg(x)(\log(c_0 + Mg(x)))^{-q(x)} + (1 + |x|)^{-n}\} \tag{2.1}$$

for all $x \in G$ and $g(y) := \Phi(y, f(y))$, whenever f is a non-negative measurable function on $\mathbb{R}^n$ with $\|f\|_{\Phi(\cdot,\cdot)(\mathbb{R}^n)} \leqslant 1$.

For later use it is convenient to note that

$$C^{-1}(1 + |x|)^{-n/p_\infty} \leqslant (1 + |x|)^{-n/p(x)} \leqslant C(1 + |x|)^{-n/p_\infty} \tag{2.2}$$

in view of (p3).

LEMMA 2.2. *Let f be a non-negative measurable function on $\mathbb{R}^n$ with $\|f\|_{\Phi(\cdot,\cdot)(\mathbb{R}^n)} \leqslant 1$. If $J \leqslant 1$, then*

$$I_1 := \frac{1}{|B(x,r)|}\int_{B(x,r)\setminus B(0,|x|/2)} f(y)\, dy \leqslant C\left\{J^{1/p(x)} + (1 + |x|)^{-n/p(x)}\right\}.$$

Proof. By condition (Φ_2), for $K > 0$ we have

$$I_1 \leqslant K + \frac{C}{|B(x,r)|}\int_{B(x,r)\setminus B(0,|x|/2)} f(y)\left(\frac{f(y)}{K}\right)^{p(y)-1}\left(\frac{\log(c_0 + f(y))}{\log(c_0 + K)}\right)^{q(y)} dy.$$

https://doi.org/10.1017/S0017089509990292 Published online by Cambridge University Press

Then we take $K := \max\{(1+|x|)^{-n/p(x)}, J^{1/p(x)}\} \leqslant 1$ and find

$$I_1 \leqslant K + CK^{-p(x)+1}J \leqslant CK,$$

since $p(y) \leqslant p(x) + C/\log(e+|x|)$ for $y \in B(x,r) \setminus B(0,|x|/2)$ by (p3). Thus the proof is complete. □

LEMMA 2.3. *Let f be a non-negative measurable function on $\mathbb{R}^n$ with $\|f\|_{\Phi(\cdot,\cdot)(\mathbb{R}^n)} \leqslant 1$. If $J \leqslant 1$, then*

$$I_2 := \frac{1}{|B(x,r)|}\int_{B(x,r)\cap B(0,|x|/2)} f(y)\,dy \leqslant C(1+|x|)^{-n/p_\infty}.$$

Proof. Since $J \leqslant 1$, we see from Lemma 2.1 that I_2 is bounded on $B(0,e)$, so that we have only to treat the case when $|x| \geqslant e$.

If $r \leqslant |x|/2$, then the integration set is empty and the claim is trivial. We will show that

$$I' := \frac{1}{|B(0,r)|}\int_{B(0,r)} f(y)\,dy \leqslant Cr^{-n/p_\infty} \tag{2.3}$$

for $r > 1$. Since $I_2 \leqslant I'$ when $r > |x|/2$, the claim then follows.

By condition (Φ_2), we have the following for a measurable function $K = K(y) > 0$:

$$\begin{aligned} I' \leqslant{}& \frac{1}{|B(0,r)|}\int_{B(0,r)} K(y)\,dy \\ &+ \frac{C}{|B(0,r)|}\int_{B(0,r)} f(y)\left(\frac{f(y)}{K(y)}\right)^{p(y)-1}\left(\frac{\log(c_0+f(y))}{\log(c_0+K(y))}\right)^{q(y)} dy. \end{aligned}$$

If $p_\infty > 1$, then we take $K := (1+|y|)^{-n/p_\infty}$ and find that

$$I' \leqslant C\big(r^{-n/p_\infty} + r^{n(p_\infty-1)/p_\infty}J'\big)$$

by use of (p3), where

$$J' := \frac{1}{|B(0,r)|}\int_{B(0,r)} \Phi(y,f(y))\,dy.$$

If $p_\infty = 1$, then we take $K := (1+|y|)^{-\beta}$ for $\beta > n$ and obtain

$$I' \leqslant C(r^{-n} + J').$$

Noting that $J' \leqslant Cr^{-n}$ completes the proof. □

LEMMA 2.4. *Let f be a non-negative measurable function on an open set Ω with $\|f\|_{\Phi(\cdot,\cdot)(\Omega)} \leqslant 1$. Set*

$$N(x) := Mg(x)^{1/p(x)}(\log(c_0+Mg(x)))^{-q(x)/p(x)},$$

where $g(y) := \Phi(y, f(y))$. *Then*

$$\int_{E_t} \Phi(x, t)\ dx \leqslant C,$$

where $E_t := \{x \in \Omega : N(x) > t, Mg(x) > C_1(1 + |x|)^{-n}\}$ *and* $C_1 := |B(0, 1/2)|^{-1}$.

Proof. By the Besicovitch covering theorem, we can find a countable family of balls $B_i = B(x_i, r_i)$ with a bounded overlap property such that $E_t \subset \cup_i B_i$,

$$t < g_{B_i}^{1/p(x_i)}(\log(c_0 + g_{B_i}))^{-q(x_i)/p(x_i)}$$

and

$$g_{B_i} > C_1(1 + |x_i|)^{-n}.$$

If $1 \leqslant g_{B_i} \leqslant |B_i|^{-1}$, then Conditions (p2) and (q2) imply that

$$g_{B_i}^{1/p(x_i)}(\log(c_0 + g_{B_i}))^{-q(x_i)/p(x_i)} \leqslant C g_{B_i}^{1/p(x)}(\log(c_0 + g_{B_i}))^{-q(x)/p(x)}$$

for $x \in B_i$; and if $C_1(1 + |x_i|)^{-n} < g_{B_i} \leqslant 1$, then $r_i \leqslant (1 + |x_i|)/2$, so that we obtain the above inequality by using (p3). A similar argument holds for changing $q(x_i)$ to $q(x)$. Thus we obtain

$$\begin{aligned}
&\Phi\big(x, g_{B_i}^{1/p(x_i)}(\log(c_0 + g_{B_i}))^{-q(x_i)/p(x_i)}\big)\\
&\quad\leqslant C\Phi\big(x, g_{B_i}^{1/p(x)}(\log(c_0 + g_{B_i}))^{-q(x)/p(x)}\big)\\
&\quad= C g_{B_i}(\log(c_0 + g_{B_i}))^{-q(x)}\big(\log(c_0 + g_{B_i}^{1/p(x)}(\log(c_0 + g_{B_i}))^{-q(x)/p(x)})\big)^{q(x)}\\
&\quad\leqslant C g_{B_i}.
\end{aligned}$$

Hence we see that

$$\begin{aligned}
\int_{E_t} \Phi(x, t)\, dx &\leqslant \sum_i \int_{B_i} \Phi(x, t)\, dx\\
&\leqslant C \sum_i \int_{B_i} g_{B_i}\, dx = C \sum_i \int_{B_i} g(y)\, dy\\
&\leqslant C \int_\Omega g(y)\, dy \leqslant C.
\end{aligned}$$

□

We are now ready for the first main result, a weak-type estimate for the maximal function. This is an extension of [**2**, Theorem 1.6] and [**12**, Theorem 3.2].

THEOREM 2.5. *Let f be a non-negative measurable function on $\mathbb{R}^n$ with $\|f\|_{\Phi(\cdot,\cdot)(\mathbb{R}^n)} \leqslant 1$. Then*

$$\int_{\{x \in \mathbb{R}^n : Mf(x) > t\}} \Phi(x, t)\ dx \leqslant C.$$

Proof. Lemmas 2.1–2.3 and (2.2) give

$$I \leqslant C\big\{J^{1/p(x)}(\log(c_0 + J))^{-q(x)/p(x)} + (1 + |x|)^{-n/p_\infty}\big\} \tag{2.4}$$

https://doi.org/10.1017/S0017089509990292 Published online by Cambridge University Press

for $x \in \mathbb{R}^n$. Hence

$$\{x \in \mathbb{R}^n : Mf(x) > t\} \subset E_t \cup \{x \in \mathbb{R}^n : (1 + |x|)^{-n/p_\infty} > t/C\}$$

with E_t as in Lemma 2.4. Note that we may assume $Mg(x) \geqslant C_1(1 + |x|)^{-n}$ in the first set, since if $Mg(x) \leqslant C_1(1 + |x|)^{-n}$ and $N(x) > t/C$, then $(1 + |x|)^{-n/p_\infty} > t/C$.

If the second set is empty, the claim follows from Lemma 2.4. If this is not the case we define $r > 0$ so that $(1 + r)^{-n/p_\infty} = t/C$. Note that t is bounded in this case. Then

$$\int_{\{x\in\mathbb{R}^n : Mf(x)>t\}} \Phi(x, t)\, dx \leqslant \int_{E_t} \Phi(x, t)\, dx + \int_{B(0,r)} \Phi(x, t)\, dx.$$

The first integral on the right-hand side is bounded by Lemma 2.4. For the second, we note that $\Phi(x, t) \leqslant Ct^{p(x)}$ since t and q are bounded. By the definition of r we have

$$\int_{B(0,r)} t^{p(x)}\, dx \leqslant C \int_{B(0,r)} (1 + r)^{-np(x)/p_\infty}\, dx \leqslant C \int_{B(0,r)} (1 + r)^{-n+(Cn/p_\infty)/\log(e+|x|)}\, dx.$$

For $0 < m < n$, noting that $(1 + r)^{-m+(Cn/p_\infty)/\log(e+t)}(1 + t)^m$ is bounded on (c_1, r) when $-m + (Cn/p_\infty)/\log(e + c_1) < 0$, we find

$$\int_{B(0,r)} t^{p(x)}\, dx \leqslant \int_{B(0,c_1)} t^{p(x)}\, dx + C(1 + r)^{m-n} \int_{B(0,r)} (1 + |x|)^{-m}\, dx \leqslant C.$$

Therefore $\int_{B(0,r)} \Phi(x, t)\, dx \leqslant C$, and so we obtain the theorem. □

REMARK 2.6. Take $\omega \in C^\infty(\mathbb{R})$ such that $0 \leqslant \omega \leqslant 1$, $\omega(r) = 0$ when $r \leqslant 0$ and $\omega(r) = 1$ when $r \geqslant 1/2$. Let

$$p(x) := 1 + \frac{a \log(e + \log(e + |x|))}{\log(e + |x|)}\, \omega\left(\frac{2x_n - |x|}{1 + |x|}\right)$$

for $x = (x_1, \ldots , x_n)$, where $a > 0$. Consider the function

$$f(x) := \begin{cases} (e + |x|)^{-n}(\log(e + |x|))^\beta & \text{if } 4x_n > 3|x| + 1, \\ 0 & \text{elsewhere.} \end{cases}$$

If $-1 < \beta < an - 1$, then $f \in L^{p(\cdot)}(\mathbb{R}^n)$. Note that

$$Mf(x) \geqslant C(e + |x|)^{-n}(\log(e + |x|))^{\beta+1}$$

for all $x \in \mathbb{R}^n$. There exists a constant $C > 0$ such that if

$$|x| \leqslant Ct^{-1/n}(\log(e + t^{-1}))^{(\beta+1)/n},$$

then $Mf(x) > t$, so that

$$\begin{aligned}\int_{\{x\in\mathbb{R}^n : Mf(x)>t\}} t^{p(x)}\, dx &\geqslant t\, |\{x \in \mathbb{R}^n : Mf(x) > t, x_n < 0\}| \\ &\geqslant C(\log(e + t^{-1}))^{\beta+1},\end{aligned}$$

which tends to ∞ as $t \to 0+$. This example shows that the assumption on the exponent in our weak-type estimate is quite sharp.

3. Weak-type inequality for Riesz potentials. For $0 < \alpha < n$, we define the Riesz potential of order α for a locally integrable function f on $\mathbb{R}^n$ by

$$I_\alpha f(x) := \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy.$$

Here it is natural to assume that

$$\int_{\mathbb{R}^n} (1+|y|)^{\alpha-n} |f(y)|\, dy < \infty, \tag{3.1}$$

which is equivalent to the condition that $I_\alpha |f| \not\equiv \infty$ (see [**16**, Theorem 1.1, Chapter 2]).

Our aim in this section is to establish weak-type estimates for Riesz potentials of functions in $L^\Phi(\mathbb{R}^n)$, when the exponent p satisfies

$$p^+ < n/\alpha.$$

Let $p^\sharp(x)$ denote the Sobolev conjugate of $p(x)$, as defined in the 'Introduction'.

LEMMA 3.1. *Suppose that $p^+ < n/\alpha$. If f is a non-negative measurable function on $\mathbb{R}^n$ with $\|f\|_{\Phi(\cdot,\cdot)(\mathbb{R}^n)} \leqslant 1$, then*

$$\int_{\mathbb{R}^n \setminus B(x,r)} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy \leqslant C\big\{r^{\alpha-n/p(x)} + (1+|x|)^{\alpha-n/p_\infty}\big\}$$

for all $x \in \mathbb{R}^n$ and $r \geqslant 1/e$.

Proof. If $|x| \leqslant r$ and $r \geqslant 1/e$, then (2.3) gives

$$\begin{aligned}\int_{\mathbb{R}^n \setminus B(x,r)} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy &\leqslant C \int_{\mathbb{R}^n} (r+|y|)^{\alpha-n} f(y)\, dy \\ &\leqslant C \int_0^\infty \left(\int_{B(0,t)} f(y)\, dy \right) (r+t)^{\alpha-n-1}\, dt \\ &\leqslant C r^{\alpha-n/p_\infty} \leqslant C(1+|x|)^{\alpha-n/p_\infty}.\end{aligned}$$

Next consider the case $|x| > r \geqslant 1/e$. Then we have

$$\int_{B(0,|x|/2)\setminus B(x,r)} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy \leqslant C|x|^{\alpha-n} \int_{B(0,|x|/2)} f(y)\, dy \leqslant C|x|^{\alpha-n/p_\infty}$$

and

$$\begin{aligned}\int_{\mathbb{R}^n \setminus B(0,2|x|)} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy &\leqslant C \int_{\mathbb{R}^n \setminus B(0,2|x|)} |y|^{\alpha-n} f(y)\, dy \\ &\leqslant C \int_{2|x|}^\infty \left(\int_{B(0,t)} f(y)\, dy \right) t^{\alpha-n-1}\, dt \\ &\leqslant C(1+|x|)^{\alpha-n/p_\infty}.\end{aligned}$$

https://doi.org/10.1017/S0017089509990292 Published online by Cambridge University Press

It remains to estimate the integral of $|x-y|^{\alpha-n}f(y)$ over the set $E := B(0, 2|x|) \setminus \{B(0, |x|/2) \cup B(x, r)\}$. By condition (Φ_2), for $K(y) := |x-y|^{-n/p(x)}$ we have

$$\begin{aligned}\int_E \frac{f(y)}{|x-y|^{n-\alpha}}\,dy &\leqslant \int_E \frac{K(y)}{|x-y|^{n-\alpha}}\,dy \\ &\quad + C\int_E \frac{f(y)}{|x-y|^{n-\alpha}}\left(\frac{f(y)}{K(y)}\right)^{p(y)-1}\left(\frac{\log(c_0+f(y))}{\log(c_0+K(y))}\right)^{q(y)} dy \\ &\leqslant Cr^{\alpha-n/p(x)} + Cr^{\alpha-n+n(p(x)-1)/p(x)}\int_E \Phi(y, f(y))\,dy \\ &\leqslant Cr^{\alpha-n/p(x)},\end{aligned}$$

since $p(y) \leqslant p(x) + C/\log|x|$ for $y \in \mathbb{R}^n \setminus B(0, |x|/2)$ by (p3) and $\alpha p^+ < n$. □

LEMMA 3.2. *Suppose that $p^+ < n/\alpha$. Let f be a non-negative measurable function on $\mathbb{R}^n$ with $\|f\|_{\Phi(\cdot,\cdot)(\mathbb{R}^n)} \leqslant 1$. Then*

$$\int_{B(x,1/e)\setminus B(x,\delta)} \frac{f(y)}{|x-y|^{n-\alpha}}\,dy \leqslant C\delta^{\alpha-n/p(x)}(\log(c_0+1/\delta))^{-q(x)/p(x)}$$

for all $x \in \mathbb{R}^n$ and $0 < \delta < 1/e$.

Proof. The proof is similar to the last case in the previous proof. Let us set $E := B(x, 1/e) \setminus B(x, \delta)$ and

$$K(y) := |x-y|^{-n/p(x)}(\log(c_0 + 1/|x-y|))^{-q(x)/p(x)}$$

for $y \in E$. By Conditions (p2), (q2) and (Φ_2), we obtain

$$\begin{aligned}\int_E \frac{f(y)}{|x-y|^{n-\alpha}}\,dy &\leqslant \int_E \frac{K(y)}{|x-y|^{n-\alpha}}\,dy \\ &\quad + C\int_E \frac{f(y)}{|x-y|^{n-\alpha}}\left(\frac{f(y)}{K(y)}\right)^{p(y)-1}\left(\frac{\log(c_0+f(y))}{\log(c_0+K(y))}\right)^{q(y)} dy \\ &\leqslant C\bigg(\delta^{\alpha-n/p(x)}(\log(c_0+1/\delta))^{-q(x)/p(x)} \\ &\quad + \int_E |x-y|^{\alpha-n/p(x)}(\log(c_0+1/|x-y|))^{-q(x)/p(x)}\Phi(y, f(y))\,dy\bigg) \\ &\leqslant C\delta^{\alpha-n/p(x)}(\log(c_0+1/\delta))^{-q(x)/p(x)}\left(1 + \int_E \Phi(y, f(y))\,dy\right) \\ &\leqslant C\delta^{\alpha-n/p(x)}(\log(c_0+1/\delta))^{-q(x)/p(x)},\end{aligned}$$

as required. □

The next lemma is a generalisation of [**18**, Theorem 2.8].

LEMMA 3.3. *Suppose that $p^+ < n/\alpha$. Let $f \in L^{\Phi}(\mathbb{R}^n)$ be non-negative with $\|f\|_{\Phi(\cdot,\cdot)(\mathbb{R}^n)} \leqslant 1$. Then*

$$I_\alpha f(x) \leqslant C\big\{Mf(x)^{p(x)/p^\sharp(x)}(\log(c_0 + Mf(x)))^{-\alpha q(x)/n} + (1+|x|)^{-n/p^\sharp_\infty}\big\}.$$

https://doi.org/10.1017/S0017089509990292 Published online by Cambridge University Press

Proof. By Lemmas 3.1 and 3.2,

$$I_\alpha f(x) = \int_{B(x,\delta)} \frac{f(y)}{|x-y|^{n-\alpha}}\,dy + \int_{\mathbb{R}^n\setminus B(x,\delta)} \frac{f(y)}{|x-y|^{n-\alpha}}\,dy$$
$$\leqslant C\big\{\delta^\alpha Mf(x) + \delta^{\alpha-n/p(x)}(\log(c_0+1/\delta))^{-q(x)/p(x)} + (1+|x|)^{\alpha-n/p_\infty}\big\}$$

for $\delta > 0$. Here, letting

$$\delta = \min\big\{Mf(x)^{-p(x)/n}(\log(c_0+Mf(x)))^{-q(x)/n}, 1+|x|\big\},$$

we find

$$I_\alpha f(x) \leqslant C\big\{Mf(x)^{p(x)/p^\sharp(x)}(\log(c_0+Mf(x)))^{-\alpha q(x)/n} + (1+|x|)^{-n/p^\sharp_\infty}\big\}.$$

□

Recall that $\Psi(x,t) = (t\log(c_0+t)^{q(x)/p(x)})^{p^\#(x)}$.

Lemma 3.4. *Suppose that $p^+ < n/\alpha$. Let f be a non-negative measurable function on an open set Ω with $\|f\|_{\Phi(\cdot,\cdot)(\Omega)} \leqslant 1$. Set*

$$N(x) := Mg(x)^{1/p^\sharp(x)}(\log(c_0+Mg(x)))^{-q(x)/p(x)},$$

where $g(y) := \Phi(y, f(y))$. Then

$$\int_{\tilde{E}_t} \Psi(x,t)\;dx \leqslant C,$$

where $\tilde{E}_t := \{x \in \Omega : N(x) > t, Mg(x) \geqslant C_1(1+|x|)^{-n}\}$ and $C_1 := |B(0,1/2)|^{-1}$.

Proof. By the Besicovitch covering theorem, we can find a countable family of balls $B_i = B(x_i, r_i)$ with a bounded overlap property such that $\tilde{E}_t \subset \cup_i B_i$,

$$t < g_{B_i}^{1/p^\sharp(x_i)}(\log(c_0+g_{B_i}))^{-q(x_i)/p(x_i)}$$

and

$$g_{B_i} > C_1(1+|x|)^{-n}.$$

As in Lemma 2.4, we obtain

$$\Psi\big(x, g_{B_i}^{1/p^\sharp(x_i)}(\log(c_0+g_{B_i}))^{-q(x_i)/p(x_i)}\big) \leqslant C\Psi\big(x, g_{B_i}^{1/p^\sharp(x)}(\log(c_0+g_{B_i}))^{-q(x)/p(x)}\big) \leqslant Cg_{B_i}$$

for $x \in B_i$. Hence obtain as before that

$$\int_{\tilde{E}_t} \Psi(x,t)\,dx \leqslant \sum_i \int_{B_i} \Psi(x,t)\,dx$$
$$\leqslant C\sum_i \int_{B_i} g_{B_i}\,dx \leqslant C\int_\Omega g(y)\,dy \leqslant C.$$

□

Now we are ready to show the weak-type estimate for Riesz potentials, as an extension of [**2**, Theorem 1.9] and [**12**, Theorem 3.4].

THEOREM 3.5. *Suppose that* $p^+ < n/\alpha$. *Let* f *be a non-negative measurable function on* $\mathbb{R}^n$ *with* $\|f\|_{\Phi(\cdot,\cdot)(\mathbb{R}^n)} \leqslant 1$. *Then*

$$\int_{\{x\in\mathbb{R}^n : I_\alpha f(x)>t\}} \Psi(x,t)\ dx \leqslant C.$$

Proof. Lemmas 3.3 and 2.1–2.3 give

$$\begin{aligned} I_\alpha f(x) &\leqslant C\big\{Mf(x)^{p(x)/p^\sharp(x)}(\log(c_0+Mf(x)))^{-\alpha q(x)/n} + (1+|x|)^{-n/p^\sharp_\infty}\big\} \\ &\leqslant C\big\{Mg(x)^{1/p^\sharp(x)}(\log(c_0+Mg(x)))^{-q(x)/p(x)} + (1+|x|)^{-n/p^\sharp_\infty}\big\} \end{aligned}$$

for $x\in\mathbb{R}^n$. Hence

$$\big\{x\in\mathbb{R}^n : I_\alpha f(x) > t\} \subset \tilde{E}_t \cup \{x\in\mathbb{R}^n : (1+|x|)^{-n/p^\sharp_\infty} > t/C\big\},$$

where $\tilde{E}_t$ is as in Lemma 3.4. If the second set is empty, then the claim follows from Lemma 3.4. If this is not the case we define $r>0$ so that $(1+r)^{-n/p^\sharp_\infty} = t/C$. Then

$$\int_{\{x\in\mathbb{R}^n : I_\alpha f(x)>t\}} \Psi(x,t)\,dx \leqslant \int_{\tilde{E}_t} \Psi(x,t)\,dx + \int_{B(0,r)} \Psi(x,t)\,dx.$$

The first integral on the right-hand side is bounded by Lemma 3.4. For the second we note that $\Psi(x,t) \leqslant Ct^{p^\sharp(x)}$, since t and $q(\cdot)$ are bounded. Thus

$$\int_{B(0,r)} t^{p^\sharp(x)}\,dx \leqslant \int_{B(0,r)} C(1+r)^{-n+(Cn/p^\sharp_\infty)/\log(e+|x|)}\,dx \leqslant c,$$

where the last step follows exactly as in the proof of Theorem 2.5. □

REMARK 3.6. Continuing with the notation of Remark 2.6, we further see that

$$I_\alpha f(x) \geqslant C(e+|x|)^{\alpha-n}(\log(e+|x|))^{\beta+1}$$

for all $x\in\mathbb{R}^n$, so that

$$\begin{aligned} \int_{\{x\in\mathbb{R}^n : I_\alpha f(x)>t\}} t^{p^\sharp(x)}\,dx &\geqslant t^{n/(n-\alpha)}|\{x\in\mathbb{R}^n : I_\alpha f(x) > t,\, x_n < 0\}| \\ &\geqslant C(\log(e+t^{-1}))^{n(\beta+1)/(n-\alpha)}, \end{aligned}$$

which tends to ∞ as $t\to 0+$.

REMARK 3.7. In view of [**17**], for each $\beta > 1$ one can find a constant $C>0$ such that

$$\int_{\mathbb{R}^n} \{I_\alpha f(x)\}^{p^\sharp(x)}(\log(e+I_\alpha f(x)))^{-\beta}(\log(e+I_\alpha f(x)^{-1}))^{-\beta}\ dx \leqslant C$$

whenever f is a non-negative measurable function on $\mathbb{R}^n$ with $\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \leqslant 1$. This gives a supplement of O'Neil [**20**, Theorem 5.3].

4. Sobolev functions. Let us consider the generalised Orlicz–Sobolev space $W^{1,\Phi}(\Omega)$ with the norm

$$\|u\|_{1,\Phi(\cdot,\cdot)(\Omega)} = \|u\|_{\Phi(\cdot,\cdot)(\Omega)} + \|\nabla u\|_{\Phi(\cdot,\cdot)(\Omega)} < \infty.$$

Further we denote by $W_0^{1,\Phi}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in the space $W^{1,\Phi}(\Omega)$ (cf. [**10**] for definitions of zero boundary value functions in the variable exponent context). To conclude the paper, we derive a Sobolev inequality for functions in $W_0^{1,\Phi}(\Omega)$ as the application of Sobolev's weak-type inequality for Riesz potentials of functions in $L^\Phi(\Omega)$. First note the following lemma.

LEMMA 4.1 *(Corollary 2.3 of [**18**]).* Set $\kappa(y,t) := t(\log(e+t))^y$ for y and $t \geqslant 0$. Then

$$\kappa(y, at) \leqslant \tau(y,a)\kappa(y,t)$$

whenever $a, t > 0$, where

$$\tau(y,a) := a\max\{(C\log(e+a))^y, (C\log(e+a^{-1}))^{-y}\}.$$

We define local versions of p^+ and p^- as follows:

$$p_\Omega^- = \operatorname{ess\,inf}_{x\in\Omega} p(x) \text{ and } p_\Omega^+ = \operatorname{ess\,sup}_{x\in\Omega} p(x).$$

Using the previous lemma we can derive a scaled version of the weak-type estimate from the previous section, which will be needed below.

LEMMA 4.2. *Let Ω be an open set in $\mathbb{R}^n$. Suppose that $p^+ < n/\alpha$. Let $f \in L^\Phi(\mathbb{R}^n)$ be non-negative with $\|f\|_{\Phi(\cdot,\cdot)(\mathbb{R}^n)} \leqslant 1$. Then for every $\varepsilon > 0$ there exists a constant $C > 0$ such that*

$$\int_{\{x\in\Omega: I_\alpha f(x)>t\}} \Psi(x,t)\ dx \leqslant C\|f\|_{\Phi(\cdot,\cdot)(\mathbb{R}^n)}^{(p^\sharp)_\Omega^- - \varepsilon},$$

for every $t > 0$.

Proof. For simplicity we denote $\|f\|_{\Phi(\cdot,\cdot)(\mathbb{R}^n)}$ by $a \in [0,1]$. The case $a = 0$ is clear; so we assume that $a > 0$. We apply Theorem 3.5 to the function f/a, whose norm is equal to 1. Thus

$$\begin{aligned}\int_{\{x\in\Omega: I_\alpha f(x)>s\}} \Psi(x,s/a)\ dx &= \int_{\{x\in\Omega: I_\alpha \frac{f}{a}(x)>t\}} \Psi(x,t)\ dx \\ &\leqslant \int_{\{x\in\mathbb{R}^n: I_\alpha \frac{f}{a}(x)>t\}} \Psi(x,t)\ dx \leqslant C.\end{aligned}$$

With κ as in the previous lemma and $r = q(x)p^\sharp(x)/p(x)$, we have $\Psi(x,t) = t^{p^\sharp(x)-1}\kappa(r,t)$. Hence the lemma implies that

$$\Psi(x,s/a) = \Psi(x,s)a^{1-p^\sharp(x)}\frac{\kappa(r,s/a)}{\kappa(r,s)} \geqslant \Psi(x,s)a^{1-p^\sharp(x)}\tau(r,a)^{-1}.$$

Since τ is logarithmic and $a \leqslant 1$, it follows that $a^{p^\sharp(x)-1}\tau(r,a) \leqslant Ca^{(p^\sharp)_\Omega^- - \varepsilon}$. Now the claim follows by combining the inequalities derived. □

https://doi.org/10.1017/S0017089509990292 Published online by Cambridge University Press

Lemma 4.3. *Suppose that* $p^+ < n$, $p^+_\Omega < (p^*)^-_\Omega$ *and* Ω *is an open set. If* $u \in W^{1,\Phi}_0(\mathbb{R}^n)$, *then there exists a constant* $c_1 > 0$ *such that*

$$\|u\|_{\Psi(\cdot,\cdot)(\Omega)} \leqslant c_1 \|\nabla u\|_{\Phi(\cdot,\cdot)(\mathbb{R}^n)}.$$

Proof. We may assume that $\|\nabla u\|_{\Phi(\cdot,\cdot)(\mathbb{R}^n)} \leqslant 1$ and u is non-negative. It follows from [**16**, Theorem 1.2, Chapter 6] that

$$|v(x)| \leqslant C(n) I_1 |\nabla v|(x)$$

for $v \in W^{1,1}_0(\mathbb{R}^n)$ and almost every $x \in \mathbb{R}^n$. For $u \in W^{1,\Phi}_0(\mathbb{R}^n)$ and each integer j, we write $U_j = \{x \in \Omega : 2^j < u(x) \leqslant 2^{j+1}\}$ and $v_j = \max\{0, \min\{u - 2^j, 2^j\}\}$. Since $v_j \in W^{1,1}_0(\Omega)$ and $v_j(x) = 2^j$ for almost every $x \in U_{j+1}$, we have

$$I_1 |\nabla v_j|(x) \geqslant C 2^j$$

for almost every $x \in U_{j+1}$. It follows that

$$\begin{aligned}
\int_\Omega \Psi(x, u(x))\, dx &\leqslant \sum_{j\in\mathbb{Z}} \int_{U_{j+1}} \Psi(x, u(x))\, dx \\
&\leqslant C \sum_{j\in\mathbb{Z}} \int_{U_{j+1}} \Psi(x, 2^{j+1})\, dx \\
&\leqslant C \sum_{j\in\mathbb{Z}} \int_{\{x\in U_{j+1} : I_1|\nabla v_j|(x) > C2^j\}} \Psi(x, C2^j)\, dx.
\end{aligned}$$

Taking $r \in (p^+, (p^*)^-_\Omega)$, we obtain by Lemma 4.2 that

$$\begin{aligned}
\sum_{j\in\mathbb{Z}} \int_{\{x\in U_{j+1} : I_1|\nabla v_j|(x) > C2^j\}} \Psi(x, C2^j)\, dx &\leqslant C \sum_{j\in\mathbb{Z}} \|\nabla v_j\|^r_{\Phi(\cdot,\cdot)(\mathbb{R}^n)} \\
&\leqslant C \sum_{j\in\mathbb{Z}} \int_{U_j} \Phi(x, |\nabla u(x)|)\, dx \leqslant C,
\end{aligned}$$

which completes the proof. □

Proof of Theorem 1.1. We may split $\mathbb{R}^n$ into a finite number of cubes $\Omega_1, \ldots, \Omega_k$ and the complement of a cube, Ω_0, in such a way that $p^+_{\Omega_i} < (p^*)^-_{\Omega_i}$ for each i. Then

$$\|u\|_{\Psi(\cdot,\cdot)(\mathbb{R}^n)} \leqslant \sum_{i=0}^{k} \|u\|_{\Psi(\cdot,\cdot)(\Omega_i)} \leqslant c_1 \sum_{i=0}^{k} \|\nabla u\|_{\Phi(\cdot,\cdot)(\mathbb{R}^n)} = (k+1) c_1 \|\nabla u\|_{\Phi(\cdot,\cdot)(\mathbb{R}^n)},$$

by the previous lemma. □

REFERENCES

1. A. Almeida and S. Samko, Pointwise inequalities in variable Sobolev spaces and applications, *Z. Anal. Anwend.* **26**(2) (2007), 179–193.

https://doi.org/10.1017/S0017089509990292 Published online by Cambridge University Press

2. C. Capone, D. Cruz-Uribe and A. Fiorenza, The fractional maximal operators on variable L^p spaces, *Rev. Mat. Iberoam.* **23**(3) (2007), 743–770.

3. B. Çekiç, R. Mashiyev and G. T. Alisoy, On the Sobolev-type inequality for Lebesgue spaces with a variable exponent, *Int. Math. Forum* **1**(25–28) (2006), 1313–1323.

4. D. Cruz-Uribe and A. Fiorenza, $L \log L$ results for the maximal operator in variable L^p spaces, *Trans. Amer. Math. Soc.* **361** (2009), 2631–2647.

5. L. Diening, Riesz potential and Sobolev embeddings of generalized Lebesgue and Sobolev spaces $L^{p(\cdot)}$ and $W^{k,p(\cdot)}$, *Math. Nachr.* **263**(1) (2004), 31–43.

6. L. Diening, P. Hästö and A. Nekvinda, Open problems in variable exponent Lebesgue and Sobolev spaces, in *FSDONA04 Proceedings*, Milovy, Czech Republic, 2004 (P. Drabek and J. Rákosník, Editors) (Academy of Sciences of the Czech Republic, Prague, 2005), 38–58.

7. D. Edmunds and J. Rákosník, Sobolev embeddings with variable exponent, *Stud. Math.* **143**(3) (2000), 267–293.

8. D. Edmunds and J. Rákosník, Sobolev embeddings with variable exponent. II, *Math. Nachr.* **246/247** (2002), 53–67.

9. T. Futamura, Y. Mizuta and T. Shimomura, Sobolev embeddings for variable exponent Riesz potentials on metric spaces, *Ann. Acad. Sci. Fenn. Math.* **31** (2006), 495–522.

10. P. Harjulehto, Variable exponent Sobolev spaces with zero boundary values, *Math. Bohem.* **132**(2) (2007), 125–136.

11. P. Harjulehto and P. Hästö, A capacity approach to the Poincaré inequality and Sobolev imbeddings in variable exponent Sobolev spaces, *Rev. Mat. Complut.* **17** (2004), 129–146.

12. P. Harjulehto and P. Hästö, Sobolev inequalities for variable exponents attaining the values 1 and n, *Publ. Mat.* **52** (2008), 347–363.

13. P. Hästö, Local-to-global results in variable exponent spaces, *Math. Res. Lett.* **16**(2) (2009), 263–278.

14. V. Kokilashvili and S. Samko, On Sobolev theorem for Riesz type potentials in the Lebesgue spaces with variable exponent, *Z. Anal. Anwend.* **22**(4) (2003), 899–910.

15. K. Kurata and N. Shioji, Compact embedding from $W_0^{1,2}(\Omega)$ to $L^{q(x)}(\Omega)$ and its application to nonlinear elliptic boundary value problem with variable critical exponent, *J. Math. Anal. Appl.* **339**(2) (2008), 1386–1394.

16. Y. Mizuta, *Potential theory in Euclidean spaces* (Gakkōtosho, Tokyo, 1996).

17. Y. Mizuta, Sobolev's inequality for Riesz potentials with variable exponents approaching 1, in Proceedings of the 16th International Conference on Finite or Infinite Dimensional Complex Analysis and Applications (Dongguk University, Gyeongju, Korea, 2008), 189–195.

18. Y. Mizuta, T. Ohno and T. Shimomura, Sobolev's inequalities and vanishing integrability for Riesz potentials of functions in the generalized Lebesgue space $L^{p(\cdot)}(\log L)^{q(\cdot)}$, *J. Math. Anal. Appl.* **345** (2008), 70–85.

19. J. Musielak *Orlicz spaces and modular spaces*, Lecture Notes in Mathematics, vol. 1034 (Springer, Berlin, 1983).

20. R. O'Neil Fractional integration in Orlicz spaces. I, *Trans. Amer. Math. Soc.* **115** (1965), 300–328.

21. S. Samko On a progress in the theory of Lebesgue spaces with variable exponent: Maximal and singular operators, *Integral Transform. Spec. Funct.* **16**(5–6) (2005), 461–482.

22. S. Samko, E. Shargorodsky and B. Vakulov Weighted Sobolev theorem with variable exponent for spatial and spherical potential operators. II, *J. Math. Anal. Appl.* **325**(1) (2007), 745–751.