SOME CHARACTERIZATIONS OF HARDY SPACES
ASSOCIATED WITH TWISTED CONVOLUTION

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Abstract

In this paper, we shall give some characterizations of the Hardy space associated with twisted convolution, including Lusin area integral, Littlewood–Paley \( g \)-function and heat maximal function.


Keywords and phrases: twisted convolution, Hardy space, atomic decomposition, Lusin area integral, Littlewood–Paley \( g \)-function.

1. Introduction

On \( \mathbb{C}^n \) consider the \( 2n \) linear differential operators

\[
Z_j = \frac{\partial}{\partial z_j} + \frac{1}{4} \bar{z}_j, \quad \bar{Z}_j = \frac{\partial}{\partial \bar{z}_j} - \frac{1}{4} z_j, \quad j = 1, 2, \ldots, n.
\]

(1)

Together with the identity they generate a Lie algebra \( h^n \) which is isomorphic to the \( 2n + 1 \)-dimensional Heisenberg algebra. The only nontrivial commutation relations are

\[
[Z_j, \bar{Z}_j] = -\frac{1}{2} I, \quad j = 1, 2, \ldots, n.
\]

(2)

The operator \( L \) defined by

\[
L = -\frac{1}{2} \sum_{j=1}^{n} (Z_j \bar{Z}_j + \bar{Z}_j Z_j)
\]

is nonnegative, self-adjoint and elliptic. Therefore it generates a diffusion semigroup \( \{T^L_t\}_{t \geq 0} = \{e^{-tL}\}_{t \geq 0} \). The operators in (1) generate a family of ‘twisted translations’
\[ \tau_w \text{ on } \mathbb{C}^n \text{ defined on measurable functions by} \]

\[
(\tau_w f)(z) = \exp \left( \frac{1}{2} \sum_{j=1}^{n} (w_j z_j + \overline{w}_j \overline{z}_j) \right) f(z) \]

\[ = f(z + w) \exp \left( \frac{i}{2} \Im(z \cdot \overline{w}) \right). \]

The ‘twisted convolution’ of two functions \( f \) and \( g \) on \( \mathbb{C}^n \) can now be defined as

\[
(f \times g)(z) = \int_{\mathbb{C}^n} f(w) \tau_{-w} g(z) \, dw \]

\[ = \int_{\mathbb{C}^n} f(z - w) g(w) \omega(z, w) \, dw, \]

where \( \omega(z, w) = \exp(\Im(z \cdot \overline{w})/2) \). More about twisted convolution can be found in \([1, 7, 9]\).

In \([8]\) the authors defined the Hardy space \( H^1_L(\mathbb{C}^n) \) associated with twisted convolution. They gave several characterizations of \( H^1_L(\mathbb{C}^n) \), such as maximal function, atomic decomposition and Riesz transform. The purpose of this paper is to consider other characterizations, including Lusin area integral and Littlewood–Paley \( g \)-function. In order to prove our result, we also give a heat maximal function characterization for \( H^1_L(\mathbb{C}^n) \).

We first give some basic notation for \( H^1_L(\mathbb{C}^n) \). Let \( B \) denote the class of \( C^\infty \)-functions \( \varphi \) on \( \mathbb{C}^n \), supported on the ball \( B(0, 1) \) such that \( \|\varphi\|_\infty \leq 1 \) and \( \|\nabla \varphi\|_\infty \leq 2 \). For \( t > 0 \), let \( \varphi_t(z) = t^{-2n} \varphi(z/t) \). Given \( \sigma > 0 \), \( 0 < \sigma \leq +\infty \) and a tempered distribution \( f \), define the grand maximal function

\[ M_\sigma f(z) = \sup_{\varphi \in B} \sup_{0 < t < \sigma} |\varphi_t \times f(z)|. \]

Then the Hardy space \( H^1_L(\mathbb{C}^n) \) can be defined by

\[ H^1_L(\mathbb{C}^n) = \{ f \in L^1(\mathbb{C}^n) \mid M_\infty f \in L^1(\mathbb{C}^n) \}. \]

We define atoms for \( H^1_L(\mathbb{C}^n) \) as follows. A function \( a(z) \) is an atom for the Hardy space \( H^1_L(\mathbb{C}^n) \) associated to a ball \( B(z_0, r) \) if the following properties hold.

1. \( \text{supp } a \subset B(z_0, r) \).
2. \( \|a\|_\infty \leq |B(z_0, r)|^{-1} \).
3. \( \int a(w) \omega(z_0, w) \, dw = 0 \).

The atomic quasi-norm in \( H^1_L(\mathbb{C}^n) \) is defined by

\[ \|f\|_{L-\text{atom}} = \inf \{ \Sigma |\lambda_j| \}, \]

where the infimum is taken over all decompositions \( f = \sum \lambda_j a_j \) and \( a_j \) are atoms.

The following result has been proved in \([8]\).
**Proposition 1.** For a tempered distribution $f$ on $\mathbb{C}^n$ the following properties are equivalent.

(i) $M_\infty f \in L^1(\mathbb{C}^n)$.

(ii) For some $\sigma$, $0 < \sigma < +\infty$, $M_\sigma f \in L^1(\mathbb{C}^n)$.

(iii) For some radial $\varphi \in S$, such that $\int \varphi(z) \, dz \neq 0$, we have

$$\sup_{0 < r < 1} |\varphi_t \times f(z)| \in L^1(\mathbb{C}^n).$$

(iv) The distribution $f$ can be decomposed as $f = \sum \lambda_j a_j$, where $a_j$ are atoms and $\sum |\lambda_j| < +\infty$.

It is well known that there are many equivalent characterizations of the classical Hardy spaces (see [4]), we shall consider other characterizations for $H^1_L(\mathbb{C}^n)$ in this paper.

We define the Lusin area integral operator by

$$(\mathcal{S}_t^\alpha f)(z) = \left( \int_0^{+\infty} \int_{|z-w| < \alpha t} |Q_t^L f(w)|^2 \frac{dw \, dt}{t^{2n+1}} \right)^{1/2},$$

where $Q_t^L f(x) = t^2 (\partial_s T_{s=t^2} f)(x)$.

**Remark 2.** It is easy to see that the definition of area integral operator is independent of $\alpha$ in the sense of $\|(S_t^\alpha f)\|_{L^p} \sim \|(S_t^\beta f)\|_{L^p}$, for $0 < \alpha < \beta < \infty$ and $0 < p < \infty$ (see [3]). In the following we use $S_L$ to denote $S^1_L$.

We can characterize $H^1_L(\mathbb{C}^n)$ as follows.

**Theorem 3.** A function $f \in H^1_L(\mathbb{C}^n)$ if and only if its area integral $S_L f \in L^1(\mathbb{C}^n)$ and $f \in L^1(\mathbb{C}^n)$. Moreover,

$$\|f\|_{H^1_L} \sim \|S_L f\|_{L^1}.$$  

The Littlewood–Paley $g$-function is defined by

$$g_L(f)(z) = \left( \int_0^{+\infty} |Q_t^L f(z)|^2 \frac{dt}{t} \right)^{1/2}.$$

The Hardy space $H^1_L(\mathbb{C}^n)$ can also be characterized by $g_L$ as in the following theorem.

**Theorem 4.** A function $f \in H^1_L(\mathbb{C}^n)$ if and only if $g_L f \in L^1(\mathbb{C}^n)$ and $f \in L^1(\mathbb{C}^n)$.

Moreover,

$$\|f\|_{H^1_L} \sim \|g_L f\|_{L^1}.$$  

We also need some basic propositions about the tent space (see [3]).
Let $0 < p < \infty$, and $1 \leq q \leq \infty$, then the tent space $T^p_q$ is defined as the space of functions $f$ on $\mathbb{C}^n \times \mathbb{R}^+$, so that

$$\left( \int_{\Gamma(z)} |f(w, t)|^q \frac{dw \, dt}{t^{2n+1}} \right)^{1/q} \in L^p(\mathbb{C}^n) \quad \text{when } 1 \leq q < \infty,$$

and

$$\sup_{(w, t) \in \Gamma(z)} |f(w, t)| \in L^p(\mathbb{C}^n) \quad \text{when } q = \infty,$$

where $\Gamma(z)$ is the standard cone whose vertex is $z \in \mathbb{C}^n$, that is,

$$\Gamma(z) = \{(w, t) : |w - z| < t \}.$$

Assume $B(z_0, r)$ is a ball in $\mathbb{C}^n$, its tent $\hat{B}$ is defined by

$$\hat{B} = \{(w, t) : |w - z_0| \leq r - t \}.$$

A function $a(z, t)$ that supported in a tent $\hat{B}$, $B$ is a ball in $\mathbb{C}^n$, is said to be an atom in the tent space $T^p_q$ if and only if it satisfies

$$\left( \int_{\hat{B}} |a(z, t)|^2 \frac{d\mu \, dt}{t} \right)^{1/2} \leq |B|^{1/2 - 1/p}.$$

The atomic decomposition of $T^p_q$ is stated as in the following proposition.

**Proposition 5.** When $0 < p \leq 1$, then any $f \in T^p_q$ can be written as $f = \sum \lambda_k a_k$, where $a_k$ are atoms and $\sum |\lambda_k|^p \leq C \|f\|_{T^p_q}^p$.

The paper is organized as follows: in Section 2, we give some estimates of the kernels; in Section 3, we prove the main results of this paper.

Throughout the article, we shall use $A$ and $C$ to denote the positive constants, which are independent of main parameters and may be different at each occurrence. By $B_1 \sim B_2$, we mean that there exists a constant $C > 1$ such that $1/C \leq B_1 / B_2 \leq C$.

## 2. Preliminaries

In this section, we give some estimates of the kernel of $Q^L_t$ that we shall use subsequently.

Let $\{T^L_t\}_{t>0}$ be the heat semigroup generated by the operator $L$, then, for $f \in L^2(\mathbb{C}^n)$, the function $e^{-tL}f$ has the special Hermite expansion (see [11])

$$e^{-tL}f(z) = (2\pi)^{-n} \sum_{k=0}^{\infty} e^{-(2k+n)t} f \times \varphi_k(z),$$

where $\varphi_k(z)$ are the Hermite polynomials.
where $\varphi_k$ is the Laguerre function. Therefore $e^{-tL}f$ is given by twisted convolution with the kernel
\begin{equation}
K_t(z) = (2\pi)^{-n} \sum_{k=0}^{\infty} e^{-(2k+n)t} \varphi_k(z).
\end{equation}

Let
\begin{equation}
L_\alpha^k(x) = \sum_{j=0}^{k} \frac{\Gamma(k + \alpha + 1)}{\Gamma(k - j + 1)\Gamma(j + \alpha + 1)} (-x)^j j!
\end{equation}
be the Laguerre polynomials of type $\alpha$ and degree $k$, then we have the following generating function for Laguerre polynomials:
\begin{equation}
\sum_{k=0}^{\infty} L_\alpha^k(x)r^k = (1 - r)^{-\alpha - 1} e^{-r/(1-r)x}.
\end{equation}

From (4) we obtain
\begin{equation}
K_t(z) = (4\pi)^{-n} (\sinh t)^{-n} e^{-\frac{1}{4}|z|^2(\coth t)}.
\end{equation}

It is easy to prove that the heat kernel $K_t(z)$ has the following estimates (see the proof of Lemma 7).

**Lemma 6.** There exists a positive constant $C > 0$ such that the following inequalities hold.
(i) $|K_t(z)| \leq Ct^{-n} e^{-C|z|^2/t}$.
(ii) $|\nabla K_t(z)| \leq Ct^{-n-1} e^{-C|z|^2/t}$.

Let $Q_t^L(z)$ be the twisted convolution kernel of $Q_t^L$, then
\begin{equation}
Q_t^L(z) = t^2 \partial_t K_s(z)|_{s=t^2}.
\end{equation}

**Lemma 7.** There exists a constant $C > 0$ such that the following inequalities hold.
(i) $|Q_t^L(z)| \leq Ct^{-2n} e^{-C t^{-1}|z|^2}$.
(ii) $|\nabla Q_t^L(z)| \leq Ct^{-2n-1} e^{-C t^{-2}|z|^2}$.

**Proof.** It is easy to see that
\begin{equation}
\partial_t K_t(z) = (4\pi)^{-n} (-n)(\sinh t)^{-n-1} (\cosh t) e^{-\frac{1}{4}|z|^2(\coth t)} \\
+ (4\pi)^{-n} (\sinh t)^{-n-2} (-\frac{1}{4}|z|^2) e^{-\frac{1}{4}|z|^2(\coth t)}.
\end{equation}
Therefore
\begin{equation}
|\partial_t K_t(z)| \leq C ((\sinh t)^{-n-1} (\cosh t) e^{-\frac{1}{4}|z|^2(\coth t)} \\
+ (\sinh t)^{-n-2} |z|^2 e^{-\frac{1}{4}|z|^2(\coth t)}).
\end{equation}
There exists a constant $C_1 > 0$ such that for all $t > 0$ we have
\[ \sinh t \geq C_1 t, \quad \coth t \geq C_1 t^{-1} \] (6) and
\[ (\sinh t)^{-n-1} \cosh t \leq \begin{cases} C_1 (\sinh t)^{-n-1} \leq C_1 t^{-n-1} & \text{if } 0 < t \leq 1, \\ C_1 (\sinh t)^{-n} \coth t \leq C_1 t^{-n-1} & \text{if } 1 < t < \infty. \end{cases} \] (7)

From (6) and (7) we have
\[ |\partial_t K_t(z)| \leq C (t^{-n-1} e^{-\frac{1}{4} t^{-1} |z|^2} + t^{-n-2} |z|^2 e^{-\frac{1}{4} t^{-1} |z|^2}) \leq C t^{-n-1} e^{-C t^{-1} |z|^2}. \]

So, we have
\[ |Q^L_t(z)| \leq C t^{-2n} e^{-C t^{-2} |z|^2}. \]
This proves part (i).

To prove part (ii), it is sufficient to prove
\[ |\partial_{z_j} \partial_t K_t(z)| \leq C t^{-n-\frac{3}{2}} e^{-C t^{-1} |z|^2}, \quad j = 1, 2, \ldots, n. \] (8)

It is easy to calculate
\[ \partial_{z_j} \partial_t K_t(z) = (4\pi)^{-n} (-n) (\sinh t)^{-n-1} (\cosh t) 2z_j (-\frac{1}{4} \cosh t) e^{-\frac{1}{4} |z|^2} (\coth t) + (4\pi)^{-n} (\sinh t)^{-n-2} (-\frac{1}{4} \cosh t) 2z_j (-\frac{1}{4} \cosh t) \times (-\frac{1}{4} |z|^2) e^{-\frac{1}{4} |z|^2} (\coth t) + (4\pi)^{-n} (\sinh t)^{-n-2} (-\frac{1}{2} z_j) e^{-\frac{1}{4} |z|^2} (\coth t). \]

By (6) and (7),
\[ |\partial_{z_j} \partial_t K_t(z)| \leq C \left( t^{-n-\frac{3}{2}} \frac{|z|}{\sqrt{t}} e^{-\frac{1}{4} t^{-1} |z|^2} + t^{-n-2} \left( \frac{|z|}{\sqrt{t}} \right)^3 e^{-\frac{1}{4} t^{-1} |z|^2} \right) \leq C t^{-n-\frac{3}{2}} e^{-C t^{-1} |z|^2}. \]

This completes the proof of (8) and so part (ii) is proved. \[ \square \]

We can also consider the following operator $Q^2_t = t^4 \partial_t^2 T^L_s |_{s=t^2}$. If we use $Q^2_t(z)$ to denote the twisted convolution kernel of $Q^2_t$, then similarly as Lemma 7, we can prove the following result.

**Lemma 8.** There exists a constant $C > 0$ such that the following inequalities hold.
(i) $|Q^2_t(z)| \leq C t^{-2n} e^{-C t^{-2} |z|^2}$.
(ii) $|\nabla Q^2_t(z)| \leq C t^{-2n-1} e^{-C t^{-2} |z|^2}$. 
3. The proofs of the main results

In this section, we shall give the proofs of the main results of this paper.

We first give the maximal function characterization for \( H^1_L(\mathbb{C}^n) \). By Proposition 1, we have the following lemma (see [8, p. 281]).

**Lemma 9.** \( f \in H^1_L(\mathbb{C}^n) \) if and only if

\[
\tilde{M} f(z) = \sup_{0 < t < 1} |K_t \times f(z)| \in L^1(\mathbb{C}^n)
\]

and \( f \in L^1(\mathbb{C}^n) \).

Let

\[
M f(z) = \sup_{t > 0} |K_t \times f(z)| \in L^1(\mathbb{C}^n),
\]

then we can characterize \( H^1_L(\mathbb{C}^n) \) by the maximal function \( M f \) as follows.

**Theorem 10.** \( f \in H^1_L(\mathbb{C}^n) \) if and only if \( M f \in L^1(\mathbb{C}^n) \) and \( f \in L^1(\mathbb{C}^n) \).

**Proof.** By Lemma 9, we know \( f \in H^1_L(\mathbb{C}^n) \) if \( M f \in L^1(\mathbb{C}^n) \) and \( f \in L^1(\mathbb{C}^n) \).

For the reverse, we just need to prove that there exists \( C > 0 \) such that for any atom \( a(z) \) of \( H^1_L(\mathbb{C}^n) \),

\[
\| Ma \|_{L^1} \leq C.
\]

Let \( a(z) \) be an atom of \( H^1_L(\mathbb{C}^n) \). By twisted translation, we can assume that \( \text{supp} \ a \subset B(0, r) \). Then, we have \( \int a(w) \, dw = 0 \).

By Lemma 6, we have

\[
\int_{|w| \leq 2r} |Ma(w)| \, dw \leq \| M_{H-L} a \|_{L^2(2r)^n} \leq C,
\]

where \( M_{H-L} \) is the Hardy–Littlewood maximal function.

For \( |z| > 2r \), we get

\[
K_t \times a(z) = \int K_t(z - w) a(w) \omega(z, w) \, dw = \int (K_t(z - w) - K_t(z)) a(w) \omega(z, w) \, dw + K_t(z) \tilde{a}
\]

\[
= I_1 + I_2.
\]

By Lemma 6 again, we can prove

\[
I_1 \leq C \int t^{-n-\frac{1}{2}} e^{-Ct^{-1}|z-w|^2} |w| |a(w)| \, dw \leq Cr |z|^{-2n-1}.
\]

Therefore

\[
\int_{|z| > 2r} |I_1| \, dz \leq C.
\]
By Lemma 6 (i) and Hardy’s inequality (see [6, p. 341, Theorem 7.22])
\[
\int_{|z|>2r} |I_2| \, dz \leq C \int_{\mathbb{C}^n} \frac{|\hat{a}(-i/2)z)|}{|z|^{2n}} \, dz \leq C.
\]
Therefore we have \( \| Ma \|_{L^1} \leq C \) and Theorem 10 is proved. \( \square \)

In order to get our results, we need the following lemma.

**Lemma 11.** The operators \( S_L \) and \( G_L \) are isometries on \( L^2(\mathbb{C}^n) \) up to constant factors. Exactly,
\[
\| G_L f \|_{L^2} = \frac{1}{2} \| f \|_{L^2}, \quad \| S_L f \|_{L^2} = \frac{\sqrt{c_n}}{2} \| f \|_{L^2}.
\]

**Proof.** The \( L^2 \) equality for \( G_L \) is established in [12, Proposition 3.1]. As a consequence we have
\[
\| S_L f \|_{L^2}^2 = \int_{\mathbb{C}^n} \int_{\mathbb{C}^n \times \mathbb{R}^+} |Q_L^f(w)|^2 \chi_{\Gamma}(z) \, dw \, dt \, dz = c_n \| G_L f \|_{L^2}^2 = \frac{c_n}{4} \| f \|_{L^2}^2.
\]

**Proof of Theorem 3.** As in the proof of Theorem 10, by Lemma 7, we can prove that there exists a constant \( C > 0 \) such that for any atom \( a(z) \) of \( H^1_L(\mathbb{C}^n) \) we have
\[
\| S_L a \|_{L^1} \leq C. \tag{9}
\]
Now we prove that \( f \in H^1_L(\mathbb{C}^n) \) when \( S_L f \in L^1(\mathbb{C}^n) \) and \( f \in L^1(\mathbb{C}^n) \).

We first assume that \( f \in L^1(\mathbb{C}^n) \cap L^2(\mathbb{C}^n) \). When \( S_L f \in L^1(\mathbb{C}^n) \), we know that \( Q_L^f \in T^1_2 \). By Proposition 5 we get
\[
Q_L^f(z) = \sum_j \lambda_j a_j(z, t),
\]
where \( a_j(z, t) \) are atoms of \( T^1_2 \) and \( \sum_j |\lambda_j| < \infty \). By the spectral theorem (see [10]), we can prove
\[
f(z) = 4 \int_0^\infty Q_L^f(Q_L^f f(z)) \frac{dt}{t}. \tag{11}
\]
By (10) and (11), we get
\[
f(z) = 4 \int_0^{+\infty} Q_L^f \left( \sum_j \lambda_j a_j(z, t) \right) \frac{dt}{t} = C \sum_j \lambda_j \int_0^{+\infty} Q_L^f a_j(z, t) \frac{dt}{t}.
\]
Therefore it is sufficient to prove that the functions
\[ \alpha_j = \int_0^{+\infty} Q_t^L a_j(z, t) \frac{dt}{t}, \quad i = 1, 2, \ldots, \]
are bounded in \( H^1_L(\mathbb{C}^n) \) uniformly; that is, there exists a constant \( C > 0 \) such that for any atom \( a(z, t) \) in \( T^1 \),
\[ \| \alpha \|_{H^1_L} = \left\| \int_0^{+\infty} Q_t^L a(z, t) \frac{dt}{t} \right\|_{H^1_L} \leq C. \]
We assume that \( a(z, t) \) is supported in \( \hat{B}(z_0, r) \), where \( \hat{B}(z_0, r) \) denotes the tent of the ball \( B(z_0, r) \), then
\[ \left\| \sup_{t > 0} |e^{-tL} \alpha(z)| \right\|_{L^1} \leq \left( \left\| \sup_{t > 0} |e^{-tL} \alpha(z)| \right\| \chi_{B^*} \right)_{L^1} + \left( \left\| \sup_{t > 0} |e^{-tL} \alpha(z)| \right\| \chi_{(B^*)^c} \right)_{L^1} = I_1 + I_2, \]
where \( B^* = B(z_0, 2r) \).
By the Hölder inequality, we get
\[ I_1 \leq |B^*|^{1/2} \left( \int_{\mathbb{C}^n} \left( \sup_{t > 0} |e^{-tL} \alpha(z)| \right)^2 \, dz \right)^{1/2} \leq |B^*|^{1/2} \| \alpha \|_{L^2}. \]
By the self-adjointness of \( Q_t^L \) and Lemma 11 we have
\[ \| \alpha \|_{L^2} = \sup_{\| \beta \|_{L^2} \leq 1} \int_{\mathbb{C}^n} \alpha(z) \overline{\beta}(z) \, dz \]
\[ = \sup_{\| \beta \|_{L^2} \leq 1} \int_{\mathbb{C}^n} \left( \int_0^{+\infty} Q_t^L a(z, t) \frac{dt}{t} \right) \overline{\beta}(z) \, dz \]
\[ = \sup_{\| \beta \|_{L^2} \leq 1} \int_0^{+\infty} \int_{\mathbb{C}^n} Q_t^L a(z, t) \overline{\beta}(z) \, dz \frac{dt}{t} \]
\[ = \sup_{\| \beta \|_{L^2} \leq 1} \left( \int_{\mathbb{C}^n} \int_0^{+\infty} |a(z, t)|^2 \frac{dz \, dt}{t} \right)^{1/2} \]
\[ \times \left( \int_{\mathbb{C}^n} \int_0^{+\infty} |Q_t^L \overline{\beta}(z)|^2 \frac{dz \, dt}{t} \right)^{1/2} \]
\[ \leq |B|^{-1/2} \| \beta \|_{L^2} \leq |B|^{-1/2}. \]
This proves \( I_1 \leq C \).
By Lemma 7 we can prove
\[
\sup_{s > 0} |e^{-sL} \int_{0}^{+\infty} Q_{s}^{t} a(z, t) \frac{dt}{t} |
\]
\[=
\sup_{s > 0} |e^{-sL} \int_{0}^{+\infty} (-L)te^{-tL} a(z, t) \frac{dt}{t} |
\]
\[=
\sup_{s > 0} \int_{0}^{+\infty} (-tL)e^{-(s+t)L} a(z, t) \frac{dt}{t} |
\]
\[=
\sup_{s > 0} \int_{0}^{+\infty} \left( \frac{t}{s+t} \right) ((s+t)L)e^{-(s+t)L} a(z, t) \frac{dt}{t} |
\]
\[=
\sup_{s > 0} \int_{0}^{+\infty} \left( \frac{t}{s+t} \right) \int_{\mathbb{C}^{n}} Q_{s+t}^{L}(z, w)a(w, t) \frac{dw dt}{t} |
\]
\[\leq \sup_{s > 0} \int_{0}^{+\infty} \frac{t}{s+t} \int_{\mathbb{C}^{n}} (s+t)^{-2n} \exp \left( -\frac{|z-w|^{2}}{(s+t)^{2}} \right) |a(w, t)| \frac{dw dt}{t} |
\]
\[\leq \sup_{s > 0} \int_{0}^{+\infty} \frac{t}{s+t} \int_{\mathbb{C}^{n}} (s+t)^{-2n} \left( 1 + \frac{|z-w|^{2}}{(s+t)^{2}} \right)^{-(n+1)} |a(w, t)| \frac{dw dt}{t} |
\]
\[\leq \sup_{s > 0} \left( \int_{0}^{r} \int_{B} (s+t)^{-4n} \left( 1 + \frac{|z-w|^{2}}{(s+t)^{2}} \right)^{-(2n+1)} \left( \frac{t}{s+t} \right)^{2} \frac{dw dt}{t} \right)^{1/2}
\]
\[\times \left( \int_{0}^{r} \int_{B} |a(w, t)|^{2} \frac{dw dt}{t} \right)^{1/2}
\]
\[\leq |B|^{-1/2} |z - z_{0}|^{2n+1} \left( \int_{0}^{r} \int_{B} t \frac{dw dt}{t} \right)^{1/2}
\]
\[\leq Cr |z - z_{0}|^{-(2n+1)}.
\]

Then we have
\[
I_{2} \leq Cr \int_{(B^{+})^{c}} |z - z_{0}|^{-(2n+1)} dz \leq C.
\]

When \( f \in L^{1}(\mathbb{C}^{n}) \) we can prove the result as in [2, Proposition 14]. In fact, we let \( f_{k} = T_{k}^{L}f, \ k \geq 0 \). Then, by \( f \in L^{1}(\mathbb{C}^{n}) \) and Lemma 6, we know \( f_{k} \in L^{2}(\mathbb{C}^{n}) \) and \( \|S_{L}f_{k}\|_{1} \leq \|S_{L}f\|. \) By the case of \( f \in L^{1} \cap L^{2} \), we get
\[
\|f_{k}\|_{L^{1}(\mathbb{C}^{n})} \lesssim \|S_{L}f_{k}\|_{L^{1}} \leq \|S_{L}f\|_{L^{1}}.
\]

By the monotone theorem, we have
\[
\|f_{k} - f_{n}\|_{L^{1}} \leq \|S_{k}^{L}(f_{k} - f_{n})\|_{L^{1}} \to 0 \quad \text{when} \ k, n \to +\infty.
\]

Therefore \( \{f_{k}\} \) is a Cauchy sequence in \( H_{L}^{1}(\mathbb{R}^{d}) \) and there exists \( g \in H_{L}^{1}(\mathbb{R}^{d}) \) such that
\[
\lim_{m \to +\infty} f_{k} = g \quad \text{in} \ H_{L}^{1}(\mathbb{R}^{d}).
\]
As
\[ \lim_{k \to +\infty} f_k = f \quad \text{in } (BMO_L)^*, \]
we know \( f = g \in H^1_L(\mathbb{R}^d) \) and \( \| f \|_{H^1_L(\mathbb{C}^n)} \lesssim \| S_L f \|_{L^1}. \)

This gives the proof of Theorem 3. \( \square \)

We define \( \widetilde{S}^\alpha_L f(z) \) by
\[ \widetilde{S}^\alpha_L(f)(z) = \left( \int_0^{+\infty} \int_{|z-w| < \alpha t} |Q_x^2 f(w)|^2 \frac{dw\,dt}{t^{2n+1}} \right)^{1/2}, \]
where \( \alpha > 0 \). Then in the same way as the proof of Theorem 3, we can prove the following result.

**Corollary 12.** A function \( f \in H^1_L(\mathbb{C}^n) \) if and only if its area integral \( \widetilde{S}^\alpha_L f \in L^1(\mathbb{C}^n) \) and \( f \in L^1(\mathbb{C}^n) \). Moreover,
\[ \| f \|_{H^1_L} \sim \| \widetilde{S}^\alpha_L f \|_{L^1}. \]

Now we can give the proof of Theorem 4.

**Proof of Theorem 4.** Firstly, by Lemma 7, we can prove that there exists a positive constant \( C \) such that for any atom \( a(z) \) of \( H^1_L(\mathbb{C}^n) \), we have
\[ \| \mathcal{G}_L a \|_{L^1} \leq C. \]

For the converse, by Corollary 12, it is sufficient to prove
\[ \| \widetilde{S}^1_L f \|_{L^1} \leq C \| \mathcal{G}_L f \|_{L^1}. \] (12)

Our proof is motivated by [5]. Let
\[ F(z)(t) = (\partial_t e^{-tL} f)(z), \quad V(z,s) = e^{-sL} F(z), \]
then
\[ V(z,s)(t) = e^{-sL} (\partial_t e^{-tL} f)(z) = (\partial_t e^{-(s+t)L} f)(z). \]

Therefore
\[ \int_0^{+\infty} |V(z,s)(t)|^2 t \, dt = \int_0^{+\infty} |(\partial_t e^{-(s+t)L} f)(z)|^2 t \, dt = \int_s^{+\infty} |(\partial_t e^{-tL} f)(z)|^2 (t-s) \, dt. \]

Hence
\[ \sup_{s > 0} \int_0^{+\infty} |V(z,s)(t)|^2 t \, dt \leq \int_0^{+\infty} |(\partial_t e^{-tL} f)(z)|^2 t \, dt = (\mathcal{G}_L f(z))^2. \]
Let $X = L^2((0, \infty), t \, dt)$, then
\[ \sup_{s > 0} \| e^{-sL} F(z) \|_X = \mathcal{G}_L f(z) \in L^1(\mathbb{C}^n). \]

Therefore $F \in H^1_X(\mathbb{C}^n)$ and here $H^1_X(\mathbb{C}^n)$ can be seen as a vector-valued Hardy space. This shows that $S^2_1 F(z) \in L^1(\mathbb{C}^n)$, where
\[ S^2_1 F(z) = \left( \int_0^{+\infty} \int_{|z-w| < 2t} \| Q_t^L F(w) \|^2_X \frac{dw \, dt}{t^{2n+1}} \right)^{1/2}. \]

By
\[ (S^2_1 F(z))^2 = \int_0^{+\infty} \int_{|z-w| < 2t} \| Q_t^L F(z) \|^2_X \frac{dw \, dt}{t^{2n+1}} \]
\[ = \int_0^{+\infty} \int_{|z-w| < 2t} \int_0^{+\infty} |t(-L)e^{-tL} F(w)(s)|^2 s \, ds \, \frac{dw \, dt}{t^{2n+1}} \]
\[ = \int_0^{+\infty} \int_{|z-w| < 2t} \int_0^{+\infty} |(-L)^2 e^{-(s+t)L} f(w)|^2 t^{1-2n} s \, dw \, dt \, ds \]
\[ = \int_0^{+\infty} \int_{|z-w| < 2t} \int_0^{+\infty} |(-L)^2 e^{-tL} f(w)|^2 (t-s)^{1-2n} s \, dw \, dt \, ds \]
\[ = \int_0^{+\infty} \int_{|z-w| < 2(t-s)} \int_0^{+\infty} |(-L)^2 e^{tL} f(w)|^2 (s-t)^{1-2n} s \, dw \, ds \, dt \]
\[ = \int_0^{+\infty} \int_{|z-w| < 2(t-s)} \int_0^{+\infty} |(-L)^2 e^{-tL} f(w)|^2 (s-t)^{1-2n} s \, dw \, ds \, dt \]
\[ \geq \int_0^{+\infty} \int_{|z-w| < t} \int_0^{+\infty} |(-L)^2 e^{-tL} f(w)|^2 t^{1-2n} s \, dw \, ds \, dt \]
\[ \geq \int_0^{+\infty} \int_{|z-w| < t} \int_0^{+\infty} |(-L)^2 e^{-tL} f(w)|^2 t^{1-2n} s \, dw \, ds \, dt \]
\[ = \frac{1}{8} \int_0^{+\infty} \int_{|z-w| < t} |(-L)^2 e^{-tL} f(w)|^2 t^{3-2n} s \, dw \, dt \]
\[ = \frac{1}{8} \int_0^{+\infty} \int_{|z-w| < t} |Q_t^L f(w)|^2 \frac{dw \, dt}{t^{2n+1}} = \frac{1}{8} (S^1_1 f(z))^2, \]
we get $\tilde{S}^1_1 f \in L^1(\mathbb{C}^n)$. Then $f \in H^1_L(\mathbb{C}^n)$ follows from Corollary 12.
This completes the proof of Theorem 4. \[ \square \]

References

Some characterizations of Hardy spaces associated with twisted convolution


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