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FINITENESS CONDITIONS FOR NEAR-RINGS

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ABSTRACT. There have been a number of papers which give necessary conditions for a ring to be finite, and a few, most notably H. E. Bell [1], which do the same for near-rings. We wish to make a contribution to this latter theme. Most of Bell's results concern distributive near-rings. Our main contribution is to extend a number of these results to weakly distributive near-rings.

We will use left near-rings, and all our near-rings will be zero-symmetric. A d.g., or distributively generated, near-ring R will often be denoted (R, S), where S is the semigroup of distributive generators. Unless otherwise stated, all our near-rings will be distributively generated. A d.g. near-ring R is said to be *weakly distributive* if there exists a series of ideals of R

$$R = I_0 > I_1 > \cdots > I_n = \{0\}$$

such that $(y+z)x - zx - yx \in I_{j+1}$ for all $x \in R$, $y, z \in I_j$, $0 \leq j \leq n-1$, or in other words all elements of R distribute over sums of elements of I_j modulo I_{j+1}. The least length of such a series is called the *weak distributivity class* of R. If R has class 1, i.e. $n = 1$, then R is a distributive near-ring. A fastest descending such distributive series exists, but we do not need it in this paper. More material on such near-rings can be obtained from Meldrum [3], Chapter 9 where we see the close relation of these near-rings to those with soluble additive group. This book also serves as a general reference on near-rings, as does Pilz [4]. We write most of our groups additively, and we will use $\delta_i(G)$ to denote the terms of the derived series of the group G.

The starting point is the following result due to Szele [5].

THEOREM 1. *If a ring R has both the ascending chain condition and descending chain condition on subrings, then R is finite.*

We state and prove a corresponding result for soluble groups. This result may well exist somewhere in the literature, but we have not been able to find it.

THEOREM 2. *If G is a soluble group with both the ascending and descending chain conditions on subgroups, then G is finite.*

The proof is accomplished by induction on the solubility class of G.

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COROLLARY 3. *A zero near-ring on a group of exponent 2 or on a soluble group which also has both chain conditions on subnear-rings is finite.*

PROOF. A group of exponent 2 is necessarily abelian. So in all cases, the underlying group is soluble. As all additive subgroups of a zero near-ring are subnear-rings, the theorem can be applied here to obtain the result.

The next situation we are going to look at is that of a tame endomorphism near-ring. We set up the situation first.

HYPOTHESIS 4. Let G be a group, S a group of automorphisms of G containing Inn G, the inner automorphisms of G. Let R denote the near-ring of mappings of G generated by S, so (R, S) is a d.g. near-ring. The condition that S is a group of automorphisms can be replaced by the condition that if $\alpha \in S$ and H/K is an S-simple homomorphic image of an S-subgroup H of G, then $(H/K)\alpha = H/K$. We will call this the *epimorphism property*.

The next series of results is aimed at showing that if such an R satisfies both chain conditions on subnear-rings, then it is finite.

REMARK 5. Under Hypothesis 4, any S-subgroup of G, i.e. a subgroup H of G such that $HS \subseteq H$, is normal in G. This follows immediately from the hypothesis that S contains the inner automorphisms of G. Hence any R-subgroup of G is an R-ideal of G.

LEMMA 6. *Assume Hypothesis 4. Let H be an S-subgroup of G. Then S induces actions on H and on G/H both of which satisfy Hypothesis 4.*

PROOF. Since H is an S-subgroup of G, and G/H is well defined as an S-group, the elements of S define actions on H and G/H. So there are homomorphisms $\theta: S \longrightarrow \operatorname{End} H$, $\varphi: S \longrightarrow \operatorname{End} G/H$, where $\operatorname{End} X$ denotes the semigroup of endomorphisms of X. The conditions that $S\theta \supseteq \operatorname{Inn} H$ and $S\varphi \supseteq \operatorname{Inn} G/H$ can be seen to be satisfied trivially. The epimorphism property is also trivially satisfied, as is the alternative property of being a group.

We next recall a result from Lyons and Meldrum [2], where it occurs as Theorem 4.3. Let G be an S-group with an S-series $G = G_0 \triangleright \cdots \triangleright G_n = \{0\}$ such that each G_i is an S-subgroup of G and let R be the d.g. near-ring of mappings of G generated by S. Then each G_i is an R-ideal of G.

THEOREM 7. *In the situation described just above, if R satisfies the descending chain condition on right ideals then all the S-simple non-abelian factors of an S-series of G are finite and there are only a finite number of them.*

We now come to the point at which we need the epimorphism property.

LEMMA 8. *Let H be an S-subgroup of G and let $C_G(H) = \{g \in G : g + h = h + g$ for all $h \in H\}$. If $H\alpha = H$ for all $\alpha \in S$ then $C_G(H)$ is also an S-subgroup of G.*

PROOF. That $C_G(H)$ is a subgroup is an elementary and well-known result. Let $g \in C_G(H)$, $\alpha \in S$. Then for all $h \in H$, there exists $h' \in H$ such that $h = h'\alpha$. Hence

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$g\alpha + h = g\alpha + h'\alpha = (g + h')\alpha = (h' + g)\alpha = h'\alpha + g\alpha = h + g\alpha$, showing that $g\alpha \in C_G(H)$ and proving our result.

The key step in the path to our main result is the following. It amounts to the ability to "move" an S-simple non-abelian factor of an S-series from below past a soluble factor.

LEMMA 9. *Let $G = G_0 \triangleright G_1 \triangleright \cdots \triangleright G_n = \{0\}$ be an S-series of G, and let Hypothesis 4 hold. Assume further that R satisfies the descending chain condition on ideals. Then we can re-arrange the series so that all the non-abelian S-simple factors come first in the series.*

PROOF. To prove this result by induction, using Theorem 7, we only need to show that we can move a non-abelian S-simple factor past a soluble factor.

Let $H \triangleright K \triangleright L \cdots$ be such that K/L is a finite non-abelian S-simple factor. Consider G/L. Then let $\theta: R \longrightarrow R/\operatorname{Ann}(G/L)$. Then $R\theta$ will also satisfy the descending chain condition for right ideals by the homomorphism theorems. Lemma 6 also shows that Hypothesis 4 holds in the new situation. So we may assume without loss of generality that $G \triangleright H \triangleright K \triangleright \{0\}$ where K is a finite non-abelian S-simple subgroup.

Lemma 8 now allows us to deduce that $C_G(K)$ is an S-group. Then $C_G(K) \cap K$ is an S-subgroup of K which is S-simple. Hence $C_G(K) \cap K$ is either K or $\{0\}$. If it is K, then K must be abelian, which contradicts the hypothesis. Thus we must have $C_G(K) \cap K = \{0\}$. Write M for $C_H(K) = H \cap C_G(K)$. Then M is an S-subgroup and $H/M = H/C_H(K)$ is isomorphic to a subgroup of $\operatorname{Aut} K$ and is, in consequence, finite. Note that $M \cap K = \{0\}$ since $C_G(K) \cap K = \{0\}$. Also $M + K/K \cong M/M \cap K \cong M$ and $M + K \subseteq H$. So $M + K/K \cong M$ is a subgroup of H/K and must be soluble. We have now $H \triangleright M \triangleright \{0\}$, with the finite factor H/M first and the soluble factor M next.

Using the appropriate form of the Jordan-Hölder theorem we know that H/M contains a subfactor isomorphic to K, the rest of the factors, if any, being soluble. By repeating the above process if necessary a finite number of times, we can move the factor isomorphic to K so that we end up with the situation $H \triangleright L \triangleright \{0\}$, where $H/L \cong K$ and L is soluble. This is enough to prove the lemma.

We are now in a position to prove the theorem.

THEOREM 10. *Let G and (R, S) satisfy Hypothesis 4. If R satisfies the ascending and descending chain conditions on subnear-rings, then R has a finite ideal I such that the underlying group of R/I is soluble.*

PROOF. By Lemma 9 we can assume that G has a soluble S-subgroup H of finite index. Define $N = \{r \in R : Gr \subseteq H, Hr = 0\}$. Then $N = \operatorname{Ann}_R(G/H) \cap \operatorname{Ann}_R H$, and $N^2 = \{0\}$, with N an ideal of R. Also, by definition, $(N, +) \subseteq H^G$, a direct power of a soluble group. Thus H^G is soluble and so is $(N, +)$. By Corollary 3, it follows that N is finite. As in Lyons and Meldrum [2], R/N is isomorphic to a subdirect product of $R/\operatorname{Ann}_R(G/H)$ and $R/\operatorname{Ann}_R(H)$. The first of these is finite since G/H is finite. The second is soluble since any near-ring of mappings of a soluble group is soluble. So $R/\operatorname{Ann}_R(H)$ is an epimorphic image of R/N with kernel of size at most $|R/\operatorname{Ann}_R(G/H)|$, i.e. finite. Since N is also finite, we are home.

As we will see shortly, $R/\operatorname{Ann}_R(H)$ is also finite. This will follow from the next theorem which widens considerably the class of near-rings which are finite.

DEFINITION 11. Denote by $\mathcal{X}$ the class of near-rings with the property that if $R \in \mathcal{X}$ and R has the ascending and descending chain conditions on subnear-rings, then R is finite.

THEOREM 12. *Let R be a near-ring such that R has a series of ideals $R = R_0 \triangleright R_1 \triangleright \cdots \triangleright R_k = \{0\}$, such that $R_i/R_{i+1} \in \mathcal{X}$ for $0 \leq i \leq k-1$. Then $R \in \mathcal{X}$.*

PROOF. We use induction on k. If $k = 1$, then the result is trivial. So assume that $k > 1$ and that the result is true for all near-rings with an $\mathcal{X}$-series of length at most $k-1$. Consider R/R_1 and R_1. Both these near-rings have both chain conditions on subnear-rings, in the first case because of the homomorphism theorems, in the second case trivially. By the induction hypothesis both are finite. Hence so is R. This completes the induction argument.

One of the problems in applying Theorem 12 lies in the fact that an ideal of a d.g. near-ring is not necessarily a d.g. near-ring. But we can still show that a d.g. near-ring with identity on a soluble group is in $\mathcal{X}$.

THEOREM 13. *A d.g. near-ring with identity whose underlying group is soluble is in $\mathcal{X}$.*

PROOF. Let R be such a d.g. near-ring and let n be the solubility class. Then $R \supset \delta_1(R) \supset \cdots \supset \delta_{n-1}(R) \supset \{0\}$, the derived series of R is a series of ideals of R (Meldrum [3], Theorem 9.45). By Meldrum [3], Lemma 9.47, $\delta_{n-1}(R)$ is a zero near-ring and as it is abelian it is a ring. So $\delta_{n-1}(R)$ is in $\mathcal{X}$. But $(R/\delta_{n-1}(R),+)$ is of solubility class $n-1$ and $R/\delta_{n-1}(R)$ is distributively generated. An induction argument using Theorem 12 gives us the result.

This result enables us to conclude that in Theorem 10 R is finite.

To sum up we now know that $\mathcal{X}$ contains finite near-rings, rings, zero near-rings on soluble groups, d.g. near-rings on soluble groups, d.g. near-rings arising as in Hypothesis 4, endomorphism near-rings on soluble groups. Also included are all the classes described in Bell [1].

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