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Functional Analytic Methods in Spacetime

When constructing a causal fermion system in Minkowski space in Section 5.5, we chose $\mathcal{H}$ as a subspace of the solution space $\mathcal{H}_m$ of the Dirac equation. In principle, one can choose $\mathcal{H}$ as one likes, and different choices give rise to different causal fermion systems. However, if one wants to describe a given physical system, one must specify the subspace $\mathcal{H} \subset \mathcal{H}_m$, and it important to do it right. It is not obvious what "right" and "wrong" should be. Generally speaking, $\mathcal{H}$ can be thought of as the "occupied states" of the physical system under consideration. If we want to describe the vacuum in Minkowski space (i.e., no particles and no interaction is present), then the natural and only physically reasonable choice is to let $\mathcal{H}$ be the subspace of all *negative-frequency solutions* of the Dirac equation. As already explained in Section 1.5 in the preliminaries, this choice corresponds to the physical concept of the Dirac sea as introduced by Dirac in 1930, which led to the prediction of antimatter (discovered shortly afterward in 1932, earning Dirac the Nobel prize in 1933). Following these physical concepts, it is also clear that if particles and/or antiparticles (but no interaction of the matter) is present, then $\mathcal{H}$ is obtained from the subspace of all negative-frequency solutions by occupying additional particle states and by creating "holes" in the sea corresponding to the antiparticle states. Once an interaction (e.g., an electromagnetic field) is present, it is no longer clear how $\mathcal{H}$ is to be chosen. The reason is that, as soon as the fields are time-dependent, the notion of positive and negative frequency solutions breaks down, so that there is no obvious decomposition of the solution space into two subspaces. But for the description of the physical system, a decomposition of the solution space is needed, and taking the "wrong" decomposition leads to artificial mathematical and physical difficulties.

We now explain a functional analytic method which gives rise to a canonical decomposition of the solution space into two subspaces, even in the time-dependent situation. In the static situation, this decomposition reduces to the canonical frequency splitting. This splitting is "right" in the sense that it gives rise to a physically sensible ground state of the system (a so-called Hadamard state, as we will learn in Chapter 19). Moreover, when performing our construction perturbatively, one can compute the singularities of $P(x, y)$ explicitly working exclusively with bounded line integrals. These explicit computations are the backbone of the analysis of the continuum limit in [45]. Before outlining the perturbative treatment (see Chapter 18), we now explain the general functional analytic construction.

15.1 General Setting and Basic Ideas

In preparation, we summarize the structures of Chapter 1 using a more general notation, which has the advantage that our setting applies just as well if Minkowski space is replaced by a globally hyperbolic spacetime. Thus the reader who is familiar with general relativity and Lorentzian geometry, in what follows may consider $(\mathscr{M}, g)$ as a globally hyperbolic Lorentzian manifold with spinor bundle $(S\mathscr{M}, \prec.|.\succ)$. The Dirac equation is written as

$$(\mathcal{D} - m)\, \psi_m = 0, \tag{15.1}$$

here the subscript m indicates the mass of the solution; this is of advantage because later on, we shall consider families of solutions with a varying mass parameter. In Minkowski space, one chooses $\mathcal{D} = i\partial\!\!\!/ + \mathcal{B}$ such as to get back to (1.39), where $\mathcal{B}$ is an arbitrary multiplication operator satisfying the symmetry condition (1.38). More generally, in a globally hyperbolic spacetime, the Dirac operator is a first-order differential operator, but the coefficients depend on the metric (for details, see Chapter 4). Next, we let $\mathscr{N}$ be any Cauchy surface. Then the scalar product (1.37) on the solutions can be written more generally as

$$(\psi_m|\phi_m)_m = 2\pi \int_{\mathscr{N}} \prec\psi_m|\nu\!\!\!/\, \phi_m\succ_x \, \mathrm{d}\mu_{\mathscr{N}}(x)\,, \tag{15.2}$$

where ν is the future-directed normal and $\mathrm{d}\mu_{\mathscr{N}}$ the volume measure given by the induced Riemannian metric on $\mathscr{N}$ (in Minkowski space and $\mathscr{N} = \{t = \text{const}\}$, the normal has the components $\nu^i = (1, 0, 0, 0)$ and $\mathrm{d}\mu_{\mathscr{N}} = \mathrm{d}^3x$, giving back (1.37)). Similar to the computation (1.35), the vector field $\prec\psi_m|\gamma^j\phi_m\succ_x$ is again divergence-free, implying that the abovementioned scalar product is independent of the choice of the Cauchy surface (for details, see [70, Section 2]). Forming the completion gives a Hilbert space denoted by $(\mathcal{H}_m, (.|.)_m)$.

For the following constructions, we again need the spacetime inner product (13.82). In order to explain the basic idea of the construction as first given in [70], let us assume for simplicity that the integral in (13.82) exists for all solutions $\psi_m, \phi_m \in \mathcal{H}_m$. This condition is not satisfied in Minkowski space because the time integral in (13.82) in general diverges. But it is indeed satisfied in spacetimes of finite lifetime (for details, see [70, Section 3.2]). Under this assumption, the spacetime inner product can be extended by continuity to a sesquilinear form

$$<.|.> \,:\, \mathcal{H}_m \times \mathcal{H}_m \to \mathbb{C}\,, \tag{15.3}$$

which is bounded, that is,

$$|<\phi_m|\psi_m>| \leq c\, \|\phi_m\|_m\, \|\psi_m\|_m \tag{15.4}$$

where $\|.\|_m = (.|.)_m^{\frac{1}{2}}$ is the norm on $\mathcal{H}_m$. Then, applying the Fréchet–Riesz theorem (as explained in the construction of the local correlation operator (5.32) in Section 5.5), we can uniquely represent this inner product on the Hilbert space $\mathcal{H}_m$ with a signature operator $\mathcal{S}$,

$$\mathcal{S} \,:\, \mathcal{H}_m \to \mathcal{H}_m \qquad \text{with} \qquad <\phi_m|\psi_m> = (\phi_m\,|\,\mathcal{S}\,\psi_m)_m\,. \tag{15.5}$$

We refer to $\mathcal{S}$ as the **fermionic signature operator**. It is obviously a symmetric operator. Moreover, it is bounded according to (15.4). Therefore, the spectral theorem for selfadjoint operators gives the spectral decomposition

$$\mathcal{S} = \int_{\sigma(\mathcal{S})} \lambda \, dE_\lambda \,, \tag{15.6}$$

where E_λ is the spectral measure (see Section 3.2 or, e.g., [131]). The spectral measure gives rise to the spectral calculus

$$f(\mathcal{S}) = \int_{\sigma(\mathcal{S})} f(\lambda) \; \mathrm{d}E_\lambda \;:\; \mathcal{H}_m \to \mathcal{H}_m \,, \tag{15.7}$$

where f is a bounded Borel function on $\sigma(\mathcal{S}) \subset \mathbb{R}$. Choosing f as a characteristic function, one obtains the operators $\chi_{(0,\infty)}(\mathcal{S})$ and $\chi_{(-\infty,0)}(\mathcal{S})$. Their images are referred to as the *positive* and *negative spectral subspace* of $\mathcal{H}_m$, respectively. In this way, one obtains the desired decomposition of the solution space into two subspaces. We remark that the fermionic signature operator also gives a setting for doing spectral geometry and index theory with Lorentzian signature. We will not enter these topics here but refer the interested reader to the papers [69, 46].

The basic shortcoming of the abovementioned construction is that in many physically interesting spacetimes (like Minkowski space) the inequality (15.4) fails to be true. The idea to bypass this problem is to make use of the fact that a typical solution $\psi \in C^\infty_{\mathrm{sc}}(\mathscr{M}, S\mathscr{M}) \cap \mathcal{H}_m$ of the Dirac equation oscillates for large times. If, instead of a single solution, we consider a *family* of solutions with a *varying mass parameter* m, then the wave functions for different values of m typically have different phases. Therefore, integrating over the mass parameter leads to dephasing (in the physics literature also referred to as destructive interference), giving rise to decay in time. In order to make this idea mathematically precise, one considers families of solutions $(\psi_m)_{m\in I}$ of the family of Dirac equations (15.1) with the mass parameter m varying in an open interval I. We need to assume that I does not contain the origin, because our methods for dealing with infinite lifetime do not apply in the massless case $m = 0$ (this seems no physical restriction because all known fermions in nature have a nonzero rest mass). By symmetry, it suffices to consider positive masses. Thus we choose I as the interval

$$I := (m_L, m_R) \subset \mathbb{R} \qquad \text{with parameters } m_L, m_R > 0 \,. \tag{15.8}$$

The masses of the Dirac particles of our physical system should be contained in I. Apart from that, the choice of I is arbitrary and, as we shall see, all our results will be independent of the choice of m_L and m_R. We always choose the family of solutions $(\psi_m)_{m\in I}$ in the class $C^\infty_{\mathrm{sc},0}(\mathscr{M}\times I, S\mathscr{M})$ of smooth solutions with spatially compact support in Minkowski space $\mathscr{M}$ which depend smoothly on m and vanish identically for m outside a compact subset of I. Then the "decay due to destructive interference" can be made precise by demanding that there is a constant $c > 0$ such that

$$\Big| <\!\int_I \phi_m \; \mathrm{d}m \mid \int_I \psi_{m'} \; \mathrm{d}m'\!> \Big| \le c \int_I \|\phi_m\|_m \, \|\psi_m\|_m \; \mathrm{d}m \tag{15.9}$$

for all families of solutions $(\psi_m)_{m\in I}, (\phi_m)_{m\in I} \in C^\infty_{\mathrm{sc},0}(\mathscr{M} \times I, S\mathscr{M})$. The point is that we integrate over the mass parameter *before* taking the spacetime inner product. Intuitively speaking, integrating over the mass parameter generates a decay of the wave function, making sure that the time integral converges. The inequality (15.9) is one variant of the so-called *mass oscillation property*. If (15.9) holds, we shall prove that there is a representation

$$<\int_I \phi_m \, \mathrm{d}m \,|\, \int_I \psi_{m'} \, \mathrm{d}m'> = \int_I (\phi_m \,|\, \tilde{\mathcal{S}}_m \, \psi_m)_m \, \mathrm{d}m \,, \tag{15.10}$$

which for every $m \in I$ uniquely defines the *fermionic signature operator* $\tilde{\mathcal{S}}_m$. This operator is bounded and symmetric with respect to the scalar product (15.2). Moreover, it does not depend on the choice of the interval I. Then the positive and negative spectral subspaces of the operator $\tilde{\mathcal{S}}_m$ again yield the desired splitting of the solution space into two subspaces.

Before entering the detailed constructions, we explain how the abovementioned integrals over the mass parameters are to be understood. At first sight, integrating over a varying mass parameter $m \in I$ may look like "smearing out" the physical mass in the Dirac equation. However, this picture is misleading. Instead, one should consider the mass integrals merely as a technical tool in order to generate decay for large times. The resulting operators $\tilde{\mathcal{S}}_m$ in (15.10) act on ψ_m with the corresponding mass $m \in I$. Choosing m again as the physical mass, the operator $\tilde{\mathcal{S}}_m$ acts on standard Dirac wave functions describing physical particles, without any smearing in the mass parameter.

15.2 The Mass Oscillation Properties

In a spacetime of infinite lifetime, the spacetime inner product $<\psi_m|\phi_m>$ of two solutions $\psi_m, \phi_m \in \mathcal{H}_m$ is in general ill defined, because the time integral in (13.82) may diverge. In order to avoid this difficulty, we shall consider families of solutions with a variable mass parameter. The so-called mass oscillation property will make sense of the spacetime integral in (13.82) after integrating over the mass parameter.

We consider the mass parameter in a bounded open interval I (15.8). For a given Cauchy surface $\mathscr{N}$, we consider a function $\psi_{\mathscr{N}}(x, m) \in S_x\mathscr{M}$ with $x \in \mathscr{N}$ and $m \in I$. We assume that this wave function is smooth and has compact support in both variables, $\psi_{\mathscr{N}} \in C^\infty_0(\mathscr{N} \times I, S\mathscr{M})$. For every $m \in I$, we let $\psi(., m)$ be the solution of the Cauchy problem for initial data $\psi_{\mathscr{N}}(., m)$,

$$(\mathcal{D} - m)\, \psi(x, m) = 0 \,, \qquad \psi(x, m) = \psi_{\mathscr{N}}(x, m) \;\; \forall\, x \in \mathscr{N} \,. \tag{15.11}$$

Since the solution of the Cauchy problem is smooth and depends smoothly on parameters, we know that $\psi \in C^\infty(\mathscr{M} \times I, S\mathscr{M})$. Moreover, due to finite propagation speed, $\psi(., m)$ has spatially compact support. Finally, the solution is clearly compactly supported in the mass parameter m. We summarize these properties by writing

$$\psi \in C^\infty_{\mathrm{sc},0}(\mathscr{M} \times I, S\mathscr{M}) \,, \tag{15.12}$$

where $C^\infty_{\text{sc},0}(\mathscr{M} \times I, S\mathscr{M})$ denotes the smooth wave functions with spatially compact support which are also compactly supported in I. We often denote the dependence on m by a subscript, $\psi_m(x) := \psi(x,m)$. Then for any fixed m, we can take the scalar product (15.2). On families of solutions $\psi, \phi \in C^\infty_{\text{sc},0}(\mathscr{M} \times I, S\mathscr{M})$ of (15.11), we introduce a scalar product by integrating over the mass parameter,

$$(\psi|\phi)_I := \int_I (\psi_m|\phi_m)_m \, \mathrm{d}m, \tag{15.13}$$

where $\mathrm{d}m$ is the Lebesgue measure. Forming the completion, we obtain the Hilbert space $(\mathcal{H}, (.|.)_I)$. It consists of measurable functions $\psi(x,m)$ such that for almost all $m \in I$, the function $\psi(.,m)$ is a weak solution of the Dirac equation which is square integrable over any Cauchy surface. Moreover, this spatial integral is integrable over $m \in I$, so that the scalar product (15.13) is well defined. We denote the norm on $\mathcal{H}$ by $\|.\|_I$.

For the applications, it is useful to introduce a subspace of the solutions of the form (15.12):

Definition 15.2.1 *We let $\mathcal{H}^\infty \subset C^\infty_{\text{sc},0}(\mathscr{M} \times I, S\mathscr{M}) \cap \mathcal{H}$ be a subspace of the smooth solutions with the following properties:*

(i) *$\mathcal{H}^\infty$ is invariant under multiplication by smooth functions in the mass parameter,*

$$\eta(m)\, \psi(x,m) \in \mathcal{H}^\infty \qquad \forall\, \psi \in \mathcal{H}^\infty,\ \eta \in C^\infty(I)\,. \tag{15.14}$$

(ii) *For every $m \in I$, the set $\mathcal{H}^\infty_m := \{\psi(.,m) \,|\, \psi \in \mathcal{H}^\infty\}$ is a dense subspace of $\mathcal{H}_m$,*

$$\overline{\mathcal{H}^\infty_m}^{(.|.)_m} = \mathcal{H}_m \qquad \forall\, m \in I\,. \tag{15.15}$$

We refer to $\mathcal{H}^\infty$ as the **domain** *for the mass oscillation property.*

The simplest choice is to set $\mathcal{H}^\infty = C^\infty_{\text{sc},0}(\mathscr{M} \times I, S\mathscr{M}) \cap \mathcal{H}$, but in some applications it is preferable to choose $\mathcal{H}^\infty$ as a proper subspace of $C^\infty_{\text{sc},0}(\mathscr{M} \times I, S\mathscr{M}) \cap \mathcal{H}$. (e.g., in [71, Section 6], the space $\mathcal{H}^\infty$ was chosen as being spanned by a finite number of angular modes, making it unnecessary to prove estimates uniform in the angular mode).

Our motivation for considering a variable mass parameter is that integrating over the mass parameter should improve the decay properties of the wave function for large times (as explained in the introduction in the vacuum Minkowski space). This decay for large times should also make it possible to integrate the Dirac operator in the inner product (13.82) by parts without boundary terms,

$$<\mathcal{D}\mathfrak{p}\psi|\mathfrak{p}\phi> = <\mathfrak{p}\psi|\mathcal{D}\mathfrak{p}\phi>\,, \tag{15.16}$$

implying that the solutions for different mass parameters should be orthogonal with respect to this inner product. Instead of acting with the Dirac operator, it is technically easier to work with the operator of multiplication by m, which we denote by

$$T : \mathcal{H} \to \mathcal{H} , \qquad (T\psi)_m = m\, \psi_m . \tag{15.17}$$

In view of property (i) in Definition 15.2.1, this operator leaves $\mathcal{H}^\infty$ invariant,

$$T|_{\mathcal{H}^\infty} : \mathcal{H}^\infty \to \mathcal{H}^\infty . \tag{15.18}$$

Moreover, T is a symmetric operator, and it is bounded because the interval I is,

$$T^* = T \in \mathrm{L}(\mathcal{H}) . \tag{15.19}$$

Finally, integrating over m gives the operation

$$\mathfrak{p} : \mathcal{H}^\infty \to C^\infty_{\mathrm{sc}}(\mathscr{M}, S\mathscr{M}) , \qquad \mathfrak{p}\psi = \int_I \psi_m \, \mathrm{d}m . \tag{15.20}$$

We point out for clarity that $\mathfrak{p}\psi$ no longer satisfies a Dirac equation. The following notions were introduced in [71], and we refer the reader to this paper for more details.

Definition 15.2.2 *The Dirac operator $\mathcal{D} = i\partial\!\!\!/ + \mathcal{B}$ on Minkowski space $\mathscr{M}$ has the* **weak mass oscillation property** *in the interval $I = (m_L, m_R)$ with domain $\mathcal{H}^\infty$ if the following conditions hold:*

(a) *For every $\psi, \phi \in \mathcal{H}^\infty$, the function $\prec\!\mathfrak{p}\phi|\mathfrak{p}\psi\!\succ$ is integrable on $\mathscr{M}$. Moreover, there is a constant $c = c(\psi)$ such that*

$$|{<}\mathfrak{p}\psi|\mathfrak{p}\phi{>}| \le c\, \|\phi\|_I \qquad \text{for all } \phi \in \mathcal{H}^\infty . \tag{15.21}$$

(b) *For all $\psi, \phi \in \mathcal{H}^\infty$,*

$$<\mathfrak{p}T\psi|\mathfrak{p}\phi> = <\mathfrak{p}\psi|\mathfrak{p}T\phi> . \tag{15.22}$$

Definition 15.2.3 *The Dirac operator $\mathcal{D} = i\partial\!\!\!/ + \mathcal{B}$ on Minkowski space $\mathscr{M}$ has the* **strong mass oscillation property** *in the interval $I = (m_L, m_R)$ with domain $\mathcal{H}^\infty$ if there is a constant $c > 0$ such that*

$$|{<}\mathfrak{p}\psi|\mathfrak{p}\phi{>}| \le c \int_I \|\phi_m\|_m \, \|\psi_m\|_m \, \mathrm{d}m \qquad \textit{for all } \psi, \phi \in \mathcal{H}^\infty . \tag{15.23}$$

15.3 The Fermionic Signature Operator

In this section we give abstract constructions based on the mass oscillation property. We first assume that the *weak mass oscillation property* of Definition 15.2.2 holds. Then, in view of the inequality (15.21), every $\psi \in \mathcal{H}^\infty$ gives rise to a bounded linear functional on $\mathcal{H}^\infty$. By continuity, this linear functional can be uniquely extended to $\mathcal{H}$. The Fréchet–Riesz theorem allows us to represent this linear functional by a vector $u \in \mathcal{H}$, that is,

$$(u|\phi)_I = <\mathfrak{p}\psi|\mathfrak{p}\phi> \qquad \forall\, \phi \in \mathcal{H} . \tag{15.24}$$

Varying ψ, we obtain the linear mapping

$$\mathcal{S} : \mathcal{H}^\infty \to \mathcal{H} , \qquad (\mathcal{S}\psi|\phi)_I = <\mathfrak{p}\psi|\mathfrak{p}\phi> \quad \forall\, \phi \in \mathcal{H} . \tag{15.25}$$

This operator is symmetric because

$$(\mathcal{S}\psi|\phi)_I = <\mathfrak{p}\psi|\mathfrak{p}\phi> = (\psi|\mathcal{S}\phi)_I \qquad \forall\, \phi, \psi \in \mathcal{H}^\infty\,. \tag{15.26}$$

Moreover, (15.22) implies that the operators $\mathcal{S}$ and T commute,

$$\mathcal{S}\, T = T\, \mathcal{S} \;:\; \mathcal{H}^\infty \to \mathcal{H}\,. \tag{15.27}$$

Thus the weak mass oscillation property makes it possible to introduce $\mathcal{S}$ as a densely defined symmetric operator on $\mathcal{H}$. It is indeed possible to construct a selfadjoint extension of the operator $\mathcal{S}^2$ (using the Friedrich's extension), giving rise to a functional calculus with corresponding spectral measure (for details, see [71, Section 3]). In this setting the operator $\mathcal{S}$ and the spectral measure are operators on the Hilbert space $\mathcal{H}$ which involves an integration over the mass parameter. In simple terms, this implies that all objects are defined only for almost all values of m (with respect to the Lebesgue measure on $I \subset \mathbb{R}$), and they can be modified arbitrarily on subsets of I of measure zero. But it does not seem possible to "evaluate pointwise in the mass" by constructing operators $\mathcal{S}_m$ which act on the Hilbert space $\mathcal{H}_m$ for fixed mass.

In view of this shortcoming, we shall not enter the spectral calculus based on the weak mass oscillation operator. Instead, we move on to the *strong mass oscillation property*, which makes life much easier because it implies that $\mathcal{S}$ is a bounded operator.

Theorem 15.3.1 *The following statements are equivalent:*

(i) *The strong mass oscillation property holds.*
(ii) *There is a constant $c > 0$ such that for all $\psi, \phi \in \mathcal{H}^\infty$, the following two relations hold:*

$$|<\mathfrak{p}\psi|\mathfrak{p}\phi>| \le c\, \|\psi\|_I\, \|\phi\|_I \tag{15.28}$$

$$<\mathfrak{p}T\psi|\mathfrak{p}\phi> = <\mathfrak{p}\psi|\mathfrak{p}T\phi>\,. \tag{15.29}$$

(iii) *There is a family of linear operators $\mathcal{S}_m \in \mathrm{L}(\mathcal{H}_m)$ which are uniformly bounded,*

$$\sup_{m\in I} \|\mathcal{S}_m\| < \infty\,, \tag{15.30}$$

such that

$$<\mathfrak{p}\psi|\mathfrak{p}\phi> = \int_I (\psi_m\,|\,\mathcal{S}_m\,\phi_m)_m\; \mathrm{d}m \qquad \forall\, \psi, \phi \in \mathcal{H}^\infty\,. \tag{15.31}$$

Proof The implication (iii)⇒(i) follows immediately from the estimate

$$\begin{aligned} |<\mathfrak{p}\psi|\mathfrak{p}\phi>| &\le \int_I \big|(\psi_m|\mathcal{S}_m\phi_m)_m\big|\; \mathrm{d}m \\ &\le \sup_{m\in I} \|\mathcal{S}_m\| \int_I \|\psi_m\|_m\, \|\phi\|_m\; \mathrm{d}m\,. \end{aligned} \tag{15.32}$$

In order to prove the implication (i)⇒(ii), we first apply the Schwarz inequality to (15.23) to obtain

$$\begin{aligned}|\mathopen{<}\mathfrak{p}\psi|\mathfrak{p}\phi\mathclose{>}| &\leq c \int_I \|\phi_m\|_m \,\|\psi_m\|_m \;\mathrm{d}m \\ &\leq c \Big(\int_I \|\phi_m\|_m^2 \;\mathrm{d}m \Big)^{\frac{1}{2}} \Big(\int_I \|\psi_m\|_m^2 \;\mathrm{d}m \Big)^{\frac{1}{2}} = c\, \|\phi\|_I \,\|\psi\|_I \,, \end{aligned} \tag{15.33}$$

proving (15.28). Next, given $N \in \mathbb{N}$ we subdivide the interval $I = (m_L, m_R)$ by choosing the intermediate points

$$m_\ell = \frac{\ell}{N}\,(m_R - m_L) + m_L \,, \qquad \ell = 0, \ldots, N \,. \tag{15.34}$$

Moreover, we choose nonnegative test functions $\eta_1, \ldots, \eta_N \in C^\infty_0(\mathbb{R})$ which form a partition of unity and are supported in small sub-intervals, meaning that

$$\sum_{\ell=1}^N \eta_\ell\big|_I = 1|_I \qquad \text{and} \qquad \operatorname{supp} \eta_\ell \subset (m_{\ell-1}, m_{\ell+1}) \,, \tag{15.35}$$

where we set $m_{-1} = m_L - 1$ and $m_{N+1} = m_R + 1$. For any smooth function $\eta \in C^\infty_0(\mathbb{R})$ we define the bounded linear operator $\eta(T) : \mathcal{H}^\infty \to \mathcal{H}^\infty$ by

$$\big(\eta(T)\psi\big)_m = \eta(m)\, \psi_m \,. \tag{15.36}$$

Then by linearity,

$$\begin{aligned} &\mathopen{<}\mathfrak{p}T\psi|\mathfrak{p}\phi\mathclose{>} - \mathopen{<}\mathfrak{p}\psi|\mathfrak{p}T\phi\mathclose{>} \\ &= \sum_{\ell,\ell'=1}^N \Big(\mathopen{<}\mathfrak{p}\, T\, \eta_\ell(T)\, \psi \,|\, \mathfrak{p}\, \eta_{\ell'}(T)\, \phi\mathclose{>} - \mathopen{<}\mathfrak{p}\, \eta_\ell(T)\, \psi \,|\, \mathfrak{p}\, T\, \eta_{\ell'}(T)\, \phi\mathclose{>} \Big) \\ &= \sum_{\ell,\ell'=1}^N \Big(\mathopen{<}\mathfrak{p}\, \big(T - m_\ell\big)\, \eta_\ell(T)\psi \,|\, \mathfrak{p}\, \eta_{\ell'}(T)\, \phi\mathclose{>} - \mathopen{<}\mathfrak{p}\, \eta_\ell(T)\, \psi \,|\, \mathfrak{p}\, \big(T - m_\ell\big)\, \eta_{\ell'}(T)\, \phi\mathclose{>} \Big). \end{aligned} \tag{15.37}$$

Taking the absolute value and applying (15.23), we obtain

$$\begin{aligned} &\big|\mathopen{<}\mathfrak{p}T\psi|\mathfrak{p}\phi\mathclose{>} - \mathopen{<}\mathfrak{p}\psi|\mathfrak{p}T\phi\mathclose{>}\big| \\ &\qquad \leq c \sum_{\ell,\ell'=1}^N \int_I |m - m_\ell|\; \eta_\ell(m)\, \eta_{\ell'}(m)\; \|\phi_m\|_m \,\|\psi_m\|_m \;\mathrm{d}m \,. \end{aligned} \tag{15.38}$$

In view of the second property in (15.35), we only get a contribution if $|\ell - \ell'| \leq 1$. Moreover, we know that $|m - m_\ell| \leq 2\,|I|/N$ on the support of η_ℓ. Thus

$$\begin{aligned} \big|\mathopen{<}\mathfrak{p}T\psi|\mathfrak{p}\phi\mathclose{>} - \mathopen{<}\mathfrak{p}\psi|\mathfrak{p}T\phi\mathclose{>}\big| &\leq \frac{6c\,|I|}{N} \sum_{\ell=1}^N \int_I \eta_\ell(m)\, \|\phi_m\|_m \,\|\psi_m\|_m \;\mathrm{d}m \\ &= \frac{6c\,|I|}{N} \int_I \|\phi_m\|_m \,\|\psi_m\|_m \;\mathrm{d}m \,. \end{aligned} \tag{15.39}$$

Since N is arbitrary, we obtain (15.29).

It remains to prove the implication (ii)⇒(iii). Combining (15.28) with the Fréchet–Riesz theorem, there is a bounded operator $\mathcal{S} \in \mathrm{L}(\mathcal{H})$ with

$$<\mathfrak{p}\psi|\mathfrak{p}\phi> = (\psi|\mathcal{S}\phi)_I \qquad \forall\, \psi, \phi \in \mathcal{H}^\infty\,. \tag{15.40}$$

The relation (15.29) implies that the operators $\mathcal{S}$ and T commute. Moreover, these two operators are obviously symmetric. Hence the spectral theorem for commuting selfadjoint operators implies that there is a spectral measure F on $\sigma(\mathcal{S}) \times I$ such that

$$\mathcal{S}^p\, T^q = \int_{\sigma(\mathcal{S})\times I} \nu^p\, m^q\, \mathrm{d}F_{\nu,m} \qquad \forall\, p, q \in \mathbb{N}\,. \tag{15.41}$$

For given $\psi, \phi \in \mathcal{H}^\infty$, we introduce the Borel measure $\mu_{\psi,\phi}$ on I by

$$\mu_{\psi,\phi}(\Omega) = \int_{\sigma(\mathcal{S})\times \Omega} \nu\ \mathrm{d}(\psi|F_{\nu,m}\phi)_I\,. \tag{15.42}$$

Then $\mu_{\psi,\phi}(I) = (\psi|\mathcal{S}\phi)_I$ and

$$\begin{aligned} \mu_{\psi,\phi}(\Omega) &= \int_{\sigma(\mathcal{S})\times I} \nu\ \mathrm{d}\big(\chi_\Omega(T)\,\psi\,\big|\,F_{\nu,m}\,\chi_\Omega(T)\,\phi\big)_I \\ &= (\chi_\Omega(T)\,\psi\,|\,\mathcal{S}\,\chi_\Omega(T)\,\phi)_I\,. \end{aligned} \tag{15.43}$$

Since the operator $\mathcal{S}$ is bounded, we conclude that

$$\begin{aligned} |\mu_{\psi,\phi}(\Omega)| &\leq c\,\|\chi_\Omega(T)\,\psi\|_I\,\|\chi_\Omega(T)\,\phi\|_I \\ &\overset{(15.13)}{=} c\left(\int_\Omega \|\psi\|_m^2\ \mathrm{d}m \int_\Omega \|\phi\|_{m'}^2\ \mathrm{d}m'\right)^{\frac{1}{2}} \\ &\leq c\,|\Omega|\,\Big(\sup_{m\in\Omega} \|\psi_m\|_m\Big)\Big(\sup_{m'\in\Omega} \|\phi_{m'}\|_{m'}\Big)\,. \end{aligned} \tag{15.44}$$

This shows that the measure μ is absolutely continuous with respect to the Lebesgue measure. The Radon–Nikodym theorem (see Theorem 12.5.2) implies that there is a unique function $f_{\psi,\phi} \in L^1(I,\,\mathrm{d}m)$ such that

$$\mu_{\psi,\phi}(\Omega) = \int_\Omega f_{\psi,\phi}(m)\ \mathrm{d}m\,. \tag{15.45}$$

Using this representation in (15.44), we conclude that for any $\varphi \in \mathbb{R}$,

$$\begin{aligned} \mathrm{Re}\left(\mathrm{e}^{\mathrm{i}\varphi} \int_\Omega f_{\psi,\phi}(m)\ \mathrm{d}m\right) &\leq \big|\mu_{\psi,\phi}(\Omega)\big| \\ &\leq c\,|\Omega|\,\Big(\sup_{m\in\Omega} \|\psi_m\|_m\Big)\Big(\sup_{m'\in\Omega} \|\phi_{m'}\|_{m'}\Big)\,. \end{aligned} \tag{15.46}$$

As a consequence, for almost all $m \in I$ (with respect to the Lebesgue measure $\mathrm{d}m$),

$$\mathrm{Re}\left(\mathrm{e}^{\mathrm{i}\varphi}\ f_{\psi,\phi}(m)\right) \leq c\,\|\psi_m\|_m\,\|\phi_m\|_m\,. \tag{15.47}$$

Since the phase factor is arbitrary, we obtain the pointwise bound

$$|f_{\psi,\phi}(m)| \leq c\,\|\psi_m\|_m\,\|\phi_m\|_m \qquad \text{for almost all } m \in I\,. \tag{15.48}$$

Using this inequality, we can apply the Fréchet–Riesz theorem to obtain a unique operator $\mathcal{S}_m \in \mathrm{L}(\mathcal{H}_m)$ such that

$$f_{\psi,\phi}(m) = (\psi_m|\mathcal{S}_m\phi_m)_m \qquad \text{and} \qquad \|\mathcal{S}_m\| \leq c\,. \tag{15.49}$$

Combining the abovementioned results, for any $\psi, \phi \in \mathcal{H}^\infty$ we obtain

$$\begin{aligned}
&<\!\mathfrak{p}\psi|\mathfrak{p}\phi\!> \overset{(15.40)}{=} (\psi|\mathcal{S}\phi)_I \overset{(15.41)}{=} \int_{\sigma(\mathcal{S})\times I} \nu\, d(\psi\,|\,F_{\nu,m}\,\phi)_I \\
&\overset{(15.42)}{=} \int_I \mathrm{d}\mu_{\psi,\phi} \overset{(15.45)}{=} \int_I f_{\psi,\phi}(m)\;\mathrm{d}m \overset{(15.49)}{=} \int_I (\psi_m|\mathcal{S}_m\phi_m)_m\;\mathrm{d}m\,.
\end{aligned} \tag{15.50}$$

This concludes the proof. □

Comparing the statement of Theorem 15.3.1 (ii) with Definition 15.2.2, we immediately obtain the following result.

Corollary 15.3.2 *The strong mass oscillation property implies the weak mass oscillation property.*

We next show uniqueness as well as the independence of the choice of the interval I.

Proposition 15.3.3 (uniqueness of $\mathcal{S}_m$) *The family $(\mathcal{S}_m)_{m\in I}$ in the statement of Theorem 15.3.1 can be chosen such that for all $\psi, \phi \in \mathcal{H}^\infty$, the expectation value $f_{\psi,\phi}(m) := (\psi_m|\mathcal{S}_m\phi_m)_m$ is continuous in m,*

$$f_{\psi,\phi} \in C^0_0(I)\,. \tag{15.51}$$

The family $(\mathcal{S}_m)_{m\in I}$ with the properties (15.31) *and* (15.51) *is unique. Moreover, choosing two intervals $\check{I}$ and I with $m \in \check{I} \subset I$ and $0 \notin \overline{I}$, and denoting all the objects constructed in $\check{I}$ with an additional check, we have*

$$\check{\mathcal{S}}_m = \mathcal{S}_m\,. \tag{15.52}$$

Proof Let us show that the function $f_{\psi,\phi}$ is continuous. To this end, we choose a function $\eta \in C^\infty_0(I)$. Then for any $\varepsilon > 0$ which is so small that $B_\varepsilon(\operatorname{supp}\eta) \subset I$, we obtain

$$\begin{aligned}
&\int_I \Big(f_{\psi,\phi}(m+\varepsilon) - f_{\psi,\phi}(m)\Big)\,\eta(m)\;\mathrm{d}m \\
&\quad= \int_I f_{\psi,\phi}(m)\,\Big(\eta(m-\varepsilon) - \eta(m)\Big)\;\mathrm{d}m \\
&\quad\overset{(*)}{=} <\!\int_I \Big(\eta(m-\varepsilon) - \eta(m)\Big)\psi_m\;\mathrm{d}m\,|\,\mathfrak{p}\phi\!> \\
&\quad= <\!\int_I \eta(m)\,\Big(\psi_{m+\varepsilon} - \psi_m\Big)\;\mathrm{d}m\,|\,\mathfrak{p}\phi\!>\,,
\end{aligned} \tag{15.53}$$

where in $(*)$ we used (15.41) and (15.42). Applying (15.28), we obtain

$$\left|\int_I \Big(f_{\psi,\phi}(m+\varepsilon) - f_{\psi,\phi}(m)\Big)\,\eta(m)\;\mathrm{d}m\right| \leq c\,\|\psi_{+\varepsilon} - \psi\|_I\,\|\phi\|_I\,\sup_I|\eta|\,, \tag{15.54}$$

where the vector $\psi_{+\varepsilon} \in \mathcal{H}^\infty$ is defined by $(\psi_{+\varepsilon})_m := \psi_{m+\varepsilon}$. Since

$$\lim_{\varepsilon \searrow 0} \|\psi_{+\varepsilon} - \psi\|_I = 0 \tag{15.55}$$

and η is arbitrary, we conclude that $f_{\psi,\phi}$ is continuous (15.51). This continuity is important because it implies that the function $f_{\psi,\phi}$ is uniquely defined pointwise (whereas in (15.45) this function could be modified arbitrarily on sets of measure zero).

In order to prove (15.52), we note that the representation (15.40) implies that

$$(\psi|\check{\mathcal{S}}\phi)_I = (\psi|\mathcal{S}\phi)_I \qquad \text{for all } \psi, \phi \in \check{\mathcal{H}}^\infty \,. \tag{15.56}$$

Using (15.42) and (15.45), it follows that

$$\int_\Omega \check{f}_{\psi,\phi}(m)\; \mathrm{d}m = \int_\Omega f_{\psi,\phi}(m)\; \mathrm{d}m \qquad \text{for all } \Omega \subset \check{I} \,. \tag{15.57}$$

Choosing $\check{f}_{\psi,\phi}(m)$ and $f_{\psi,\phi}(m)$ as continuous functions, we conclude that they coincide for every $m \in \check{I}$. It follows from (15.31) that the operators $\check{\mathcal{S}}_m$ and $\mathcal{S}_m$ coincide. This concludes the proof. □

15.4 The Unregularized Kernel of the Fermionic Projector

We now explain how the fermionic signature operator can be used for the construction of the so-called fermionic projector. This will give a direct connection to the kernel of the fermionic projector introduced abstractly for causal fermion systems in Chapter 5 (see (5.45)). We will explain this connection, which will be elaborated on further in Section 21.

It follows directly from by its defining equation (15.31) that the operator $\mathcal{S}_m$ is symmetric. Thus the spectral theorem gives rise to the spectral decomposition

$$\mathcal{S}_m = \int_{\sigma(\mathcal{S}_m)} \nu \; \mathrm{d}E_\nu \,, \tag{15.58}$$

where E_ν is the spectral measure (see, e.g., [131]). The spectral measure gives rise to the spectral calculus

$$f(\mathcal{S}_m) = \int_{\sigma(\mathcal{S}_m)} f(\nu) \; \mathrm{d}E_\nu \,, \tag{15.59}$$

where f is a bounded Borel function.

Definition 15.4.1 *Assume that the Dirac operator $\mathcal{D}$ on $(\mathscr{M}, g)$ satisfies the strong mass oscillation property (see Definition 15.2.3). We define the operators*

$$P_\pm \;:\; C_0^\infty(\mathscr{M}, S\mathscr{M}) \to \mathcal{H}_m \tag{15.60}$$

by

$$P_+ = \chi_{[0,\infty)}(\mathcal{S}_m)\, k_m \qquad \text{and} \qquad P_- = -\chi_{(-\infty,0)}(\mathcal{S}_m)\, k_m \tag{15.61}$$

(where χ denotes the characteristic function). The **fermionic projector** *P is defined by $P = P_-$.*

Proposition 15.4.2 *For all $\phi, \psi \in C_0^\infty(\mathscr{M}, S\mathscr{M})$, the operators $P_\pm$ are symmetric,*

$$<P_\pm \phi \,|\, \psi> = <\phi \,|\, P_\pm \psi> \,. \tag{15.62}$$

Moreover, the image of $P_\pm$ is the positive respectively negative spectral subspace of $\mathcal{S}_m$, that is,

$$\begin{aligned} \overline{P_+(C_0^\infty(\mathscr{M}, S\mathscr{M}))} &= E_{(0,\infty)}(\mathcal{H}_m) \,, \\ \overline{P_-(C_0^\infty(\mathscr{M}, S\mathscr{M}))} &= E_{(-\infty,0)}(\mathcal{H}_m) \,. \end{aligned} \tag{15.63}$$

Proof According to Proposition 13.4.4,

$$\begin{aligned} <P_- \phi \,|\, \psi> &= (P_- \phi \,|\, k_m \psi)_m = -\big(\chi_{(-\infty,0)}(\mathcal{S}_m)\, k_m \phi \,\big|\, k_m \psi\big)_m \\ &= -\big(k_m\, \phi \,\big|\, \chi_{(-\infty,0)}(\mathcal{S}_m)\, k_m \psi\big)_m = <\phi \,|\, P_- \psi> \,. \end{aligned} \tag{15.64}$$

The proof for P_+ is similar. The relations (15.63) follow immediately from the fact that $k_m(C_0^\infty(\mathscr{M}, S\mathscr{M}))$ is dense in $\mathcal{H}_m$. □

As in [70, Theorem 3.12], the fermionic projector can be represented by a two-point distribution on $\mathscr{M}$. As usual, we denote the space of test functions (with the Fréchet topology) by $\mathcal{D}$ and define the space of distributions $\mathcal{D}'$ as its dual space.

Theorem 15.4.3 *Assume that the strong mass oscillation property holds. Then there is a unique distribution $\mathcal{P} \in \mathcal{D}'(\mathscr{M} \times \mathscr{M})$ such that for all $\phi, \psi \in C_0^\infty(\mathscr{M}, S\mathscr{M})$,*

$$<\phi | P \psi> = \mathcal{P}(\phi \otimes \psi) \,. \tag{15.65}$$

Proof According to Proposition 13.4.4 and Definition 15.4.1,

$$<\phi | P \psi> = (k_m \phi \,|\, P \psi)_m = -(k_m \phi \,|\, \chi_{(-\infty,0)}(\mathcal{S}_m)\, k_m \psi)_m \,. \tag{15.66}$$

Since the norm of the operator $\chi_{(-\infty,0)}(\mathcal{S}_m)$ is bounded by one, we conclude that

$$|<\phi | P \psi>| \leq \|k_m \phi\|_m \, \|k_m \psi\|_m = \big(<\phi | k_m \phi> <\psi | k_m \psi>\big)^{\frac{1}{2}} \,, \tag{15.67}$$

where in the last step we again applied Proposition 13.4.4. As $k_m \in \mathcal{D}'(\mathscr{M} \times \mathscr{M})$, the right-hand side is continuous on $\mathcal{D}(\mathscr{M} \times \mathscr{M})$. We conclude that also the functional $<\phi | P \psi>$ is continuous on $\mathcal{D}(\mathscr{M} \times \mathscr{M})$. The result now follows from the Schwartz kernel theorem (see [105, Theorem 5.2.1], keeping in mind that this theorem applies just as well to bundle-valued distributions on a manifold simply by working with the components in local coordinates and a local trivialization). □

Exactly as explained in [70, Section 3.5], it is convenient to use the standard notation with an integral kernel $P(x, y)$,

$$<\phi | P \psi> = \iint_{\mathscr{M} \times \mathscr{M}} \prec \phi(x) \,|\, P(x,y)\, \psi(y) \succ_x \, d\mu_{\mathscr{M}}(x)\, d\mu_{\mathscr{M}}(y) \tag{15.68}$$

$$(P\psi)(x) = \int_{\mathscr{M}} P(x,y)\, \psi(y)\, d\mu_{\mathscr{M}}(y) \tag{15.69}$$

(where $P(.,.)$ coincides with the distribution $\mathcal{P}$ above). In view of Proposition 15.4.2, we know that the last integral is not only a distribution, but a function which is square integrable over every Cauchy surface. Moreover, the symmetry of P shown in Proposition 15.4.2 implies that

$$P(x,y)^* = P(y,x)\,, \tag{15.70}$$

where the star denotes the adjoint with respect to the spin inner product.

We next specify the normalization of the fermionic projector. We introduce an operator Π by

$$\begin{aligned} &\Pi \,:\, \mathcal{H}_m \to \mathcal{H}_m\,, \\ &(\Pi\,\psi_m)(x) = -2\pi \int_{\mathcal{N}} P(x,y)\,\slashed{\nu}\,(\psi_m)|_{\mathcal{N}}(y)\;\mathrm{d}\mu_{\mathcal{N}}(y)\,, \end{aligned} \tag{15.71}$$

where $\mathcal{N}$ is any Cauchy surface.

Proposition 15.4.4 **(spatial normalization)** *The operator* Π *is a projection operator on* $\mathcal{H}_m$*.*

Proof According to Theorem 13.4.2, the spatial integral in (15.71) can be combined with the factor k_m in (15.61) to give the solution of the corresponding Cauchy problem. Thus

$$\Pi \,:\, \mathcal{H}_m \to \mathcal{H}_m\,, \qquad (\Pi\,\psi_m)(x) = \chi_{(-\infty,0)}(\mathcal{S}_m)\,\psi_m\,, \tag{15.72}$$

showing that Π is a projection operator. □

Instead of the spatial normalization, one could also consider the *mass normalization* (for details on the different normalization methods see [75]). To this end, one needs to consider families of fermionic projectors P_m indexed by the mass parameter. Then for all $\phi, \psi \in C_0^\infty(\mathcal{M}, S\mathcal{M})$, we can use (15.31) and Proposition 13.4.4 to obtain

$$\begin{aligned} <\mathfrak{p}(P_m\phi)\,|\,\mathfrak{p}(P_{m'}\psi)> &= \int_I (P_m\phi\,|\,\mathcal{S}_m P_m\psi)_m\;\mathrm{d}m \\ &= \int_I (k_m\phi\,|\,\mathcal{S}_m \chi_{(-\infty,0)}(\mathcal{S}_m)\,k_m\psi)_m\;\mathrm{d}m \\ &= \int_I <\phi\,|\,\mathcal{S}_m \chi_{(-\infty,0)}(\mathcal{S}_m)\,k_m\psi>\;\mathrm{d}m = -<\phi\,|\,\mathfrak{p}(\mathcal{S}_m P_m\psi)>\,, \end{aligned} \tag{15.73}$$

which can be written in a compact formal notation as

$$P_m\,P_{m'} = \delta(m-m')\,(-\mathcal{S}_m)\,P_m\,. \tag{15.74}$$

Due to the factor $(-\mathcal{S}_m)$ on the right, in general the fermionic projector does *not* satisfy the mass normalization condition. The mass normalization condition could be arranged by modifying the definition (15.61) to

$$\mathcal{S}_m^{-1}\,\chi_{(-\infty,0)}(\mathcal{S}_m)\,k_m\,. \tag{15.75}$$

Here we prefer to work with the spatial normalization. For a detailed discussion of the different normalization methods we refer to [75, Section 2].

Finally, the spatial normalization property of Proposition 15.4.4 makes it possible to obtain a representation of the fermionic projector in terms of one-particle states. To this end, one chooses an orthonormal basis $(\psi_j)_{j\in\mathbb{N}}$ of the subspace $\chi_{(-\infty,0)}(\mathcal{S}_m) \subset \mathcal{H}_m$. Then

$$P(x,y) = -\sum_{j=1}^{\infty} |\psi_j(x)\succ\prec\psi_j(y)| \tag{15.76}$$

with convergence in $\mathcal{D}'(\mathscr{M} \times \mathscr{M})$ (for details, see [70, Proposition 3.13]).

This formulas is reminiscent of the decomposition of the kernel of the fermionic projector into physical wave functions in (5.58). Indeed, these formulas can be understood as being completely analogous, with the only difference that (15.76) is formed of wave functions in Minkowski space, whereas in (5.58) one works abstractly with the physical wave functions of a general causal fermion system. The connection can be made more precise if one identifies the structures of the causal fermion system with corresponding structures in Minkowski space. In order to avoid technicalities and too much overlap with [45], here we shall not enter the details of these identifications (which are worked out in [45, Section 1.2]). Instead, we identify (15.76) with (5.58) as describing the same object, on one side in Minkowski space, and on the other side as abstract object of the corresponding causal fermion system. With this identification, the Hilbert space $\mathcal{H}$ of the causal fermion system corresponds to the negative spectral subspace of the fermionic signature operator $\mathcal{S}_m$. In the vacuum, this gives us back the subspace of all negative frequency solutions as considered in the example of Exercise 5.14. However, the above identification has one shortcoming: the wave functions in (15.76) have not yet been regularized. This is why we refer to $P(x,y)$ as the *unregularized* kernel. In order to get complete agreement between (15.76) and (5.14), one needs to introduce an ultraviolet regularization. To this end, one proceeds as explained in the example in Section 5.5: One introduces regularization operators $(\mathfrak{R}_\varepsilon)_{\varepsilon>0}$, computes the local correlation operators $F^\varepsilon(x)$ and defines the measure ρ as the push-forwards $\mathrm{d}\rho = F^\varepsilon_* \mathrm{d}\mu_{\mathscr{M}}$. We will come back to this construction in Chapter 21.

15.5 Exercises

Exercise 15.1 Let $\mathscr{M}$ be the "spacetime strip"

$$\mathscr{M} = \{(t,\vec{x}) \in \mathbb{R}^{1,3} \text{ with } 0<t<T\}\,. \tag{15.77}$$

Show that for any solution $\psi \in C^\infty_{\text{sc}}(\mathscr{M}, S\mathscr{M}) \cap \mathcal{H}_m$ of the Dirac equation, the following inequality holds,

$$\big|{<}\psi|\phi{>}\big| \leq T\, \|\psi\|_m\, \|\phi\|_m\,. \tag{15.78}$$

This estimate illustrates why in spacetimes of finite lifetime, the spacetime inner product is a bounded sesquilinear form on $\mathcal{H}_m$.

Exercise 15.2 Let $\mathscr{M}$ again be the "spacetime strip" of the previous exercise. Let $\psi, \phi \in \mathcal{H}^\infty := \mathcal{H} \cap C^\infty_{\text{sc},0}(\mathscr{M} \times I, S\mathscr{M})$ be families of smooth Dirac solutions of spatially compact support, with compact support in the mass parameter. Moreover, we again define the operators $\mathfrak{p}$ and T as in (15.17) and (15.20). Does the equation

$$<\mathfrak{p}T\psi|\mathfrak{p}\phi> = <\mathfrak{p}\psi|\mathfrak{p}T\phi> \tag{15.79}$$

(which appears in the weak mass oscillation property) in general hold? Justify your answer by a proof or a counter example.

Exercise 15.3 Let $\mathscr{M}$ again be the "spacetime strip" of the previous exercises. Moreover, as in Exercise 5.10 we again let $\mathcal{H} \subset \mathcal{H}_m$ be a finite-dimensional subspace of the Dirac solution space $\mathcal{H}_m$, consisting of smooth wave functions of spatially compact support, that is,

$$\mathcal{H} \subset C^\infty_{\mathrm{sc}}(\mathscr{M}, S\mathscr{M}) \cap H_m \qquad \text{finite-dimensional}\,. \tag{15.80}$$

Show that the fermionic signature operator $\mathcal{S} \in \mathrm{L}(\mathcal{H})$ defined by

$$<\psi|\phi> = (\psi|\mathcal{S}\phi)_m \qquad \text{for all } \psi, \phi \in \mathcal{H} \tag{15.81}$$

can be expressed within the causal fermion system by

$$\mathcal{S} = -\int_M x \, \mathrm{d}\rho(x) \tag{15.82}$$

(where ρ is again the push-forward of $\mathrm{d}\mu_{\mathscr{M}}$).

Exercise 15.4 Let E be the Banach space $E = C^0([0,1], \mathbb{C})$ and $\Lambda : E \times E \to \mathbb{C}$ be sesquilinear, bounded and positive semi-definite.

(a) Assume that Λ satisfies for a suitable constant $c > 0$ and all $f, g \in E$ the inequality

$$\big|\Lambda(f,g)\big| \leq c \sup_{x\in[0,1]} \big|f(x)\, g(x)\big|\,. \tag{15.83}$$

Show that there is a regular bounded Borel measure μ such that

$$\Lambda(f,g) = \int_0^1 \overline{f(x)}\, g(x)\, \mathrm{d}\mu(x)\,. \tag{15.84}$$

(b) Now make the stronger assumption that Λ satisfies for a suitable constant $\tilde{c} > 0$ and all $f, g \in E$ the inequality

$$\big|\Lambda(f,g)\big| \leq \tilde{c} \int_0^1 \big|f(x)\, g(x)\big| \, \mathrm{d}x\,. \tag{15.85}$$

Show that μ is absolutely continuous w.r.to the Lebesgue measure. Show that there is a nonnegative function $h \in L^1([0,1], \mathrm{d}x)$ such that

$$\Lambda(f,g) = \int_0^1 \overline{f(x)}\, g(x)\, h(x) \, \mathrm{d}x\,. \tag{15.86}$$

Show that h is pointwise bounded by c.

(c) In order to clarify the different assumptions in this exercise, give an example for a sesquilinear, bounded and positive semi-definite functional Λ which violates (15.83). Give an example which satisfies (15.83) but violates (15.85).

Exercise 15.5 *(Toward the mass oscillation property - part 1)* This exercise illustrates the mass oscillation property. Let $0 < m_L < m_R$ and $\eta \in C_0^\infty((m_L, m_R))$. Show that the function f given by

$$f(t) = \int_{m_L}^{m_R} \eta(m)\, \mathrm{e}^{-\mathrm{i}\sqrt{1+m^2}\, t}\, \mathrm{d}m \tag{15.87}$$

has rapid decay. Does this result remain valid if m_L and m_R are chosen to have opposite signs? Justify your finding by a proof or a counter example.

Exercise 15.6 *(Toward the mass oscillation property - part 2)* Let R_T be the "spacetime strip"

$$R_T = \{(t, \vec{x}) \in \mathbb{R}^{1,3} \text{ with } 0 < t < T\} \,. \tag{15.88}$$

Show that for any solutions $\psi, \phi \in C^\infty_{\text{sc}}(\mathbb{R}^4, \mathbb{C}^4) \cap \mathcal{H}_m$ of the Dirac equation, the following inequality holds,

$$\begin{aligned} \big| {<}\psi|\phi{>}_T \big| &\leq T\, \|\psi\|_m\, \|\phi\|_m \,, \quad \text{where} \\ {<}\psi|\phi{>}_T &:= \int_{R_T} \prec \psi(x)|\phi(x) \succ \mathrm{d}^4x \,. \end{aligned} \tag{15.89}$$

This estimate illustrates how in spacetimes of finite lifetime, the spacetime inner product is a bounded sesquilinear form on $\mathcal{H}_m$.

Exercise 15.7 *(Toward the mass oscillation property - part 3)* Let R_T again be the "spacetime strip" of the previous exercises. Moreover, we again let $\mathcal{H} \subset \mathcal{H}_m$ be a finite-dimensional subspace of the Dirac solution space $\mathcal{H}_m$, consisting of smooth wave functions of spatially compact support, that is,

$$\mathcal{H} \subset C^\infty_{\text{sc}}(\mathbb{R}^4, \mathbb{C}^4) \cap \mathcal{H}_m \qquad \text{finite-dimensional} \,. \tag{15.90}$$

Show that the fermionic signature operator $\mathcal{S} \in \mathrm{L}(\mathcal{H})$ defined by

$${<}\psi|\phi{>}_T = (\psi | \mathcal{S}\phi)_m \qquad \text{for all } \psi, \phi \in \mathcal{H} \tag{15.91}$$

can be expressed within the causal fermion system by

$$\mathcal{S} = -\int_{R_T} x\, \mathrm{d}\rho(x) \tag{15.92}$$

(where ρ is again the push-forward of d^4x).

Exercise 15.8 *(The external field problem)* In physics, the notion of "particle" and "antiparticle" is often introduced as follows: Solutions of the Dirac equation with positive frequency are called "particles" and solutions with negative frequency "antiparticles." In this exercise, we will check in how far this makes sense.
To this end, take a look at the Dirac equation in an external field:

$$(\mathrm{i}\partial\!\!\!/ + \mathcal{B} - m)\psi = 0. \tag{15.93}$$

Assume that $\mathcal{B}$ is time-dependent and has the following form:

$$\mathcal{B}(t, x) = V\, \Theta(t - t_0)\Theta(t_1 - t), \tag{15.94}$$

where $V \in \mathbb{R}$, Θ denotes the Heaviside step function and $t_0 = 0$, $t_1 = 1$. In order to construct a solution thereof, for a given momentum $\vec{k}$, we use plane wave solutions of the Dirac equation,

$$\psi(t, \vec{x}) = \mathrm{e}^{-\mathrm{i}\omega t + \mathrm{i}\vec{k}\vec{x}} \chi_{\vec{k}}, \tag{15.95}$$

where $\chi_{\vec{k}}$ is a spinor $\in \mathbb{C}^4$, and patch them together suitably. (The quantity ω is called the "frequency" or "energy," and $\vec{k}$ the "momentum.") To simplify the calculation, we set $\vec{k} = (k_1, 0, 0)^T$. Proceed as follows:

(a) First, take a look at the region $t < t_0$. Reformulate (15.93) such that there is only the time derivative on the left-hand side. (Hint: Multiply by γ^0.)

(b) Insert the plane wave ansatz with $\vec{k} = (k_1, 0, 0)^T$ into the equation. Your equation now has the form $\omega\psi = H(k_1)\psi$. Show that the eigenvalues of $H(k_1)$ are $\pm\omega_0$ with $\omega_0 := \sqrt{(k_1)^2 + m^2}$.

(c) Show that one eigenvector belonging to $+\omega_0$ is $\chi_0^+ := (\frac{m+\omega_0}{k1}, 0, 0, 1)^T$ and that one eigenvector belonging to $-\omega_0$ is $\chi_0^- := (\frac{m-\omega_0}{k1}, 0, 0, 1)^T$. (Both eigenvalues have multiplicity two, but we do not need the other two eigenvectors here.)

(d) With this, you have constructed plane wave solutions $\mathrm{e}^{-\mathrm{i}(\pm\omega_0)t + \mathrm{i}\vec{k}\vec{x}} \chi_0^{\pm}$ for $t < t_0$ and also for $t > t_1$. By transforming $m \to (m - V)$, you immediately obtain plane wave solutions also for $t_0 < t < t_1$. Denote the respective quantities by ω_1 and $\chi_1^{\pm}$.

(e) Assume that for $t < t_0$ there is one "particle" present, that is, set

$$\psi(t, \vec{x}) = \mathrm{e}^{-\mathrm{i}\omega_0 t + \mathrm{i}\vec{k}\vec{x}} \chi_0^+ \quad \text{for } t < t_0. \tag{15.96}$$

Assume that the solution for $t_0 < t < t_1$ takes the form

$$A\mathrm{e}^{-\mathrm{i}\omega_1 t + \mathrm{i}\vec{k}\vec{x}} \chi_1^+ + B\mathrm{e}^{-\mathrm{i}(-\omega_1)t + \mathrm{i}\vec{k}\vec{x}} \chi_1^- \quad \text{with } A, B \in \mathbb{R}. \tag{15.97}$$

Calculate A and B for the case $k_1 = 1$ and $V = m$ by demanding continuity of the solution at $t = t_0$.

(f) Assume that for $t > t_1$ the solution takes the form

$$C\mathrm{e}^{-\mathrm{i}\omega_0 t + \mathrm{i}\vec{k}\vec{x}} \chi_0^+ + D\mathrm{e}^{-\mathrm{i}(-\omega_0)t + \mathrm{i}\vec{k}\vec{x}} \chi_0^- \quad \text{with } C, D \in \mathbb{C}. \tag{15.98}$$

Calculate C and D for $m = 2$ by demanding continuity of the solution at $t = t_1$ (here you may want to use computer algebra).

(g) Interpret what you have found. Why could this be called the "external field *problem*"?