INEQUALITIES FOR THE SCHATTEN p-NORM

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(Received 9 February, 1984)

Let H be a separable, infinite dimensional complex Hilbert space, and let $B(H)$ denote the algebra of all bounded linear operators on H. Let $K(H)$ denote the ideal of compact operators on H. For any compact operator A let $|A|=(A^*A)^{1/2}$ and $s_1(A), s_2(A), \ldots$ be the eigenvalues of $|A|$ in decreasing order and repeated according to multiplicity. If, for some $1 \leq p \leq \infty$, $\sum_{i=1}^{\infty} s_i(A)^p < \infty$, we say that A is in the Schatten p-class C_p and $\|A\|_p = \left(\sum_{i=1}^{\infty} s_i(A)^p\right)^{1/p}$ is the p-norm of A. Hence, C_1 is the trace class, C_2 is the Hilbert–Schmidt class, and C_∞ is the ideal of compact operators $K(H)$.

If $A \in C_1$ and $\{e_i\}$ is any orthonormal basis of H then the trace of A, denoted by $\operatorname{tr} A = \sum_{i=1}^{\infty} (Ae_i, e_i)$ is independent of the choice of $\{e_i\}$. If $A \in C_p$ and $B \in C_q$, then $|\operatorname{tr}(AB)| \leq \|A\|_p \|B\|_q$ whenever $1/p+1/q=1$. If $\{e_i\}$ and $\{f_i\}$ are two orthonormal sets in H, then for $A \in C_p$, $\|A\|_p^p \geq \sum_{i=1}^{\infty} |(Ae_i, f_i)|^p$. We refer to [**2**] or [**4**] for further properties of the Schatten p-classes.

In their investigation on the traces of commutators of integral operators J. Helton and R. Howe [**1**, Lemma 1.3] proved that if A is a self-adjoint operator and X is a compact operator, then $AX-XA \in C_1$ implies that $\operatorname{tr}(AX-XA)=0$. Our first inequality is a generalization of this result.

THEOREM 1. *If $X \in C_p$ $(1 \leq p \leq \infty)$ and A is an operator such that $AX-XA^* \in C_1$, then* $|\operatorname{tr}(AX-XA^*)| \leq \|X\|_p \|A-A^*\|_q$ $(1/p+1/q=1)$.

Proof. There is nothing to prove if $A-A^*$ is not in C_q, so let us assume that $A-A^* \in C_q$. Thus $X(A^*-A) \in C_1$ and so $AX-XA = AX-XA^*+X(A^*-A) \in C_1$. Now $AX-XA^* \in C_1$ implies when taking adjoints that $X^*A^*-AX^* \in C_1$. Add and subtract to get $AY-YA^* \in C_1$ and $AZ-ZA^* \in C_1$ where $X=Y+iZ$ is the cartesian decomposition of X. Since $A-A^* \in C_q$, it follows that $AY-YA \in C_1$ and $AZ-ZA \in C_1$. But Y and Z being compact self-adjoint operators (diagonalizable) implies that $\operatorname{tr}(AY-YA)=0$ and $\operatorname{tr}(AZ-ZA)=0$ (just evaluate the traces using the eigenvectors of Y and Z respectively). Therefore $\operatorname{tr}(AX-XA)=0$ and so $\operatorname{tr}(AX-XA^*)=\operatorname{tr}(AX-XA)+\operatorname{tr}(X(A-A^*))=\operatorname{tr}(X(A-A^*))$. Hence $|\operatorname{tr}(AX-XA^*)| \leq \|X\|_p \|A-A^*\|_q$ by Holder's inequality for C_p.

If A is an operator such that $\sigma(A) \cap \sigma(A^*) = \emptyset$ ($\sigma(A)$ denotes the spectrum of A) then by Rosenblum's theorem [**3**] no non-zero operator X can intertwine A and A^* i.e., $AX=XA^*$ implies $X=0$. The following inequality is related to this result.

Glasgow Math. J. **26** (1985) 141–143.

THEOREM 2. *Let* $A \in B(H)$ *with* $\operatorname{Im} A = \dfrac{A - A^*}{2i} \geq a \geq 0$. *Then* $\|AX - XA^*\| \geq a \|X\|$ *for all* $X \in B(H)$.

Proof. Let $X = Y + iZ$ be the cartesian decomposition of X. We will show that $\|AY - YA^*\| \geq 2a \|Y\|$ and $\|AZ - ZA^*\| \geq 2a \|Z\|$. Now let $|y_0| = \|Y\|$; then there is a sequence $\{f_n\}$ of unit vectors in H such that $\|(Y - y_0)f_n\| \to 0$ as $n \to \infty$. Therefore,

$$\begin{aligned}\|AY - YA^*\| &\geq |((AY - YA^*)f_n, f_n)| \\ &= |(A(Y - y_0)f_n, f_n) - ((Y - y_0)A^*f_n, f_n) + y_0(Af_n, f_n) - y_0(A^*f_n, f_n)| \\ &\geq |y_0|\,|((A - A^*)f_n, f_n)| - |(A(Y - y_0)f_n, f_n)| - |((Y - y_0)A^*f_n, f_n)| \\ &\geq 2\,|y_0|\,a - \text{term which goes to zero as } n \to \infty.\end{aligned}$$

Thus $\|AY - YA^*\| \geq 2a \|Y\|$. Similarly we get $\|AZ - ZA^*\| \geq 2a \|Z\|$. Since $AY - YA^* = i \operatorname{Im}(AX - XA^*)$ and $AZ - ZA^* = -i \operatorname{Re}(AX - XA^*)$ it follows that $2\|AX - XA^*\| \geq \|AY - YA^*\| + \|AZ - ZA^*\| \geq 2a(\|Y\| + \|Z\|) \geq 2a \|X\|$. Hence $\|AX - XA^*\| \geq a \|X\|$ as required.

COROLLARY. *Let* $A \in B(H)$ *with* $\operatorname{Im} A > a > 0$. *If* X *is an operator such that* $AX = XA^*$, *then* $X = 0$.

REMARK. The corollary above can be deduced from Rosenblum theorem, after we establish the following lemma.

LEMMA. *Let* $A \in B(H)$ *with* $\operatorname{Im} A > a > 0$, *then* $\sigma(A) \subset \{z : \operatorname{Im} z > a\}$. *In particular,* $\sigma(A) \cap \sigma(A^*) = \varnothing$.

Proof. $\operatorname{Im} A > a$ implies that $W(A) \subset \{z : \operatorname{Im} z > a\}$, where $W(A)$ denotes the numerical range of A. Thus $\sigma(A) \subset$ closure of $W(A) \subset \{z : \operatorname{Im} z \geq a\}$. It is now sufficient to show that $\sigma(A) \cap \{z : \operatorname{Im} z = a\} = \varnothing$. Let $\lambda = b + ia$, and let $A = B + iC$ be the cartesian decomposition of A. Then $A - \lambda = (B - b) + i(C - a)$. Since $C - a$ is positive and invertible, it follows that $(C - a)^{-1/2}(A - \lambda)(C - a)^{-1/2} = (C - a)^{-1/2}(B - b)(C - a)^{-1/2} + i$. Since $(C - a)^{-1/2}(B - b)(C - a)^{-1/2}$ is self-adjoint, it follows that $(C - a)^{-1/2}(A - \lambda)(C - a)^{-1/2}$ is invertible which implies that $A - \lambda$ is invertible. In fact if P is the inverse of $(C - a)^{-1/2}(A - \lambda)(C - a)^{-1/2}$, then $(C - a)^{-1/2}P(C - a)^{-1/2}$ is the inverse of $A - \lambda$. Thus we conclude that $\lambda \notin \sigma(A)$ which means $\sigma(A) \subset \{z : \operatorname{Im} z > a\}$.

We conclude with the following C_p version of Theorem 2.

THEOREM 3. *Let* $A \in B(H)$ *with* $\operatorname{Im} A \geq a \geq 0$. *Then* $\|AX - XA^*\|_p \geq a \|X\|_p$ *for all* $X \in B(H)$ *and* $1 \leq p \leq \infty$.

Proof. We assume that $AX - XA^* \in C_p$, otherwise the result is trivial. Hence $AX - XA^*$ is compact and so $\pi(A)\pi(X) = \pi(X)\pi(A)^*$ where $\pi : B(H) \to B(H)/K(H)$ is the canonical projection onto the Calkin algebra. Applying the corollary above, noting that $\sigma(\pi(A)) \subset \sigma(A)$, we get $\pi(X) = 0$ (there is nothing to prove if $a = 0$). Thus X is compact. Let $X = Y + iZ$. Now Y and Z are diagonalizable as they are compact and self-adjoint. Let $Ye_n = \lambda_n e_n$ where $\{e_n\}$ is an orthonormal basis for H. Therefore

$$\begin{aligned}\|AY - YA^*\|_p &= \left(\sum_{n=1}^{\infty} |((AY - YA^*)e_n, e_n)|^p\right)^{1/p} \\ &= \left(\sum_{n=1}^{\infty} |\lambda_n((A - A^*)e_n, e_n)|^p\right)^{1/p} \\ &\geq 2a\left(\sum_{n=1}^{\infty} |\lambda_n|^p\right)^{1/p} \\ &= 2a\,\|Y\|_p.\end{aligned}$$

Similarly we obtain (using the eigenvectors of Z) that $\|AZ - ZA^*\|_p \geq 2a\,\|Z\|_p$. Hence by an argument similar to the one in the proof of Theorem 2 we obtain that $\|AX - XA^*\|_p \geq a\,\|X\|_p$ as required.

REFERENCES

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