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New Characterizations of the Weighted Composition Operators Between Bloch Type Spaces in the Polydisk

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Abstract. We give some new characterizations for compactness of weighted composition operators uC_φ acting on Bloch-type spaces in terms of the power of the components of φ, where φ is a holomorphic self-map of the polydisk $\mathbb{D}^n$, thus generalizing the results obtained by Hyvärinen and Lindström in 2012.

1 Introduction

Let $\mathbb{D}^n$ be the polydisk of $\mathbb{C}^n$ with boundary $\partial\mathbb{D}^n$. The class of all holomorphic functions on the domain $\mathbb{D}^n$ will be denoted by $H(\mathbb{D}^n)$. Let $\varphi(z) = (\varphi_1(z), \dots, \varphi_n(z))$ be a holomorphic self-map of $\mathbb{D}^n$, and let $u(z) \in H(\mathbb{D}^n)$. The weighted composition operator is defined by

$$uC_\varphi(f)(z) = u(z)f\big(\varphi(z)\big),$$

for any $f \in H(\mathbb{D}^n)$ and $z \in \mathbb{D}^n$. When $u = 1$, it is the composition operator, and we often write C_φ instead.

For $p > 0$, the Bloch type space $\mathcal{B}^p$ consists of those $f \in H(\mathbb{D}^n)$ such that

$$\|f\|_p = |f(0)| + \sup_{z\in\mathbb{D}^n} \sum_{k=1}^{n} (1 - |z_k|^2)^p \Big| \frac{\partial f}{\partial z_k}(z) \Big| < \infty;$$

with this norm, $\mathcal{B}^p$ becomes a Banach space. When $p = 1$, it is the classical Bloch space.

It is important to provide function-theoretic characterizations of when φ induces a bounded or compact weighted composition operator on various spaces. For general references on the theory of weighted composition operators, we refer the interested reader to [1, 11, 15]. Recently, there has been great interest in studying composition operators between Bloch type spaces. For example, see [2, 5, 6, 8–12, 14]. There are still many unsolved problems that are of interest to numerous mathematicians.

In [7], Manhas and Zhao obtained new estimates of essential norms of weighted composition operators between $\mathcal{B}^p (p \neq 1)$ and $\mathcal{B}^q$ in the unit disk. And a few

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months later, by using different methods, Hyvärinen and Lindström [3] obtained similar estimates. They were the first to give the estimate of essential norm for the case $p = 1$. They characterized the essential norm in terms of two different integral operators and φ^m.

The goal of this paper is to extend the results in the unit disk to the polydisk. The higher dimensional case will be a little bit different. Some properties are not easily managed; we need some new calculating techniques. The proofs in this paper are partially based on [3, 7].

Throughout the remainder of this paper, let $\mathbb{N}$ be the set of positive integers, $I = \{i \in \mathbb{N} : \|\varphi_i\|_\infty = 1\}$ and $J = \{j \in \mathbb{N} : \|\varphi_j\|_\infty < 1\}$. Constants C are positive and may differ from one occurrence to another.

2 Some Lemmas

In this section, we present some lemmas that will be used in the proofs of our main results in the next section. The proof of the following lemma can be found in [13, Lemma 2.2].

Lemma 2.1 *Let $p > 0, m \in \mathbb{N}$ and $0 \le x \le 1$. And let*

$$H_{m,p}(x) = x^{m-1}(1-x^2)^p, \quad r_m = \Big(\frac{m-1}{m-1+2p}\Big)^{1/2}(m \ge 2).$$

Then for $m \ge 2$, $H_{m,p}$ has the following properties:

(i)

$$\max_{0\le x\le 1} H_{m,p}(x) = H_{m,p}(r_m) = (\frac{2p}{m-1+2p})^p(\frac{m-1}{m-1+2p})^{(m-1)/2},$$

that is,

$$\lim_{m\to\infty} m^p \max_{x\in[0,1]} H_{m,p}(x) = \Big(\frac{2p}{e}\Big)^p;$$

(ii) *$H_{m,p}(x)$ is decreasing on $[r_m, r_{m+1}]$, and*

$$\lim_{m\to\infty} m^p \min_{x\in[r_m,r_{m+1}]} H_{m,p}(x) = \Big(\frac{2p}{e}\Big)^p.$$

We proceed as in [3, Lemma 2.1(d)]; the following lemma is obtained, and we sketch the proof for the sake of completeness.

Lemma 2.2 *Let $m \in \mathbb{N}$ and $0 < x < 1$. And let*

$$G_m(x) = \frac{x^m}{\log\frac{e}{1-x^2}}, \quad r_m = \sqrt{\frac{\frac{m}{2}\log\frac{m}{2}}{1+\frac{m}{2}\log\frac{m}{2}}}, \quad \widetilde{r_m} = \sqrt{\frac{\frac{me}{2}\log\frac{me}{2}}{1+\frac{me}{2}\log\frac{me}{2}}}$$

(note that $r_m < r_{m+1} < \widetilde{r_m}$). Then for $m \ge 9$, G_m has the following properties:

(i) $$\lim_{m\to\infty} \log(m) \max_{r_9\le x<1} G_m(x) = 1,$$

(ii) $$\lim_{m\to\infty} \log(m) \min_{r_m\le x\le\widetilde{r_m}} G_m(x) = 1.$$

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Proof For $0 < x < 1$, we denote $G_m(x)$ defined in the assumption. As in [3, Lemma 2.1], it is easy to see that whenever $f'(x) = 0$ and $m \geq 9$, $r_m < x < \widetilde{r_m}$. Therefore, for any $r_m \leq x \leq \widetilde{r_m}$, we have

$$\log(m)G_m(x) \geq \log(m)\Big(\frac{\frac{m}{2}\log(\frac{m}{2})}{1+\frac{m}{2}\log(\frac{m}{2})}\Big)^{\frac{m}{2}}\Big(\log\Big(e+\frac{me^2}{2}\log(\frac{m}{2})\Big)\Big)^{-1},$$
$$\log(m)G_m(x) \leq \log(m)\Big(\frac{\frac{me}{2}\log(\frac{m}{2})}{1+\frac{me}{2}\log(\frac{m}{2})}\Big)^{\frac{m}{2}}\Big(\log\Big(e+\frac{me}{2}\log(\frac{m}{2})\Big)\Big)^{-1}.$$

Straightforward calculation shows that the two estimates tend to 1 as $m \to \infty$. This completes the proof of the lemma. ■

Lemma 2.3 ([10, Theorem 2]) *Let $p, q > 0$, $u \in H(\mathbb{D}^n)$. Suppose that $\varphi = (\varphi_1, \dots, \varphi_n)$ is a holomorphic self-map of $\mathbb{D}^n$, and $uC_\varphi \colon \mathcal{B}^p \to \mathcal{B}^q$ is bounded. Then the following statements are true:*

(i) $0 < p < 1$, *$uC_\varphi \colon \mathcal{B}^p \to \mathcal{B}^q$ is compact if and only if*

$$\lim_{\varphi_l(z)\to\partial\mathbb{D}} |u(z)| \sum_{k=1}^{n} \Big|\frac{\partial\varphi_l}{\partial z_k}(z)\Big| \frac{(1-|z_k|^2)^q}{(1-|\varphi_l(z)|^2)^p} = 0 \tag{2.1}$$

for each $l \in \{1, \dots, n\}$;

(ii) $p = 1$, *$uC_\varphi \colon \mathcal{B}^p \to \mathcal{B}^q$ is compact if and only if*

$$\lim_{\varphi(z)\to\partial\mathbb{D}^n} \sum_{k,l=1}^{n} \Big|\frac{\partial u}{\partial z_k}(z)\Big| (1-|z_k|^2)^q \log\frac{e}{1-|\varphi_l(z)|^2} = 0,$$

and

$$\lim_{\varphi(z)\to\partial\mathbb{D}^n} |u(z)| \sum_{k,l=1}^{n} \Big|\frac{\partial\varphi_l}{\partial z_k}(z)\Big| \frac{(1-|z_k|^2)^q}{1-|\varphi_l(z)|^2} = 0; \tag{2.2}$$

(iii) $p > 1$, *$uC_\varphi \colon \mathcal{B}^p \to \mathcal{B}^q$ is compact if and only if*

$$\lim_{\varphi(z)\to\partial\mathbb{D}^n} \sum_{k,l=1}^{n} \Big|\frac{\partial u}{\partial z_k}(z)\Big| \frac{(1-|z_k|^2)^q}{(1-|\varphi_l(z)|^2)^{p-1}} = 0,$$
$$\lim_{\varphi(z)\to\partial\mathbb{D}^n} |u(z)| \sum_{k,l=1}^{n} \Big|\frac{\partial\varphi_l}{\partial z_k}(z)\Big| \frac{(1-|z_k|^2)^q}{(1-|\varphi_l(z)|^2)^p} = 0.$$

3 Main Theorems

In [10], the authors characterized the boundedness and compactness of the composition operators between different Bloch type spaces in the polydisk. In this section, we provide some new characterizations of compactness of weighted composition operators between different Bloch type spaces in the polydisk.

Before stating our main results, we give the following conditions:

(a) $\max_{i\in I} \limsup_{m\to\infty} m^p \sup_{z\in\mathbb{D}^n} |u(z)| \sum_{k=1}^n |\varphi_i(z)|^{m-1} \big|\frac{\partial\varphi_i}{\partial z_k}(z)\big| (1-|z_k|^2)^q = 0$;

(b) $\max_{j\in J} \lim_{\varphi(z)\to\partial\mathbb{D}^n} |u(z)| \sum_{k=1}^n \big|\frac{\partial\varphi_j}{\partial z_k}(z)\big| (1-|z_k|^2)^q = 0$;

(c) $\lim_{\varphi(z)\to\partial\mathbb{D}^n}\sum_{k=1}^n\left|\frac{\partial u}{\partial z_k}(z)\right|(1-|z_k|^2)^q=0$;

(d) $\max_{i\in I}\limsup_{m\to\infty}\log(m)\sup_{z\in\mathbb{D}^n}\sum_{k=1}^n|\varphi_i(z)|^m\left|\frac{\partial u}{\partial z_k}(z)\right|(1-|z_k|^2)^q=0$;

(e) $\max_{i\in I}\limsup_{m\to\infty}m^{p-1}\sup_{z\in\mathbb{D}^n}\sum_{k=1}^n|\varphi_i(z)|^{m-1}\left|\frac{\partial u}{\partial z_k}(z)\right|(1-|z_k|^2)^q=0$.

Theorem 3.1 *Let $p>1, q>0$ and $\varphi=(\varphi_1,\dots,\varphi_n)$ be a holomorphic self-map of $\mathbb{D}^n$. If $uC_\varphi\colon\mathcal{B}^p\to\mathcal{B}^q$ is bounded, then uC_φ is compact if and only if conditions* (a), (b), (c), *and* (e) *hold.*

Proof For any $l\in\{1,2,\dots,n\}$, set

$$a_l=\lim_{\varphi(z)\to\partial\mathbb{D}^n}|u(z)|\sum_{k=1}^n\left|\frac{\partial\varphi_l}{\partial z_k}(z)\right|\frac{(1-|z_k|^2)^q}{(1-|\varphi_l(z)|^2)^p},$$

$$b_l=\lim_{\varphi(z)\to\partial\mathbb{D}^n}\sum_{k=1}^n\left|\frac{\partial u}{\partial z_k}(z)\right|\frac{(1-|z_k|^2)^q}{(1-|\varphi_l(z)|^2)^{p-1}}.$$

We divide $\{1,2,\dots,n\}$ into I and J, which is defined in the introduction.

Case 1. If $j\in J$ with $\|\varphi_j\|_\infty<1$, we have

$$\begin{aligned}\lim_{\varphi(z)\to\partial\mathbb{D}^n}|u(z)|\sum_{k=1}^n\left|\frac{\partial\varphi_j}{\partial z_k}(z)\right|(1-|z_k|^2)^q&\le a_j\\&\le\lim_{\varphi(z)\to\partial\mathbb{D}^n}|u(z)|\sum_{k=1}^n\left|\frac{\partial\varphi_j}{\partial z_k}(z)\right|\frac{(1-|z_k|^2)^q}{(1-\|\varphi_j\|_\infty^2)^p},\end{aligned}$$

and

$$\begin{aligned}\lim_{\varphi(z)\to\partial\mathbb{D}^n}\sum_{k=1}^n\left|\frac{\partial u}{\partial z_k}(z)\right|(1-|z_k|^2)^q&\le b_j\\&\le\lim_{\varphi(z)\to\partial\mathbb{D}^n}\sum_{k=1}^n\left|\frac{\partial u}{\partial z_k}(z)\right|\frac{(1-|z_k|^2)^q}{(1-\|\varphi_j\|_\infty^2)^{p-1}}.\end{aligned}$$

Therefore, condition (b) is equivalent to

$$\lim_{\varphi(z)\to\partial\mathbb{D}^n}|u(z)|\sum_{l\in J}\sum_{k=1}^n\left|\frac{\partial\varphi_l}{\partial z_k}(z)\right|\frac{(1-|z_k|^2)^q}{(1-|\varphi_l(z)|^2)^p}=0,$$

and condition (c) is equivalent to

$$\lim_{\varphi(z)\to\partial\mathbb{D}^n}\sum_{l\in J}\sum_{k=1}^n\left|\frac{\partial u}{\partial z_k}(z)\right|\frac{(1-|z_k|^2)^q}{(1-|\varphi_l(z)|^2)^{p-1}}=0.$$

Case 2. For any $i\in I$ with $\|\varphi_i\|_\infty=1$, and for each $m\ge2$, let

$$A_{m,i}=\{z=(z_1,\dots,z_n)\in\mathbb{D}^n:r_m\le|z_i|\le r_{m+1}\},$$

where $r_m=(\frac{m-1}{m-1+2\alpha})^{1/2}\quad(m\ge2)$.

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For each $i \in I$ and for any $\varepsilon > 0$, there exists a δ_0 with $0 < \delta_0 < 1$ such that

$$|u(z)| \sum_{k=1}^{n} \Big| \frac{\partial \varphi_i}{\partial z_k}(z) \Big| \frac{(1-|z_k|^2)^q}{(1-|\varphi_i(z)|^2)^p} > a_i - \varepsilon$$

whenever $\operatorname{dist}(\varphi(z), \partial \mathbb{D}^n) < \delta_0$.

Since $r_m \to 1$ as $m \to \infty$, we may choose sufficiently large m such that $r_m > 1 - \delta_0$. If $\varphi(z) \in A_{m,i}$, $r_m \le |\varphi_i(z)| \le r_{m+1}$, then $1 - r_{m+1} < 1 - |\varphi_i(z)| < 1 - r_m < \delta_0$. Hence $\operatorname{dist}(\varphi_i(z), \partial \mathbb{D}) < \delta_0$. There exists w_i with $|w_i| = 1$ such that

$$\operatorname{dist}\big(\varphi_i(z), w_i\big) = \operatorname{dist}\big(\varphi_i(z), \partial \mathbb{D}\big) < \delta_0.$$

Let $w = (\varphi_1(z), \dots, \varphi_{i-1}(z), w_i, \varphi_{i+1}(z), \dots, \varphi_n(z))$, then

$$\operatorname{dist}\big(\varphi(z), \partial \mathbb{D}^n\big) \le \operatorname{dist}\big(\varphi(z), w\big) \le \operatorname{dist}\big(\varphi_i(z), w_i\big) < \delta_0.$$

So we have

$$\sup_{\varphi(z) \in A_{m,i}} |u(z)| \sum_{k=1}^{n} \Big| \frac{\partial \varphi_i}{\partial z_k}(z) \Big| \frac{(1-|z_k|^2)^q}{(1-|\varphi_i(z)|^2)^p} > a_i - \varepsilon.$$

Let $\varepsilon \to 0$ and by Lemma 2.1, we have

$$\begin{aligned}
a_i &\le \lim_{m\to\infty} \sup_{\varphi(z) \in A_{m,i}} |u(z)| \sum_{k=1}^{n} \Big| \frac{\partial \varphi_i}{\partial z_k}(z) \Big| \frac{(1-|z_k|^2)^q}{(1-|\varphi_i(z)|^2)^p} \\
&\le \lim_{m\to\infty} \sup_{\varphi(z) \in A_{m,i}} |u(z)| \sum_{k=1}^{n} \Big| \frac{\partial \varphi_i}{\partial z_k}(z) \Big| \frac{m^p |\varphi_i(z)|^{m-1} (1-|z_k|^2)^q}{m^p |\varphi_i(z)|^{m-1} (1-|\varphi_i(z)|^2)^p} \\
&\le \frac{\lim_{m\to\infty} \sup_{\varphi(z) \in A_{m,i}} |u(z)| \sum_{k=1}^{n} \Big| \frac{\partial \varphi_i}{\partial z_k}(z) \Big| m^p |\varphi_i(z)|^{m-1} (1-|z_k|^2)^q}{\lim_{m\to\infty} \inf_{\varphi(z) \in A_{m,i}} m^p |\varphi_i(z)|^{m-1} (1-|\varphi_i(z)|^2)^p} \\
&\le C \limsup_{m\to\infty} \; m^p \sup_{z \in \mathbb{D}^n} |u(z)| \sum_{k=1}^{n} |\varphi_i(z)|^{m-1} \Big| \frac{\partial \varphi_i}{\partial z_k}(z) \Big| (1-|z_k|^2)^q.
\end{aligned}$$

On the other hand,

$$m^p \sup_{z \in \mathbb{D}^n} |u(z)| \sum_{k=1}^{n} |\varphi_i(z)|^{m-1} \Big| \frac{\partial \varphi_i}{\partial z_k}(z) \Big| (1-|z_k|^2)^q \le I_1 + I_2,$$

where

$$\begin{aligned}
I_1 &= m^p \sup_{|||\varphi(z)||| \le s} |u(z)| \sum_{k=1}^{n} |\varphi_i(z)|^{m-1} \Big| \frac{\partial \varphi_i}{\partial z_k}(z) \Big| (1-|z_k|^2)^q, \\
I_2 &= m^p \sup_{|||\varphi(z)||| > s} |u(z)| \sum_{k=1}^{n} |\varphi_i(z)|^{m-1} \Big| \frac{\partial \varphi_i}{\partial z_k}(z) \Big| (1-|z_k|^2)^q,
\end{aligned}$$

and $0 < s < 1$. Here $|||\varphi(z)||| = \max_{1\leq i\leq n} |\varphi_i(z)|$. By Lemma 2.1(i), we have

$$I_2 = m^p \sup_{|||\varphi(z)|||>s} |u(z)| \sum_{k=1}^{n} |\varphi_i(z)|^{m-1} \Big| \frac{\partial \varphi_i}{\partial z_k}(z) \Big| (1-|z_k|^2)^q \frac{(1-|\varphi_i(z)|^2)^p}{(1-|\varphi_i(z)|^2)^p}$$
$$\leq \max_{|\varphi_i(z)|\in[0,1]} H_{m,p}(|\varphi_i(z)|) \sup_{|||\varphi(z)|||>s} |u(z)| \sum_{k=1}^{n} \Big| \frac{\partial \varphi_i}{\partial z_k}(z) \Big| \frac{(1-|z_k|^2)^q}{(1-|\varphi_i(z)|^2)^p}.$$

Using standard methods, it is easy to see that I_1 tends to 0 as $m \to \infty$ for any fixed s. Then let $s \to 1$, and we obtain that condition (a) is equivalent to

$$\lim_{\varphi(z)\to\partial\mathbb{D}^n} |u(z)| \sum_{l\in I} \sum_{k=1}^{n} \Big| \frac{\partial \varphi_l}{\partial z_k}(z) \Big| \frac{(1-|z_k|^2)^q}{(1-|\varphi_l(z)|^2)^p} = 0.$$

To complete the proof of the theorem, we only need to show condition (e) is equivalent to

$$\lim_{\varphi(z)\to\partial\mathbb{D}^n} \sum_{l\in I} \sum_{k=1}^{n} \Big| \frac{\partial u}{\partial z_k}(z) \Big| \frac{(1-|z_k|^2)^q}{(1-|\varphi_l(z)|^2)^{p-1}} = 0.$$

Using $p-1$ and $|\frac{\partial u}{\partial z_k}(z)|$ instead of p and $|u(z)||\frac{\partial \varphi_i}{\partial z_k}(z)|$, respectively, in the discussion above completes the proof of the theorem. ■

Theorem 3.2 *Let $0 < p < 1, q > 0, u \in H(\mathbb{D}^n)$, and let $\varphi = (\varphi_1, \dots, \varphi_n)$ be a holomorphic self-map of $\mathbb{D}^n$. Assume that $uC_\varphi \colon \mathcal{B}^p \to \mathcal{B}^q$ is bounded. Then uC_φ is compact if and only if condition* (a) *holds.*

Proof Note that the above discussion in Theorem 3.1 is still valid for the case $0 < p < 1$. We can safely conclude that condition (a) is equivalent to (2.1) for $l \in I$, and notice that whenever $l \in J$, condition (2.1) is obviously satisfied, since the set $\{z \in \mathbb{D}^n : |\varphi_l(z)| > s, l \in J\}$ is empty when s is large enough. ■

Theorem 3.2 gives rise to the following corollary.

Corollary 3.3 *Let $0 < p < 1, q > 0$ and let $\varphi = (\varphi_1, \dots, \varphi_n)$ be a holomorphic self-map of $\mathbb{D}^n$. Assume that $C_\varphi \colon \mathcal{B}^p \to \mathcal{B}^q$ is bounded. Then C_φ is compact if and only if $\limsup_{m\to\infty} m^{p-1}\|\varphi_i^m\|_q = 0$ for any $i \in I$.*

Theorem 3.4 *Let $p = 1, q > 0$ and let $\varphi = (\varphi_1, \dots, \varphi_n)$ be a holomorphic self-map of $\mathbb{D}^n$. If $uC_\varphi \colon \mathcal{B}^1 \to \mathcal{B}^q$ is bounded, then uC_φ is compact if and only if conditions* (a), (b), (c), *and* (d) *hold.*

Proof The equivalence between conditions (a), (b), and condition (2.2) is similar as proved in Theorem 3.1, so we omit the details here. For any $l \in \{1, 2, \dots, n\}$, set

$$c_l = \lim_{\varphi(z)\to\partial\mathbb{D}^n} \sum_{k=1}^{n} \Big| \frac{\partial u}{\partial z_k}(z) \Big| (1-|z_k|^2)^q \log \frac{e}{(1-|\varphi_l(z)|^2)}.$$

Case 1. If $j \in J$ with $\|\varphi_j\|_\infty < 1$, we have

$$\lim_{\varphi(z)\to\partial\mathbb{D}^n} \sum_{k=1}^{n} \Big| \frac{\partial u}{\partial z_k}(z) \Big| (1-|z_k|^2)^q \le c_j$$
$$\le \lim_{\varphi(z)\to\partial\mathbb{D}^n} \sum_{k=1}^{n} \Big| \frac{\partial u}{\partial z_k}(z) \Big| (1-|z_k|^2)^q \log \frac{e}{1-\|\varphi_j\|_\infty^2}.$$

Therefore, condition (c) is equivalent to

$$\lim_{\varphi(z)\to\partial\mathbb{D}^n} \sum_{l\in J} \sum_{k=1}^{n} \Big| \frac{\partial u}{\partial z_k}(z) \Big| (1-|z_k|^2)^q \log \frac{e}{(1-|\varphi_l(z)|^2)} = 0.$$

Case 2. For any $i \in I$ with $\|\varphi_i\|_\infty = 1$, and for each $m \ge 9$, let

$$B_{m,i} = \{z = (z_1, \dots, z_n) \in \mathbb{D}^n : r_m \le |z_i| \le \widetilde{r_m}\},$$

where

$$r_m = \sqrt{\frac{\frac{m}{2}\log\frac{m}{2}}{1+\frac{m}{2}\log\frac{m}{2}}}, \quad \widetilde{r_m} = \sqrt{\frac{\frac{me}{2}\log\frac{me}{2}}{1+\frac{me}{2}\log\frac{me}{2}}}$$

(note that $r_m < r_{m+1} < \widetilde{r_m}$). As discussed above, we get that for any $\varepsilon > 0$, we can choose m large enough such that

$$\sup_{\varphi(z)\in B_{m,i}} \sum_{k=1}^{n} \Big| \frac{\partial u}{\partial z_k}(z) \Big| (1-|z_k|^2)^q \log \frac{e}{1-|\varphi_i(z)|^2} > c_i - \varepsilon.$$

Let $\varepsilon \to 0$ and by Lemma 2.2, we have

$$\begin{aligned}
c_i &\le \lim_{m\to\infty} \sup_{\varphi(z)\in B_{m,i}} \sum_{k=1}^{n} \Big| \frac{\partial u}{\partial z_k}(z) \Big| (1-|z_k|^2)^q \log \frac{e}{1-|\varphi_i(z)|^2} \\
&\le \lim_{m\to\infty} \sup_{\varphi(z)\in B_{m,i}} \sum_{k=1}^{n} \Big| \frac{\partial u}{\partial z_k}(z) \Big| \frac{|\varphi_i(z)|^m (1-|z_k|^2)^q \log \frac{e}{1-|\varphi_i(z)|^2} \log(m)}{|\varphi_i(z)|^m \log(m)} \\
&\le \frac{\lim_{m\to\infty} \log(m) \sup_{\varphi(z)\in B_{m,i}} \sum_{k=1}^{n} \Big| \frac{\partial u}{\partial z_k}(z) \Big| |\varphi_i(z)|^m (1-|z_k|^2)^q}{\lim_{m\to\infty} \log(m) \inf_{\varphi(z)\in B_{m,i}} |\varphi_i(z)|^m (\log \frac{e}{1-|\varphi_i(z)|^2})^{-1}} \\
&\le \limsup_{m\to\infty} \log(m) \sup_{z\in\mathbb{D}^n} \sum_{k=1}^{n} \Big| \frac{\partial u}{\partial z_k}(z) \Big| |\varphi_i(z)|^m (1-|z_k|^2)^q.
\end{aligned}$$

Conversely,

$$\log(m) \sup_{z\in\mathbb{D}^n} \sum_{k=1}^{n} \Big| \frac{\partial u}{\partial z_k}(z) \Big| |\varphi_i(z)|^m (1-|z_k|^2)^q \le K_1 + K_2$$

where

$$K_1 = \log(m) \sup_{|||\varphi(z)||| \le s} \sum_{k=1}^{n} \Big| \frac{\partial u}{\partial z_k}(z) \Big| |\varphi_i(z)|^m (1-|z_k|^2)^q,$$

$$K_2 = \log(m) \sup_{|||\varphi(z)||| > s} \sum_{k=1}^{n} \Big| \frac{\partial u}{\partial z_k}(z) \Big| |\varphi_i(z)|^m (1-|z_k|^2)^q,$$

and $0 < r_9 < s < 1$. Here r_9 is as defined in Lemma 2.2. By Lemma 2.2(i), we have

$$\begin{aligned} K_2 &= \log(m) \sup_{|||\varphi(z)||| > s} \sum_{k=1}^{n} \Big| \frac{\partial u}{\partial z_k}(z) \Big| \frac{|\varphi_i(z)|^m \log \frac{e}{1-|\varphi_i(z)|^2} (1-|z_k|^2)^q}{\log \frac{e}{1-|\varphi_i(z)|^2}} \\ &\le \log(m) \max_{|\varphi_i(z)| \in [r_9, 1)} G_m(|\varphi_i(z)|) \sup_{|||\varphi(z)||| > s} \sum_{k=1}^{n} \Big| \frac{\partial u}{\partial z_k}(z) \Big| (1-|z_k|^2)^q \log \frac{e}{1-|\varphi_i(z)|^2} \end{aligned}$$

It is easy to check K_1 tends to 0 as $m \to \infty$ for any fixed s. Letting $s \to 1$, we obtain that condition (d) is equivalent to

$$\lim_{\varphi(z) \to \partial \mathbb{D}^n} \sum_{l \in I} \sum_{k=1}^{n} \Big| \frac{\partial u}{\partial z_k}(z) \Big| (1-|z_k|^2)^q \log \frac{e}{(1-|\varphi_l(z)|^2)} = 0.$$

This completes the proof of the theorem. ∎

By Theorems 3.1 and 3.4, we immediately gain the following corollary.

Corollary 3.5 *Let $p \ge 1, q > 0$ and $\varphi = (\varphi_1, \dots, \varphi_n)$ let be a holomorphic self-map of $\mathbb{D}^n$. If $C_\varphi \colon \mathcal{B}^p \to \mathcal{B}^q$ is bounded, then C_φ is compact if and only if the following two conditions hold:*

(i) $\limsup_{m \to \infty} m^{p-1} \|\varphi_i^m\|_q = 0$ *for any* $i \in I$;

(ii) $\lim_{\varphi(z) \to \partial \mathbb{D}^n} \sum_{k=1}^{n} \Big| \frac{\partial \varphi_j}{\partial z_k}(z) \Big| (1-|z_k|^2)^q = 0$ *for any* $j \in J$.

For an analytic function $u \in \mathbb{D}$, we define two integral operators by

$$I_u f = \int_0^z f'(\xi) u(\xi)\, d\xi, \quad J_u f = \int_0^z f(\xi) u'(\xi)\, d\xi$$

for all $f \in H(\mathbb{D})$ and $z \in \mathbb{D}$.

Now, combining the three theorems above, we obtain the following corollary, which is [3, Corollary 4.4].

Corollary 3.6 *Let $\varphi \colon \mathbb{D} \to \mathbb{D}$ be analytic, $u \in H(\mathbb{D})$ and $0 < q < \infty$.*

(i) *If $0 < p < 1$ and $uC_\varphi \colon \mathcal{B}^p \to \mathcal{B}^q$ is bounded, then uC_φ is compact if and only if*

$$\limsup_{m \to \infty} m^{p-1} \|I_u(\varphi^n)\|_q = 0.$$

(ii) *If $p = 1$ and $uC_\varphi \colon \mathcal{B}^p \to \mathcal{B}^q$ is bounded, then uC_φ is compact if and only if*

$$\limsup_{m \to \infty} \|I_u(\varphi^n)\|_q = 0 \quad \text{and} \quad \limsup_{m \to \infty} \log(m) |J_u(\varphi^n)\|_q = 0.$$

(iii) *If* $p > 1$ *and* $uC_\varphi\colon \mathcal{B}^p \to \mathcal{B}^q$ *is bounded, then* uC_φ *is compact if and only if*

$$\limsup_{m\to\infty} m^{p-1}\|I_u(\varphi^n)\|_q = 0 \quad \textit{and} \quad \limsup_{m\to\infty} m^{p-1}\|J_u(\varphi^{n-1})\|_q = 0.$$

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References

[1] C. C. Cowen and B. D. MacCluer, *Composition operators on spaces of analytic functions.* Studies in Advanced Mathematics, CRC Press, Boca Raton , FL, 1995.
[2] ———, *Essential norms of composition operators between Bloch type spaces in the polydisk.* Arch. Math. (Basel) **99**(2012), no. 6, 547–556. http://dx.doi.org/10.1007/s00013-012-0457-0
[3] O. Hyvärinen and M. Lindström, *Estimates of essential norms of weighted composition operators between Bloch-type spaces.* J. Math. Anal. Appl. **393**(2012), no. 1, 38–44. http://dx.doi.org/10.1016/j.jmaa.2012.03.059
[4] B. MacCluer and R. Zhao, *Essential norms of weighted composition operators between Bloch-type spaces.* Rocky Mountain J. Math. **33**(2003), no. 4, 1437–1458. http://dx.doi.org/10.1216/rmjm/1181075473
[5] K. M. Madigan, *Composition operators on analytic Lipschitz spaces.* Proc. Amer. Math. Soc. **119**(1993), no. 2, 465–473. http://dx.doi.org/10.1090/S0002-9939-1993-1152987-6
[6] K. Madigan and A. Matheson, *Compact composition operators on the Bloch space.* Trans. Amer. Math. Soc. **347**(1995), no. 7, 2679–2687. http://dx.doi.org/10.1090/S0002-9947-1995-1273508-X
[7] J. S. Manhas and R. Zhao, *New estimates of essential norms of weighted composition operators between Bloch type spaces.* J. Math. Anal. Appl. **389**(2012), no. 1, 32–47. http://dx.doi.org/10.1016/j.jmaa.2011.11.039
[8] A. Montes-Rodriguez, *The essential norm of a composition operator on the Bloch spaces.* Pacific J. Math. **188**(1999), no. 2, 339–351. http://dx.doi.org/10.2140/pjm.1999.188.339
[9] ———, *Weighted composition operators on weighted Banach spaces of analytic functions.* J. London Math. Soc. **61**(2000), no. 3, 872–884. http://dx.doi.org/10.1112/S0024610700008875
[10] S. Stević, R. Y. Chen, and Z. H. Zhou, *Weighted composition operators between Bloch-type spaces in the polydisk.* Sb. Math. **201**(2010), no. 1–2, 289–319. http://dx.doi.org/10.4213/sm4514
[11] J. H. Shapiro, *Composition operators and classical function theory.* Universitext: Tracts in Mathematics, Springer-Verlag, New York, 1993.
[12] J. Xiao, *Composition operators associated with Bloch-type spaces.* Complex Variables Theory Appl. **46**(2001), no. 2, 109–121. http://dx.doi.org/10.1080/17476930108815401
[13] R. Zhao, *Essential norms of composition operators between Bloch type spaces.* Proc. Amer. Math. Soc. **138**(2010), no. 7, 2537–2546. http://dx.doi.org/10.1090/S0002-9939-10-10285-8
[14] Z. H. Zhou and J. H. Shi, *Composition operators on the Bloch space in polydisk.* Complex Variables Theory Appl. **46**(2001), no. 1, 73–88. http://dx.doi.org/10.1080/17476930108815398
[15] K. H. Zhu, *Operator theory in function spaces.* Monographs and Textbooks in Pure and Applied Mathematics, 139, Marcel Dekker, New York, 1990.

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