REPRESENTATION OF TYPE A MONOIDS

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A semigroup $T$ consisting of one-one mapping between certain principal left ideals in a type $A$ semigroup $S$ is constructed. $T$ is shown to be a type $A$ semigroup. A representation of $S$ by $T$ is then obtained which is analogous to Vagner-Preston's results on inverse semigroups.

1. INTRODUCTION

Many results are now available in the literature on type $A$ semigroups; some of which are analogous to those on inverse semigroups; see for example Fountain [6, 5], Asibong-Ibe [2, 3, 4], Armstrong [1] and Fountain and Lawson [7]. Because of the close relationship which exists between a type $A$ semigroup and an inverse semigroup, each type $A$ being basically a special type of subsemigroup of an inverse semigroup via an embedding, it is natural to ask whether a representation exists for a type $A$ semigroup similar to Vagner-Preston's for inverse semigroup. This paper answers this question.

Let us recall a few definitions. Let $S$ be a semigroup and $a, b \in S$. Then $(a, b) \in L^*$ if and only if $a \text{Cb}$ is an oversemigroup of $S$. The relation $L^*$ which properly contains the Green's relation $L$ on $S$ has the following equivalent characterisation, see [10].

**Lemma 1.1.** Let $S$ be a semigroup and $a, b \in S$. The following are equivalent:

(i) $(a, b) \in L^*$,

(ii) for all $x, y \in S$, $ax = ay$ if and only if $bx = by$,

(iii) there exists an $S$-isomorphism $\lambda: aS^1 \rightarrow bS^1$ such that $a\lambda = b$.

**Lemma 1.2.** Let $S$ be a semigroup and $e$ an idempotent in $S$. Then for any $a$ in $S$, the following are equivalent:

(i) $(e, a) \in L^*$,

(ii) $ae = a$, and for all $x, y \in S$, $ax = ay$ if and only if $ex = ey$.

$R^*$ is dual to $L^*$ and the above definition and properties of $L^*$ apply in a dual manner to $R^*$.

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Let $S$ be a semigroup with a semilattice $E(S)$ of idempotents. Then $S$ is said to be an adequate semigroup if each $L^*$-class and each $R^*$-class contains an idempotent.

An adequate semigroup $S$ is said to be a type $A$ semigroup if for each $a$ in $S$ and $e$ in $E(S)$, $ea = a(ea)^+$ and $ae = (ae)^+a$, where $x^+$ and $x^-$ are respectively idempotents in the $L^*$ and $R^*$ classes $L_e^*$ and $R_e^*$. A type $A$ semigroup has been characterised in the following way in [5].

**Theorem 1.3.** Let $S$ be an adequate semigroup. Then for $a \in S$, $e \in E(S)$, the following are equivalent:

(i) $S$ is a type $A$ semigroup,
(ii) $eS^1 \cap aS^1 = eaS^1$ and $S^1e \cap S^1a = S^1ae$, and
(iii) there exist embeddings $\lambda_1: S \rightarrow S_1$, and $\lambda_2: S \rightarrow S_2$ into inverse semigroups $S_1$, $S_2$ such that $a^+\lambda_1 = (a\lambda_1)^{-1}(a\lambda_1)$ and $a^+\lambda_2 = (a\lambda_2)(a\lambda_2)^{-1}$.

2. Type A Semigroup of Mappings

In this and subsequent sections, the term semigroup $S$ will refer to a type $A$ semigroup $S$ with $E(S)$ as its set of idempotents. Other notation used here agrees with that of [9] and [5].

Let $a \in S$; then $a^+$, $a^* \in E(S)$, and $aa^* = a^+a = a$. Consider the left principal ideals $Sa^+$ and $Sa^*$ and let $x_1 \in Sa^+$. Then for some $x \in S$, $x_1 = xa^+ \in Sa^+$ and $x_1a = x_a^+a = xa = xaa^* \in Sa^*$. Evidently for every $s$ in $S$, $sa^* = s(aa^*) = sa \in Sa^*$. Let us define a mapping $a_\alpha: Sa^+ \rightarrow Sa^*$ by putting for every $x$ in $S$, $xa_\alpha = xa$, where $a \in S$. Since $aa^* = a$, $Sa = Saa^* \subseteq Sa^*$, so for $x \in S$, $xa = xa^+ a = (xa^+)a_\alpha \in (Sa^+)a_\alpha$ so evidently $(Sa^+)a_\alpha = Sa \subseteq Sa^*$. Thus ran $a_\alpha = Sa$. However, if $a$ is regular then $Sa = Sa^*$, thus in this case ran $a_\alpha = Sa^*$. Let us show that each $a_\alpha$, $a \in S$ is a one-to-one mapping.

**Lemma 2.1.** For each $a \in S$, $a_\alpha$ is a one-one mapping from $Sa^+$ into $Sa^*$. Also $a_\alpha$ is onto if and only if $a$ is regular.

**Proof:** Consider the mapping $a_\alpha: Sa^+ \rightarrow Sa^*$, and let $xa = ya$ for $x, y$ in $S$. Then $(xa^+)a_\alpha = xa = ya = (ya^+)a_\alpha$. But $aR^*a^+$, so $xa = ya$ if and only if $xa^+ = ya^+$ for all $x, y$ in $S$. Consequently, $a_\alpha$ is a one-one mapping.

Now if $a_\alpha$ is onto then $(Sa^+)a_\alpha = Sa^*$. Thus $(Sa^+)a_\alpha = Sa^+ a = Sa = Sa^*$; consequently $aLa^*$, and $a$ must be regular. Conversely, if $a$ is regular, $aa^{-1}a = a$, $a^* = a^{-1}a$ and clearly $Sa^* = Sa$, so $La$ is onto.

**Corollary 2.2.** For each $a \in S$, $a_\alpha$ has inverse $a_\alpha^{-1}$ if and only if $a$ is regular.

**Proof:** If $a_\alpha^{-1} = a_\alpha^{-1}$ then $xa^+ = (xa)a_\alpha^{-1} = (xa)a_\alpha^{-1} = xaa^{-1}$. So $xa^+a =
zaa−1a = za and bijectivity of αa forces a+ = aa−1 so aa−1a = a. Conversely if a is regular αa is bijective so α−1a exists and obviously α−1a = αa−1.

Now let a be a non-regular element in S. Let λ: Sa → Sa+ be an S-system isomorphism with aλ = a+. Thus given αa: Sa+ → Sa* we can define α−1a | Sa → Sa+ by putting α−1a(za)λ = (za)aλ = za+ for z ∈ S. One checks that if x ∈ Sa+, xaa−1αa−1 = (xa)α−1a = za+ = z and for each y = za, we have ya−1αa = za+αa = za = y. Observe that Sa ≠ Sa* because an equality implies regularity of a, which is a contradiction to our assumption.

Now let us consider the subset T of I(S), the symmetric inverse semigroup where

\[ T = \{ \alpha_a | a ∈ S, \alpha_a : Sa^+ → Sa^* \} \]

and impose the condition that α−1a ∈ T if and only if α−1a = αa−1, that is if and only if a is regular. Thus the domain and codomain of elements of T are respectively the principal left ideals generated by a+ and a* for any a ∈ S.

An important fact is there is closure in T with respect to the product of its elements. Let us show this as follows. Consider the mappings αa: Sa+ → Sa*, αb: Sb+ → Sb*. Now Sa* ∩ Sb+ = Sαb+*, and a*b+ = (a+b)*. Consequently ab+ = aa*b+ = (a(ab+))a; hence a(ab+)* = ab+ = (ab+)+a. Since Sa ⊆ Sa* then Sa ∩ Sb+ = Sαb ⊆ Sαb+* so that Sαb+ = S(ab+)+a = S(ab)+a = S(ab)+αa. But Sαb+ ⊆ Sαb+, and hence (Sαb+)αb ⊆ (Sαb+)αb, and (Sαb+)αb = S(ab)+αaαb = S(ab). Indeed, since (Sαb+)*αb = Sαb* and a*b = b(a*b)* = b(ab)*, one checks that Sαb+ = S(ab)+* ⊆ S(ab)*. With Sαb ⊆ S(ab)*, it is clear that the codomain of αaαb = S(ab)∗ and its domain is S(ab)+. Evidently, it follows from these facts that αaαb = αab, showing closure property in T. It is then clear that T is a semigroup.

Let a, b be regular elements in S. Then (ab) is regular with inverse (ab)−1 ∈ S. Also αa, αb are regular in T and evidently αab = αaαb is regular in T with inverse α−1(αab)−1 ∈ T, α−1(αab)−1 = α−1a−1αb−1 = αa−1αb−1 = αa−1αb−1 = α(b−1a−1) = α(ab)−1 ∈ T. Let us now show below that T is a type A monoid.

**Theorem 2.3.** For a type A semigroup S, the set T = {αa | a ∈ S, αa : Sa+ → Sa*} such that for each x in S, xαa = xa, is a type A monoid.

We will prove this fact through the following lemmas.

**Lemma 2.4.**

(i) \((αa, αb) ∈ L^∗(T) if and only if (a, b) ∈ L^∗(S), and

(ii) \((αa, αb) ∈ R^∗(T) if and only if (a, b) ∈ R^∗(S).

**Proof:** Let \((αa, αb) ∈ L^∗ for αa, αb in T. Then for all αc, αd in T we have that

\[ αaαc = αaαd \]

if and only if αbαc = αbαd.
Let \( \alpha_a \alpha_c = \alpha_a \alpha_d \). Then \((\text{dom} \alpha_a \alpha_c) \alpha_a = \text{ran} \alpha_a \cap \text{dom} \alpha_c = (\text{ran} \alpha_a \cap \text{dom} \alpha_d) \alpha_a\). Also \((\text{ran} \alpha_a \cap \text{dom} \alpha_c) \alpha_c = (\text{ran} \alpha_a \cap \text{dom} \alpha_d) \alpha_d\). Now if \( xa_a \in \text{ran} \alpha_a \cap \text{dom} \alpha_c \), then the equality \( \alpha_a \cap \text{dom} \alpha_c = \text{ran} \alpha_a \cap \text{dom} \alpha_d \) implies that for all \( x \) in \( S \), \( x \alpha_a = \alpha_a \alpha_d \). That is, \( \alpha_a = x \alpha_d \) and in particular for \( x = a^+ \), \( a^+ \alpha_a = a^+ \alpha_d \). Thus if \( \alpha_a \alpha_c = \alpha_a \alpha_d \), \( \alpha_a = \alpha_d \). If \( b \in S \), \( \alpha_a \alpha_c = \alpha_a \alpha_d \) implies \( \alpha_b \alpha_c = \alpha_b \alpha_d \). Hence for all \( c, d \in S \), we can deduce that \( \alpha_a \alpha_c = \alpha_a \alpha_d \) implies \( \alpha_b \alpha_c = \alpha_b \alpha_d \). Since this is true for all \( \alpha_c, \alpha_d \in T \), then \( (\alpha_a, \alpha_b) \in L^*(T) \), which completes the proof of (i). The proof of (ii) is similar, so the lemma is proved.

From the above lemma we have the following.

**Corollary 2.5.** Let \( \alpha_a, \alpha_b \in T \). Then

(i) \((\alpha_a, \alpha_b) \in H^*(T) \) if and only if \((a, b) \in H^*(S)\),

(ii) \((\alpha_a, \alpha_b) \in D^*(T) \) if and only if \((a, b) \in D^*(S)\).

**Proof:** (i) If \((\alpha_a, \alpha_b) \in H^*(T)\), then obviously \((\alpha_a, \alpha_b) \in L^*(T) \) and \((\alpha_a, \alpha_b) \in R^*(T) \) and by Lemma 2.4 \((a, b) \in L^* \cap R^* \) = \( H^* \). Conversely, if \((a, b) \in H^* \), then \((\alpha_a, \alpha_b) \in H^*(T) \) holds from Lemma 2.4.

(ii) For \((\alpha_a, \alpha_b) \in D^*(T) \), there exist \( a_{x_1}, x_{x_2}, \ldots, x_{x_n} \in T \) such that

\[
\alpha_a L^* a_{x_1} R^* a_{x_2} L^* \ldots a_{x_n} R^* a_b.
\]

But Lemma 2.4 implies that in \( S \), \( a L^* a_{x_1} R^* a_{x_2} L^* \ldots a_{x_n} R^* a_b \) whence \((a, b) \in D^* \). The converse can also be shown using Lemma 2.4.

To identify idempotent elements in \( T \), observe that if \( a \) in \( S \) is an idempotent then \( a^+ = a^* = a \). If \( x \in Se \), \( xe = x \) so that \( x \alpha_e = xe, \alpha_e = 1_{Se} \).

**Lemma 2.6.** An element \( \alpha_a \in T \) is an idempotent if and only if \( a \) in \( S \) is an idempotent. Moreover, \( E(T) \) is a semilattice.

**Proof:** If \( \alpha_a \) is an idempotent then \( \alpha_a^2 = \alpha_a \) implies \( \text{dom} \alpha_a^2 = (\text{ran} \alpha_a \cap \text{dom} \alpha_a) \alpha_a^{-1} = \text{dom} \alpha_a \), that is, \( \text{ran} \alpha_a \cap \text{dom} \alpha_a = \text{ran} \alpha_a \) so that \( \text{ran} \alpha_a \subseteq \text{dom} \alpha_a \). Also \( \text{ran} \alpha_a^2 = (\text{ran} \alpha_a \cap \text{dom} \alpha_a) \alpha_a = \text{ran} \alpha_a \) hence \( \text{dom} \alpha_a, \subseteq \text{ran} \alpha_a \). From both inclusions, \( \text{dom} \alpha_a = \text{ran} \alpha_a \). Thus \( Sa = Sa^+ \) and for \( z \in \text{dom} \alpha_a \), \( z \alpha_a^2 = z \alpha_a \), that is \( za^2 = za \), so in particular, for \( z = a^+ \), \( a^2 = a^+ a^2 = a^+ a = a \). Therefore \( a \) is an idempotent in \( S \).

Conversely if \( a \) is an idempotent in \( S \) then \( a^* = a^+ \) so that \( Sa^+ = Sa^* \) and quite clearly \( \text{dom} \alpha_a^2 = \text{dom} \alpha_a = \text{ran} \alpha_a = \text{ran} \alpha_a^2 \) and for all \( x \in Sa^+ \), \( xa^2 = za \). Hence for all \( x \in Sa^+ \), \( xa^2 = z \alpha_a \), so \( \alpha_a^2 = \alpha_a \).
Let $\alpha_e, \alpha_f \in E(T)$, the set of idempotents of $T$. Now $\alpha_e \alpha_f = \alpha_f \alpha_e = \alpha_f / \alpha_e$ and if $e \leq f$, $ef = fe = e$, so $\alpha_e \alpha_f = \alpha_f / \alpha_e = \alpha_e$. This completes the proof of the lemma.

For $a \in S, a^* \in L_a^*$, $a^+ \in R_a^*$ and $\alpha_a \alpha_a^* = \alpha_a$ and $\alpha_a + \alpha_a = \alpha_a + a = \alpha_a$. Evidently $(\alpha_a, \alpha_a^*) \in L^*(T)$ by Lemma 2.4, so we have

**Lemma 2.7.** For each $\alpha_a \in T$

(i) $(\alpha_a, \alpha_a^*) \in L^*(T)$ and

(ii) $(\alpha_a, \alpha_a^+) \in R^*(T)$.

Let $L_{\alpha_a}^*$ and $R_{\alpha_a}^*$ be the $L^*(T)$ and $R^*(T)$ classes containing $\alpha_a$. Let us denote by $\alpha_a^*$ and $\alpha_a^+$ the unique idempotents in $L_{\alpha_a}^*$ and $R_{\alpha_a}^*$ respectively. Now for $a \in S$, $e \in E(S)$, $ea = a(ea)^+$, $ae = (ae)^+ a$, and consequently $\alpha_e \alpha_a = \alpha_a = \alpha_a (ae)^+ = \alpha_a \alpha_e^* \alpha_a = \alpha_a \alpha_e^* = \alpha_a (\alpha_e \alpha_a)^*$ and similarly $\alpha_a \alpha_e = (\alpha_a \alpha_e)^+ \alpha_a$. Thus we have proved that

**Lemma 2.8.** For $\alpha_a, \alpha_e \in T$,

(i) $\alpha_e \alpha_a = \alpha_a (\alpha_e \alpha_a)^*$ and

(ii) $\alpha_a \alpha_e = (\alpha_a \alpha_e)^+ \alpha_a$.

These last observations together with Lemmas 2.4 to 2.7 complete the proof of Theorem 2.3.

Let $\beta_a : a^* S \rightarrow a^+ S$, $a \in S$ where $z \beta_a = ax$ for $x \in S$; using methods similar to the above, $\beta_a$ is a one-to-one mapping satisfying Lemmas 2.4 to 2.8 and

**Corollary 2.9.** $T^* = \{ \beta_a \mid a \in S \}$ is a type A semigroup.

### 3. Representation of Type A Monoid

We show here that there is a Vagner-Preston type representation from a type A semigroup $S$ into a type A semigroup of mappings on a set $X$. Let $X = S$, $a \in S$, and let $\varphi : S \rightarrow T$ be a mapping such that $a \varphi = a_a$, where $T = \{ \alpha_a \mid a \in S \}$ is the type A semigroup in Theorem 2.3 above.

**Theorem 3.1.** The mapping $\varphi : S \rightarrow T$, where $a \varphi = a_a$, is an isomorphism from $S$ onto $T$.

**Proof:** If $a, b \in S$, then $(ab) \varphi = a_{ab} = a_a a_b = a \varphi . b \varphi$. Also $a \varphi = b \varphi$ implies $a_a = a_b$, which in turn implies that $Sa^+ = Sb^+$, $Sa = Sb$, the domains and ranges of $a_a$ and $a_b$, respectively, and for all $x \in Sa^+$, $x \alpha_a = x \alpha_b$. Now $Sa^+ = Sb^+$ implies $a^+ Lb^+$ and hence $a^+ = b^+$. Similarly $Sa^+ = Sb^+$ implies $a^* = b^*$. But $x \alpha_a = x \alpha_b$ implies that $xa = xb$ for all $x \in Sa^+$; hence for $x = a^+$, $a = a^+ a = a^+ b = b^+ b = b$. Thus if $a_a = a_b$ then $a = b$, showing that $\varphi$ is a one-to-one homomorphism. By definition of $T$, $\varphi$ is onto, so the proof is complete. □
From Corollary 2.9, \( T' = \{ \beta_a \mid a \in S \} \) in type \( A \) semigroup and so

**COROLLARY 3.2.** Let \( \psi: S \to T' \) be a mapping given by \( a\psi = \beta_a \), for \( a \in S \). Then \( \psi \) is an isomorphism.

**PROOF:** As in Theorem 3.1 above, \( (ab)\psi = \beta_{ab} = \beta_a \beta_b = (a\psi)(b\psi) \), so \( \psi \) is a one-to-one homomorphism from \( S \) onto \( T' \). This completes the proof.

Let \( S \) be a left type \( A \) monoid and \( T = \{ \alpha_a \mid a \in S, \alpha_a: Sa^+ \to Sa^* \} \) where \( \alpha_a^{-1} \in T \) if and only if \( \alpha_a^{-1} = \alpha_{a-1} \), that is, if and only if \( a \) is regular.

**THEOREM 3.3.** \( T \) is a left adequate semigroup.

**PROOF:** Consider \( \alpha_a: Sa^+ \to Sa^* \), \( \alpha_b: Sb^+ \to Sb^* \) as defined earlier, where \( a, b \in S \) are non-regular. Now \( \text{ran } \alpha_a = S \alpha_a \neq Sa^* \) and \( \text{ran } \alpha_a \cap \text{dom } \alpha_b = S \alpha_a \cap Sb^+ = S(ab)^+a = (\text{dom } \alpha_a) \alpha_a \). Also \( (\text{ran } \alpha_a \cap \text{dom } \alpha_b) \alpha_b = Sab^+b = Sab \subseteq S(ab)^* \), so that \( (\text{ran } \alpha_a \cap \text{dom } \alpha_b) \alpha_a = \text{ran } \alpha_{ab} \). Since \( \text{ran } \alpha_a \cap \text{dom } \alpha_b = \text{ran } \alpha_{ab} \), by the previous lemma, \( T \) is a semigroup.

The proof of the theorem is complete by noting that the relevant aspects of Lemmas 2.4 - 2.7 above hold for \( T \) as well.

In fact \( T \) is a left type \( A \) semigroup since for \( \alpha_a, \alpha_e \in T \), \( \alpha_a \alpha_e = (\alpha_a \alpha_e)^+ \alpha_a \), which is true by Lemma 2.8 since \( S \) is a left type \( A \) monoid, for \( a \in S \), \( e \in E(S) \), \( ae = (ae)^+ a \).

Since \( ea \neq a(ea)^* \) does not hold in general for a left type \( A \) semigroup \( S \) with \( a \in S \), and \( e \) an idempotent, in general the equality \( \alpha_e \alpha_a = \alpha_a (\alpha_e \alpha_a)^* \) does not hold. However, we show below an example in which \( S \) is left type \( A \) and \( T \) a type \( A \) monoid.

**EXAMPLE:** Consider the semigroup \( S \) with the following multiplication table:

<table>
<thead>
<tr>
<th>.</th>
<th>( e )</th>
<th>( f )</th>
<th>( z )</th>
<th>( a )</th>
<th>( c )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( e )</td>
<td>( e )</td>
<td>( z )</td>
<td>( z )</td>
<td>( c )</td>
<td>( c )</td>
</tr>
<tr>
<td>( f )</td>
<td>( z )</td>
<td>( f )</td>
<td>( z )</td>
<td>( z )</td>
<td>( z )</td>
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<td>( z )</td>
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</tr>
<tr>
<td>( a )</td>
<td>( z )</td>
<td>( a )</td>
<td>( z )</td>
<td>( z )</td>
<td>( z )</td>
</tr>
<tr>
<td>( S )</td>
<td>( z )</td>
<td>( a )</td>
<td>( z )</td>
<td>( z )</td>
<td>( z )</td>
</tr>
</tbody>
</table>

The \( L^* \) classes of \( S \) are \( \{ f, a, c \} \), \( \{ z \} \), \( \{ e \} \) and the \( R^* \) classes are \( \{ e, a, c \} \), \( \{ f \} \), \( \{ z \} \).

It is easy to check that for each idempotent \( u \in E(S) \) and each \( z \in S \), \( zu = (zu)^+z \), and that \( c = ea \neq a(ea)^* = af = a \), hence \( S \) is left type \( A \) but not a right type \( A \) monoid.

Now define \( \alpha_a: Sa^+ \to Sa^* \) as usual. So \( T = \{ \alpha_e, \alpha_f, \alpha_z, \alpha_a, \alpha_c \} \), with \( L^* \)-classes \( \{ \alpha_a, \alpha_c, \alpha_f \} \), \( \{ \alpha_e \} \), \( \{ \alpha_z \} \) and \( R^* \)-classes: \( \{ \alpha_a, \alpha_e, \alpha_c \} \), \( \{ \alpha_f \} \) and \( \{ \alpha_z \} \). It is straightforward to verify that \( \alpha_e, \alpha_f \) are the only elements with \( \alpha_e^{-1} = \alpha_{e-1} = \alpha_e \), \( \alpha_f^{-1} = \alpha_{f-1} = \alpha_f \) so \( \alpha_{e-1}, \alpha_{f-1} \in T \). Now for all \( u \in \{ e, f \} \), \( z \in S \),
\[ \alpha_z \alpha_u = (\alpha_z \alpha_u)^+ \alpha_z \] but while \( ea \neq a(ea)^* \), we have \( \alpha_e \alpha_a = \alpha_{ea} = \alpha_e \), and
\[ \alpha_a \alpha_{(ea)^*} = \alpha_a (\alpha_e \alpha_a)^* = \alpha_a \alpha_e = \alpha_a \alpha_f = \alpha_a \] and for all \( x \in S \), \( x \alpha_e = x \alpha_a \), hence \( \alpha_e = \alpha_a \), since \( Sc^+ = Sa^+ \), and \( Sc = Sa \). One also finds that \( \alpha_e \alpha_c = \alpha_e (\alpha_e \alpha_c)^* \), and in general \( \alpha_u \alpha_c = \alpha_u (\alpha_u \alpha_c)^* \) so \( T \) is a type \( A \), with \( E = \{ \alpha_e, \alpha_f, \alpha_z \} \) as a semilattice.

From all the foregoing we have for the left type \( A \) semigroup in the table:

**Theorem 3.4.** \( S \) is isomorphic to a left type \( A \) semigroup of one-to-one mappings on \( S \).

Let us consider an arbitrary left type \( A \) semigroup \( S \) and \( T \), the semigroup of one-to-one mappings \( \alpha_a, a \in S \). The following result holds.

**Theorem 3.5.** Let \( S \) be a left type \( A \) semigroup; then \( T \) is a left type \( A \) semigroup. Moreover \( S \) is isomorphic to \( T \).

To see this clearly, consider an arbitrary left type \( A \) monoid \( S \) and \( T = \{ \alpha_a \mid a \in S, \alpha_a : Sa^+ \to Sa^* \} \) where \( \alpha_a : Sa^+ \to Sa^* \) is defined by putting

\[ x \alpha_a = xa, \quad \text{for every} \quad x \in S, \]
and \( \alpha_a^{-1} \in T \), \( a \in S \) if and only if \( \alpha_a^{-1} = \alpha_a^{-1} \). Then \( Sa \cap Sb^+ = Sab^+ \), for \( a \in S \), \( b^+ \in E(S) \), and if \( \alpha_a : Sa^+ \to Sa^* \), \( \alpha_b : Sb^+ \to Sb^* \) and \( a, b \in S \) have no inverses in \( S \), ran \( \alpha_a = Sa \neq Sa^* \), ran \( \alpha_b = Sb \neq Sb^* \). Also dom \( \alpha_a \alpha_b = S(ab)^+ = \text{dom} \alpha_{ab} \) and ran \( \alpha_a \alpha_b = Sab = \text{ran} \alpha_{ab} \) and \( T \) is a semigroup.

\[ Sa = Sa^* \] if and only if \( S \) is regular and in such cases \( \alpha_a \) is bijective and \( \alpha_a^{-1} = \alpha_a^{-1} \).

That \( T \) is a left type \( A \) semigroup is shown in Theorem 3.3 together with Lemmas 2.4 – 2.6 and the following lemmas.

**Lemma 3.6.** \( (\alpha_a, \alpha_a^+) \in R^*(T) \) for all \( a \in S \), \( a^+ \in E(S) \).

**Lemma 3.7.** \( \alpha_a \alpha_e = (\alpha_a \alpha_e)^+ \alpha_a \) for all \( a \in S \), \( e \in E(S) \).

**Proof:** \( \alpha_a \alpha_e = \alpha_{ae} = (\alpha_{ae})^+ a = \alpha_{ae}^+ \alpha_a = (\alpha_a \alpha_e)^+ \alpha_a \), since \( ae = (ae)^+ a \).

The proof of Theorem 3.5 is complete by noting that if \( \psi : S \to T \) is a mapping where \( \psi \) is defined by \( a \psi = \alpha_a \) for \( a \in S \), then for all \( a, b \) in \( S \)

\[ (ab \psi) = (a \psi)(b \psi) \]
and \( \psi \) is one-to-one and onto.

If \( S \) is an adequate semigroup which is not type \( A \), the above result may not hold. Now for \( a, b \in S \) suppose that \( z \in Sa \cap Sb^+ \). Then \( z = sa = tb^+ \) for some \( s, t \in S \).
and since $z = zb^+ = sab^+ \in Sab^+$ then $Sa \cap Sb^+ \subseteq Sab^+$. To understand the situation clearly, let $S = C \cup D \cup \{1\}$ where $C = \langle a \rangle$ is the free semigroup on $a$ and $D = \langle b \rangle$ the free monoid generated by $b$, with multiplication in $S$ defined by $a^m b^n = b^{m+n}$, $b^m a^n = a^{m+n}$, for $m > 0$, $n \geq 0$, $b^0 = e$, and 1 is the identity in $S$. The $L^*$- and $R^*$-classes of $S$ are respectively $C \cup \{1\}$, $D$ and $\{1\}$, $C \cup D$. For $a, b \in S$, $a^* = 1$, $a^+ = e$, $b^* = b^+ = e$, $Sa \cap Sb^+ = \emptyset$, $Sab^+ = D \setminus \{e\}$, so $Sa \cap Sb^+ \neq Sab^+$. Moreover, $a_a : Sa^+ \rightarrow Sa^*$ is not one-to-one since for $z = a^t$, $y = b^t$, $z a_a = y a_a$ but $z \neq y$.

**References**