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RESEARCH ARTICLE

On the extension of positive maps to Haagerup noncommutative L^p-spaces

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Abstract
Let M be a von Neumann algebra, let φ be a normal faithful state on M and let $L^p(M,\varphi)$ be the associated Haagerup noncommutative L^p-spaces, for $1 \leq p \leq \infty$. Let $D \in L^1(M,\varphi)$ be the density of φ. Given a positive map $T\colon M \to M$ such that $\varphi \circ T \leq C_1\varphi$ for some $C_1 \geq 0$, we study the boundedness of the L^p-extension $T_{p,\theta}\colon D^{\frac{1-\theta}{p}} M D^{\frac{\theta}{p}} \to L^p(M,\varphi)$ which maps $D^{\frac{1-\theta}{p}} x D^{\frac{\theta}{p}}$ to $D^{\frac{1-\theta}{p}} T(x) D^{\frac{\theta}{p}}$ for all $x \in M$. Haagerup–Junge–Xu showed that $T_{p,\frac{1}{2}}$ is always bounded and left open the question whether $T_{p,\theta}$ is bounded for $\theta \neq \frac{1}{2}$. We show that for any $1 \leq p < 2$ and any $\theta \in [0, 2^{-1}(1-\sqrt{p-1})] \cup [2^{-1}(1+\sqrt{p-1}), 1]$, there exists a completely positive T such that $T_{p,\theta}$ is unbounded. We also show that if T is 2-positive, then $T_{p,\theta}$ is bounded provided that $p \geq 2$ or $1 \leq p < 2$ and $\theta \in [1-p/2, p/2]$.

Contents

1 Introduction 1
2 The extension problem 2
2.1 Haagerup noncommutative L^p-spaces . . . 2
2.2 Extension of maps $M \to M$ . . . 4
3 Computing $\|T_{p,\theta}\|$ on semifinite von Neumann algebras 6
4 Extension results 8
5 The use of infinite tensor products 12
6 Nonextension results 14
References 19

1. Introduction

Let M be a von Neumann algebra equipped with a normal faithful state φ. Let $T\colon M \to M$ be a positive map such that $\varphi \circ T \leq C_1\varphi$ on the positive cone M^+, for some constant $C_1 \geq 0$. Assume first that φ is a trace (that is, $\varphi(xy) = \varphi(yx)$ for all $x, y \in M$) and consider the associated noncommutative L^p-spaces $\mathcal{L}^p(M,\varphi)$ (see, e.g., [6, 19] or [10, Chapter 4]). Let $C_\infty = \|T\|$. Then for all $1 \leq p < \infty$, T extends to a bounded map on $\mathcal{L}^p(M,\varphi)$, with

$$\left\|T\colon \mathcal{L}^p(M,\varphi) \longrightarrow \mathcal{L}^p(M,\varphi)\right\| \leq C_\infty^{1-\frac{1}{p}} C_1^{\frac{1}{p}}; \tag{1.1}$$

see [16, Lemma 1.1]. This extension result plays a significant role in various aspects of operator theory on noncommutative L^p-spaces, in particular for the study of diffusion operators or semigroups on those spaces; see, for example, [1, 7, 11] or [14, Chapter 5].

Let us now drop the tracial assumption on φ. For any $1 \leq p \leq \infty$, let $L^p(M,\varphi)$ denote the Haagerup noncommutative L^p-space $L^p(M,\varphi)$ associated with φ [8, 9, 10, 22]. These spaces extend the tracial noncommutative L^p-spaces $\mathcal{L}^p(\cdots)$ in a very beautiful way and many topics in operator theory which had been first studied on tracial noncommutative L^p-spaces were/are investigated on Haagerup noncommutative L^p-spaces. This has led to several major advances; see in particular [9], [16, Section 7], [4], [2] and [13].

The question of extending a positive map $T\colon M \to M$ to $L^p(M,\varphi)$ was first considered in [16, Section 7] and [9, Section 5]. Let $D \in L^1(M,\varphi)$ be the density of φ, let $1 \leq p < \infty$ and let $\theta \in [0,1]$. Let $T_{p,\theta}\colon D^{\frac{1-\theta}{p}} M D^{\frac{\theta}{p}} \to L^p(M,\varphi)$ be defined by

$$T_{p,\theta}\Big(D^{\frac{1-\theta}{p}} x D^{\frac{\theta}{p}}\Big) = D^{\frac{1-\theta}{p}} T(x) D^{\frac{\theta}{p}}, \qquad x \in M. \tag{1.2}$$

(See Section 2 for the necessary background on D and the above definition.) Then [9, Theorem 5.1] shows that if $\varphi \circ T \leq C_1 \varphi$, then $T_{p,\frac{1}{2}}$ extends to a bounded map on $L^p(M,\varphi)$, with

$$\|T_{p,\frac{1}{2}}\colon L^p(M,\varphi) \longrightarrow L^p(M,\varphi)\| \leq C_\infty^{1-\frac{1}{p}} C_1^{\frac{1}{p}}.$$

This extends the tracial case (1.1); see Remark 2.5. Furthermore, [9, Proposition 5.5] shows that if T commutes with the modular automorphism group of φ, then $T_{p,\theta} = T_{p,\frac{1}{2}}$ for all $\theta \in [0,1]$.

In addition to the above results, Haagerup–Junge–Xu stated as an open problem the question whether $T_{p,\theta}$ is always bounded for $\theta \neq \frac{1}{2}$ (see [9, Section 5]). The main result of the present paper is a negative answer to this question. More precisely, we show that if $1 \leq p < 2$ and if either $0 \leq \theta < 2^{-1}(1-\sqrt{p-1})$ or $2^{-1}(1+\sqrt{p-1}) < \theta \leq 1$, then there exists M, φ as above and a unital completely positive map $T\colon M \to M$ such that $\varphi \circ T = \varphi$ and $T_{p,\theta}$ is unbounded; see Theorem 6.1.

We also show that for any M, φ as above and for any 2-positive map $T\colon M \to M$ such that $\varphi \circ T \leq C_1 \varphi$ for some $C_1 \geq 0$, then $T_{p,\theta}$ is bounded for all $p \geq 2$ and all $\theta \in [0,1]$; see Theorem 4.1. In other words, the Haagerup–Junge–Xu problem has a positive solution for $p \geq 2$, provided that we restrict to 2-positive maps. We also show, under the same assumptions, that $T_{p,\theta}$ is bounded for all $1 \leq p \leq 2$ and all $\theta \in [1-p/2, p/2]$; see Theorem 4.3.

Section 2 contains preliminaries on the $L^p(M,\varphi)$ and on the question whether $T_{p,\theta}$ is bounded. Section 3 presents a way to compute $\|T_{p,\theta}\|$ in the case when $M = M_n$ is a matrix algebra, which plays a key role in the last part of the paper. Section 4 contains the extension results stated in the previous paragraph. Finally, Sections 5 and 6 are devoted to the construction of examples for which $T_{p,\theta}$ is unbounded.

2. The extension problem

Throughout we consider a von Neumann algebra M and we let M_* denote its predual. We let M^+ and M_*^+ denote the positive cones of M and M_*, respectively.

2.1. Haagerup noncommutative L^p-spaces

Assume that M is σ-finite, and let φ be a normal faithful state on M. We shall briefly recall the definition of the Haagerup noncommutative L^p-spaces $L^p(M,\varphi)$ associated with φ, as well as some of their main features. We refer the reader to [8], [9, Section 1], [10, Chapter 9], [19, Section 3] and [22] for details

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and complements. We note that $L^p(M,\varphi)$ can actually be defined when φ is any normal faithful weight on M. The assumption that φ is a state makes the description below a little simpler.

Let $(\sigma_t^\varphi)_{t\in\mathbb{R}}$ be the modular automorphism group of φ [20, Chapter VIII], and let

$$\mathcal{R} = M \rtimes_{\sigma^\varphi} \mathbb{R} \subset M\overline{\otimes}B(L^2(\mathbb{R}))$$

be the resulting crossed product; see, for example, [20, Chapter X]. If $M \subset B(H)$ for some Hilbert space H, then we have $\mathcal{R} \subset B(L^2(\mathbb{R}; H))$. Let us regard M as a sub-von Neumann algebra of $\mathcal{R}$ in the natural way. Then $(\sigma_t^\varphi)_{t\in\mathbb{R}}$ is given by

$$\sigma_t^\varphi(x) = \lambda(t)x\lambda(t)^*, \qquad t \in \mathbb{R},\ x \in M, \tag{2.1}$$

where $\lambda(t) \in B(L^2(\mathbb{R}; H))$ is defined by $[\lambda(t)\xi](s) = \xi(s-t)$ for all $\xi \in L^2(\mathbb{R}; H)$. This is a unitary. For any $t \in \mathbb{R}$, define $W(t) \in B(L^2(\mathbb{R}; H))$ by $[W(t)\xi](s) = e^{-its}\xi(s)$ for all $\xi \in L^2(\mathbb{R}; H)$. Then the dual action $\widehat{\sigma}^\varphi : \mathbb{R} \longrightarrow \mathrm{Aut}(\mathcal{R})$ of σ^φ is defined by

$$\widehat{\sigma}_t^\varphi(x) = W(t)xW(t)^*, \qquad t \in \mathbb{R},\ x \in \mathcal{R}.$$

(See [20, §VIII.2].) A remarkable fact is that for any $x \in \mathcal{R}$, $\widehat{\sigma}_t^\varphi(x) = x$ for all $t \in \mathbb{R}$ if and only if $x \in M$.

There exists a unique normal semifinite trace τ_0 on $\mathcal{R}$ such that

$$\tau_0 \circ \widehat{\sigma}_t^\varphi = e^{-t}\tau_0, \qquad t \in \mathbb{R};$$

see, for example, [10, Theorem 8.15]. This trace gives rise to the $*$-algebra $L^0(\mathcal{R}, \tau_0)$ of τ_0-measurable operators [10, Chapter 4]. Then for any $1 \le p \le \infty$, the Haagerup L^p-space $L^p(M,\varphi)$ is defined as

$$L^p(M,\varphi) = \left\{ y \in L^0(\mathcal{R}, \tau_0) \ :\ \widehat{\sigma}_t^\varphi(y) = e^{-\frac{t}{p}}y \text{ for all } t \in \mathbb{R} \right\}.$$

At this stage, this is just a $*$-subspace of $L^0(\mathcal{R}, \tau_0)$ (with no norm). One defines its positive cone as

$$L^p(M,\varphi)^+ = L^p(M,\varphi) \cap L^0(\mathcal{R}, \tau_0)^+.$$

It follows from above that $L^\infty(M,\varphi) = M$.

Let $\psi \in M_*^+$, that we regard as a normal weight on M, and let $\widehat{\psi}$ be its dual weight on $\mathcal{R}$[20, §VIII.1]. Let h_ψ be the Radon–Nikodym derivative of $\widehat{\psi}$ with respect to τ_0. That is, h_ψ is the unique positive operator affiliated with $\mathcal{R}$ such that

$$\widehat{\psi}(y) = \tau_0\Big(h_\psi^{\frac12} y h_\psi^{\frac12}\Big), \qquad y \in \mathcal{R}_+.$$

It turns out that h_ψ belongs to $L^1(M,\varphi)^+$ for all $\psi \in M_*^+$ and that the mapping $\psi \mapsto h_\psi$ is a bijection from M_*^+ onto $L^1(M,\varphi)^+$. This bijection readily extends to a linear isomorphism $M_* \longrightarrow L^1(M,\varphi)$, still denoted by $\psi \mapsto h_\psi$. Then $L^1(M,\varphi)$ is equipped with the norm $\|\cdot\|_1$ inherited from M_*, that is, $\|h_\psi\|_1 = \|\psi\|_{M_*}$ for all $\psi \in M_*$. Next, for any $1 \le p < \infty$ and any $y \in L^p(M,\varphi)$, the positive operator $|y|$ belongs to $L^p(M,\varphi)$ as well (thanks to the polar decomposition) and hence $|y|^p$ belongs to $L^1(M,\varphi)$. This allows to define $\|y\|_p = \||y|^p\|^{\frac1p}$ for all $y \in L^p(M,\varphi)$. Then $\|\cdot\|_p$ is a complete norm on $L^p(M,\varphi)$.

The Banach spaces $L^p(M,\varphi)$, $1 \le p \le \infty$, satisfy the following version of Hölder's inequality (see, e.g., [10, Proposition 9.17]).

Lemma 2.1. *Let $1 \le p, q, r \le \infty$ such that $p^{-1} + q^{-1} = r^{-1}$. Then for all $x \in L^p(M,\varphi)$ and all $y \in L^q(M,\varphi)$, the product xy belongs to $L^r(M,\varphi)$ and $\|xy\|_r \le \|x\|_p\|y\|_q$.*

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Let D be the Radon–Nikodym derivative of $\widehat{\varphi}$ with respect to τ_0, and recall that $D \in L^1(M, \varphi)^+$. This operator is called the density of φ. Recall that we regard M as a sub-von Neumann algebra of $\mathcal{R}$. Then $D^{it} = \lambda(t)$ is a unitary of $\mathcal{R}$ for all $t \in \mathbb{R}$ and

$$\sigma_t^\varphi(x) = D^{it} x D^{-it}, \qquad t \in \mathbb{R},\ x \in M. \tag{2.2}$$

Let $\mathrm{Tr}\colon L^1(M, \varphi) \to \mathbb{C}$ be defined by $\mathrm{Tr}(h_\psi) = \psi(1)$ for all $\psi \in M_*$. This functional has two remarkable properties. First, for all $x \in M$ and all $\psi \in M_*$, we have

$$\mathrm{Tr}(h_\psi x) = \psi(x). \tag{2.3}$$

Second if $1 \leq p, q \leq \infty$ are such that $p^{-1} + q^{-1} = 1$, then for all $x \in L^p(M, \varphi)$ and all $y \in L^q(M, \varphi)$, we have

$$\mathrm{Tr}(xy) = \mathrm{Tr}(yx).$$

This tracial property will be used without any further comment in the paper.

It follows from the definition of $\|\cdot\|_1$ and equation (2.3) that the duality pairing $\langle x, y\rangle = \mathrm{Tr}(xy)$ for $x \in M$ and $y \in L^1(M, \varphi)$ yields an isometric isomorphism

$$L^1(M, \varphi)^* \simeq M. \tag{2.4}$$

As a special case of equation (2.3), we have

$$\varphi(x) = \mathrm{Tr}(Dx), \qquad x \in M. \tag{2.5}$$

We note that $L^2(M, \varphi)$ is a space for the inner product $(x|y) = \mathrm{Tr}(y^* x)$. Moreover, by equation (2.5), we have

$$\varphi(x^* x) = \|x D^{\frac{1}{2}}\|_2^2 \quad \text{and} \quad \varphi(x x^*) = \|D^{\frac{1}{2}} x\|_2^2, \qquad x \in M. \tag{2.6}$$

We finally mention a useful tool. Let $M_a \subset M$ be the subset of all $x \in M$ such that $t \mapsto \sigma_t^\varphi(x)$ extends to an entire function $z \in \mathbb{C} \mapsto \sigma_z^\varphi(x) \in M$. (Such elements are called analytic). It is well known that M_a is a w^*-dense $*$-subalgebra of M [20, Section VIII.2]. Furthermore,

$$\sigma_{i\theta}(x) = D^{-\theta} x D^{\theta}, \tag{2.7}$$

for all $x \in M_a$ and all $\theta \in [0, 1]$, and $M_a D^{\frac{1}{p}} = D^{\frac{1}{p}} M_a$ is dense in $L^p(M, \varphi)$, for all $1 \leq p < \infty$. See [15, Lemma 1.1] and its proof for these properties.

2.2. Extension of maps $M \to M$

Given any linear map $T\colon M \to M$, we say that T is positive if $T(M^+) \subset M^+$. This implies that T is bounded. For any $n \geq 1$, we say that T is n-positive if the tensor extension map $I_{M_n} \otimes T\colon M_n \overline{\otimes} M \to M_n \overline{\otimes} M$ is positive. (Here, M_n is the algebra of $n \times n$ matrices.) Next, we say that T is completely positive if T is n-positive for all $n \geq 1$. See, for example, [18] for basics on these notions.

Consider any $\theta \in [0, 1]$ and $1 \leq p < \infty$. It follows from Lemma 2.1 that $D^{\frac{1-\theta}{p}} x D^{\frac{\theta}{p}}$ belongs to $L^p(M, \varphi)$ for all $x \in M$. We set

$$\mathcal{A}_{p,\theta} = D^{\frac{(1-\theta)}{p}} M D^{\frac{\theta}{p}} \subset L^p(M, \varphi). \tag{2.8}$$

It turns out that this is a dense subspace; see [15, Lemma 1.1].

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Let $T: M \to M$ be any bounded linear map. For any (p,θ) as above, define a linear map $T_{p,\theta}: \mathcal{A}_{p,\theta} \to \mathcal{A}_{p,\theta}$ by equation (1.2). The question we consider in this paper is whether $T_{p,\theta}$ extends to a bounded map $L^p(M,\varphi) \to L^p(M,\varphi)$ in the case when T is 2-positive and $\varphi \circ T \leq \varphi$ on M_+. More precisely, we consider the following:

Question 2.2. Determine the pairs $(p,\theta) \in [1,\infty) \times [0,1]$ such that

$$T_{p,\theta}: L^p(M,\varphi) \longrightarrow L^p(M,\varphi)$$

is bounded for all (M,φ) as above and all 2-positive maps $T: M \to M$ satisfying $\varphi \circ T \leq \varphi$ on M_+.

As in the introduction, we could consider maps such that $\varphi \circ T \leq C_1\varphi$ for some $C_1 \geq 0$. However, by an obvious scaling, there is no loss in considering $C_1 = 1$ only.

Remark 2.3. Question 2.2 originates from the Haagerup–Junge–Xu paper [9]. In Section 5 of the latter paper, the authors consider two von Neumann algebras M, N, and normal faithful states $\varphi \in M_*$ and $\psi \in N_*$ with respective densities $D_\varphi \in L^1(M,\varphi)$ and $D_\psi \in L^1(N,\psi)$. Then they consider a positive map $T: M \to N$ such that $\psi \circ T \leq C_1\varphi$ for some $C_1 > 0$. Given any $(p,\theta) \in [1,\infty) \times [0,1]$, they define $T_{p,\theta}: D_\varphi^{\frac{1-\theta}{p}} M D_\varphi^{\frac{\theta}{p}} \to L^p(N,\psi)$ by

$$T_{p,\theta}\Big(D_\varphi^{\frac{1-\theta}{p}} x D_\varphi^{\frac{\theta}{p}}\Big) = D_\psi^{\frac{1-\theta}{p}} T(x) D_\psi^{\frac{\theta}{p}}, \qquad x \in M.$$

In [9, Theorem 5.1], they show that $T_{p,\frac{1}{2}}$ is bounded and that setting $C_\infty = \|T\|$, we have $\|T_{p,\frac{1}{2}}: L^p(M,\varphi) \to L^p(N,\psi)\| \leq C_\infty^{1-\frac{1}{p}} C_1^{\frac{1}{p}}$. Then after the statement of [9, Proposition 5.4], they mention that the boundedness of $T_{p,\theta}$ for $\theta \neq \frac{1}{2}$ is an open question.

Remark 2.4. We wish to point out a special case which will be used in Section 5. Let B be a von Neumman algebra equipped with a normal faithful state ψ. Let $A \subset B$ be a sub-von Neumann algebra which is stable under the modular automorphism group of ψ (i.e., $\sigma_t^\psi(A) \subset A$ for all $t \in \mathbb{R}$). Let $\varphi = \psi_{|A}$ be the restriction of ψ to A. Let $D \in L^1(A,\varphi)$ and $\Delta \in L^1(B,\psi)$ be the densities of φ and ψ, respectively. On the one hand, it follows from [9, Theorem 5.1] (see Remark 2.3) that there exists, for every $1 \leq p < \infty$, a contraction

$$\Lambda(p): L^p(A,\varphi) \longrightarrow L^p(B,\psi)$$

such that $[\Lambda(p)](D^{\frac{1}{2p}} x D^{\frac{1}{2p}}) = \Delta^{\frac{1}{2p}} x \Delta^{\frac{1}{2p}}$ for all $x \in A$.

On the other hand, there exists a unique normal conditional expectation $E: B \to A$ such that $\psi = \varphi \circ E$ on B by [20, Theorem IX.4.2]. Moreover, it is easy to check that under the natural identifications $L^1(A,\varphi)^* \simeq A$ and $L^1(B,\psi)^* \simeq B$ (see equation (2.4) and the discussion preceding it), we have

$$\Lambda(1)^* = E.$$

Now, using [9, Theorem 5.1] again, there exists, for every $1 \leq p < \infty$, a contraction $E(p): L^p(B,\psi) \to L^p(A,\varphi)$ such that $[E(p)](\Delta^{\frac{1}{2p}} y \Delta^{\frac{1}{2p}}) = D^{\frac{1}{2p}} E(y) D^{\frac{1}{2p}}$ for all $y \in B$. It is clear that $E(p) \circ \Lambda(p) = I_{L^p(A,\varphi)}$. Consequently, $\Lambda(p)$ is an isometry.

We refer to [15, Section 2] for more on this.

Remark 2.5. Let $T: M \to M$ be a positive map, and let φ, D as in Subsection 2.1. Assume that φ is tracial and for any $1 \leq p < \infty$, let $\mathcal{L}^p(M,\varphi)$ be the (classical) noncommutative L^p-space with respect to the trace φ [10, Section 4.3]. That is, $\mathcal{L}^p(M,\varphi)$ is the completion of M for the norm

$$\|x\|_{\mathcal{L}^p(M,\varphi)} = \big(\varphi(|x|^p)\big)^{\frac{1}{p}}, \qquad x \in M.$$

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In this case, D commutes with M and

$$\|D^{\frac{1}{p}}x\|_{L^p(M,\varphi)} = \|x\|_{\mathcal{L}^p(M,\varphi)}, \qquad x \in M;$$

see, for example, [10, Example 9.11]. Hence, $T_{p,\theta} = T_{p,0}$ for all $1 \leq p < \infty$ and all $\theta \in [0, 1]$ and moreover, $T_{p,0}$ is bounded if and only if T extends to a bounded map $\mathcal{L}^p(M,\varphi) \to \mathcal{L}^p(M,\varphi)$. Thus, in the tracial case, the fact that $T_{p,0}$ is bounded under the assumption $\varphi \circ T \leq C_1\varphi$ is equivalent to the result mentionned in the first paragraph of Section 1; see (equation 1.1).

3. Computing $\|T_{p,\theta}\|$ on semifinite von Neumann algebras

As in the previous section, we let M be a von Neumann algebra equipped with a normal faithful state φ and we let $D \in L^1(M,\varphi)^+$ be the density of φ. We assume further that M is semifinite, and we let τ be a distinguished normal semifinite faithful trace on M. For any $1 \leq p \leq \infty$, we let $\mathcal{L}^p(M,\tau)$ be the noncommutative L^p-space with respect to τ. Although $\mathcal{L}^p(M,\tau)$ is isometrically isomorphic to the Haagerup L^p-space $L^p(M,\tau)$, it is necessary for our purpose to consider $\mathcal{L}^p(M,\tau)$ as such.

Let us give a brief account, for which we refer, for example, to [10, Section 4.3]. Let $\mathcal{L}^0(M,\tau)$ be the $*$-algebra of all τ-measurable operators on M. For any $p < \infty$, $\mathcal{L}^p(M,\tau)$ is the Banach space of all $x \in \mathcal{L}^0(M,\tau)$ such that $\tau(|x|^p) < \infty$, equipped with the norm

$$\|x\|_{\mathcal{L}^p(M,\tau)} = \big(\tau(|x|^p)\big)^{\frac{1}{p}}, \qquad x \in \mathcal{L}^p(M,\tau).$$

Moreover, $\mathcal{L}^\infty(M,\tau) = M$. The following analogue of Lemma 2.1 holds true: Whenever $1 \leq p, q, r \leq \infty$ are such that $p^{-1} + q^{-1} = r^{-1}$, then for all $x \in \mathcal{L}^p(M,\tau)$ and $y \in \mathcal{L}^q(M,\tau)$, xy belongs to $\mathcal{L}^r(M,\tau)$, with $\|xy\|_r \leq \|x\|_p\|x\|_q$ (Hölder's inequality). Furthermore, we have an isometric identification

$$\mathcal{L}^1(M,\tau)^* \simeq M \tag{3.1}$$

for the duality pairing given by $\langle x, y\rangle = \tau(yx)$ for all $x \in M$ and $y \in \mathcal{L}^1(M,\tau)$.

Let $\gamma \in \mathcal{L}^1(M,\tau)$ be associated with φ in the identification (3.1), that is,

$$\varphi(x) = \tau(\gamma x), \qquad x \in M. \tag{3.2}$$

Then γ is positive and it is clear from Hölder's inequality that for any $1 \leq p < \infty$, $\theta \in [0, 1]$ and $x \in M$, the product $\gamma^{\frac{1-\theta}{p}}x\gamma^{\frac{\theta}{p}}$ belongs to $\mathcal{L}^p(M,\tau)$.

It is well known that $\mathcal{L}^p(M,\tau)$ and $L^p(M,\varphi)$ are isometrically isomorphic (apply Remark 9.10 and Example 9.11 in [10]). The following lemma provides concrete isometric isomorphisms between these two spaces.

Lemma 3.1. *Let $1 \leq p < \infty$ and $\theta \in [0, 1]$. Then for all $x \in M$, we have*

$$\big\|\gamma^{\frac{1-\theta}{p}}x\gamma^{\frac{\theta}{p}}\big\|_{\mathcal{L}^p(M,\tau)} = \big\|D^{\frac{1-\theta}{p}}xD^{\frac{\theta}{p}}\big\|_{L^p(M,\varphi)}.$$

Before giving the proof of this lemma, we recall a classical tool. For any $\theta \in [0, 1]$, define an embedding $J_\theta\colon M \to L^1(M,\varphi)$ by letting

$$J_\theta(x) = D^{1-\theta}xD^\theta, \qquad x \in M.$$

Consider $(J_\theta(M), L^1(M,\varphi))$ as an interpolation couple, the norm on $J_\theta(M)$ being given by the norm on M, that is,

$$\big\|D^{1-\theta}xD^\theta\big\|_{J_\theta(M)} = \|x\|_M, \qquad x \in M. \tag{3.3}$$

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For any $1 \le p \le \infty$, let

$$C(p,\theta) = \big[J_\theta(M), L^1(M,\varphi)\big]_{\frac{1}{p}} \tag{3.4}$$

be the resulting interpolation space provided by the complex interpolation method [3, Chapter 4]. Regard $C(p,\theta)$ as a subspace of $L^1(M,\varphi)$ in the natural way. Then Kosaki's theorem [17, Theorem 9.1] (see also [10, Theorem 9.36]) asserts that $C(p,\theta)$ is equal to $D^{\frac{1-\theta}{p'}} L^p(M,\varphi) D^{\frac{\theta}{p'}}$ and that

$$\big\|D^{\frac{1-\theta}{p'}} y D^{\frac{\theta}{p'}}\big\|_{C(p,\theta)} = \|y\|_{L^p(M,\varphi)}, \qquad y \in L^p(M,\varphi). \tag{3.5}$$

Here, p' is the conjugate index of p so that $D^{\frac{1-\theta}{p'}} y D^{\frac{\theta}{p'}}$ belongs to $L^1(M,\varphi)$ provided that y belongs to $L^p(M,\varphi)$.

Likewise, let $j_\theta \colon M \to \mathcal{L}^1(M,\tau)$ be defined by $j_\theta(x) = \gamma^{1-\theta} x \gamma^\theta$ for all $x \in M$. Consider $(j_\theta(M), \mathcal{L}^1(M,\tau))$ as an interpolation couple, the norm on $j_\theta(M)$ being given by the norm on M, and set

$$c(p,\theta) = [j_\theta(M), \mathcal{L}^1(M,\tau)]_{\frac{1}{p}}, \tag{3.6}$$

regarded as a subspace of $\mathcal{L}^1(M,\tau)$. Then arguing as in the proof of [17, Theorem 9.1], one obtains that $c(p,\theta)$ is equal to $\gamma^{\frac{1-\theta}{p'}} \mathcal{L}^p(M,\tau) \gamma^{\frac{\theta}{p'}}$ and that

$$\big\|\gamma^{\frac{1-\theta}{p'}} y \gamma^{\frac{\theta}{p'}}\big\|_{c(p,\theta)} = \|y\|_{\mathcal{L}^p(M,\tau)}, \qquad y \in \mathcal{L}^p(M,\tau). \tag{3.7}$$

Proof of Lemma 3.1. We fix some $\theta \in [0,1]$. We start with the case $p = 1$. Let $x \in M$. For any $x' \in M$, we have $\tau(\gamma x x') = \mathrm{Tr}(D x x')$ and hence $|\tau(\gamma x x')| = |\mathrm{Tr}(D x x')|$, by equations (2.5) and (3.2). Taking the supremum over all $x' \in M$ with $\|x'\|_M \le 1$, it therefore follows from equations (2.4) and (3.1) that

$$\big\|\gamma x\big\|_{\mathcal{L}^1(M,\tau)} = \big\|D x\big\|_{L^1(M,\varphi)}, \qquad x \in M. \tag{3.8}$$

Now, assume that $x \in M_a$ (the space of analytic elements of M). According to equation (2.7), we have $D\sigma^\varphi_{i\theta}(x) = D^{1-\theta} x D^\theta$. Likewise, $\sigma^\varphi_t(x) = \gamma^{it} x \gamma^{-it}$ for all $t \in \mathbb{R}$, by [20, Theorem VIII.2.11], hence $\sigma^\varphi_{i\theta}(x) = \gamma^{-\theta} x \gamma^\theta$. Hence, we have $\gamma\sigma^\varphi_{i\theta}(x) = \gamma^{1-\theta} x \gamma^\theta$. Applying equation (3.8) with $\sigma^\varphi_{i\theta}(x)$ in place of x, we deduce that

$$\big\|\gamma^{(1-\theta)} x \gamma^\theta\big\|_{\mathcal{L}^1(M,\tau)} = \big\|D^{(1-\theta)} x D^\theta\big\|_{L^1(M,\varphi)}. \tag{3.9}$$

Consider the standard representation $M \hookrightarrow B(L^2(M,\varphi))$, and consider an arbitrary $x \in M$. Assume that $\theta \ge \frac{1}{2}$. There exists a net $(x_i)_i$ in M_a such that $x_i \to x$ strongly. Then $x_i D^{\frac{1}{2}} \to x D^{\frac{1}{2}}$ in $L^2(M,\varphi)$. Applying Lemma 2.1 (Hölder's inequality), we deduce that $D^{1-\theta} x_i D^\theta = D^{1-\theta}(x_i D^{\frac{1}{2}}) D^{\theta-\frac{1}{2}}$ converges to $D^{1-\theta} x D^\theta$ in $L^1(M,\varphi)$. (This result can also be formally deduced from [12, Lemma 2.3].) Likewise, $\gamma^{1-\theta} x_i \gamma^\theta$ converges to $\gamma^{1-\theta} x \gamma^\theta$ in $\mathcal{L}^1(M,\tau)$. Consequently, equation (3.9) holds true for x. Changing x into x^*, we obtain this result as well if $\theta < \frac{1}{2}$. This proves the result when $p = 1$.

We further note that the proof that $\mathcal{A}_{1,\theta} = D^{(1-\theta)} M D^\theta$ is dense in $L^1(M,\varphi)$ shows as well that the space $\gamma^{1-\theta} M \gamma^\theta$ is dense in $\mathcal{L}^1(M,\tau)$. Thus, equation (3.9) provides an isometric isomorphism

$$\Phi \colon L^1(M,\varphi) \longrightarrow \mathcal{L}^1(M,\tau)$$

such that

$$\Phi\big(D^{1-\theta} x D^\theta\big) = \gamma^{1-\theta} x \gamma^\theta, \qquad x \in M.$$

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Now, let $p > 1$ and consider the interpolation spaces $C(p,\theta)$ and $c(p,\theta)$ defined by equations (3.4) and (3.6). Since $j_\theta = \Phi \circ J_\theta$, the mapping Φ restricts to an isometric isomorphism from $C(p,\theta)$ onto $c(p,\theta)$. Let $x \in M$. Applying equations (3.7) and (3.5), we deduce that

$$\begin{aligned}\big\|\gamma^{\frac{1-\theta}{p}} x \gamma^{\frac{\theta}{p}}\big\|_{\mathcal{L}^p(M,\tau)} &= \big\|\gamma^{1-\theta} x \gamma^{\theta}\big\|_{c(p,\theta)}\\ &= \big\|D^{1-\theta} x D^{\theta}\big\|_{C(p,\theta)}\\ &= \big\|D^{\frac{1-\theta}{p}} x D^{\frac{\theta}{p}}\big\|_{L^p(M,\varphi)},\end{aligned}$$

which proves the result. ◻

The following is a straightforward consequence of Lemma 3.1. Given any $T\colon M \to M$, it provides a concrete way to compute the norm of the operator $T_{p,\theta}$ associated with φ. Note that in this statement, this norm may be infinite.

Corollary 3.2. *Let $1 \le p < \infty$, let $\theta \in [0,1]$, and let $T\colon M \to M$ be any bounded map. Then*

$$\|T_{p,\theta}\| = \sup\Big\{\big\|\gamma^{\frac{1-\theta}{p}} T(x) \gamma^{\frac{\theta}{p}}\big\|_p \,:\, x \in M,\ \big\|\gamma^{\frac{1-\theta}{p}} x \gamma^{\frac{\theta}{p}}\big\|_p \le 1\Big\}.$$

Let $n \ge 1$ be an integer, and consider the special case when $M = M_n$, equipped with its usual trace tr. For any φ and $T\colon M_n \to M_n$ as above, $T_{p,\theta}$ is trivially bounded for all $1 \le p < \infty$ and θ since $L^p(M_n,\varphi)$ is finite-dimensional. However, we will see in Sections 5 and 6 that finding (lower) estimates of the norm of $T_{p,\theta}$ in this setting will be instrumental to devise counterexamples on infinite dimensional von Neumann algebras. This is why we give a version of the preceding corollary in this specific case.

For any $1 \le p < \infty$, let $S_n^p = \mathcal{L}^p(M_n, \mathrm{tr})$ denote the p-Schatten class over M_n.

Proposition 3.3. *Let $\Gamma \in M_n$ be a positive definite matrix such that $\mathrm{tr}(\Gamma) = 1$ and let φ be the faithful state on M_n associated with Γ, that is, $\varphi(X) = \mathrm{tr}(\Gamma X)$ for all $X \in M_n$. Let $T\colon M_n \to M_n$ be any linear map. For any $p \in [1,\infty)$ and $\theta \in [0,1]$, let $U_{p,\theta}\colon S_n^p \to S_n^p$ be defined by*

$$U_{p,\theta}(Y) = \Gamma^{\frac{1-\theta}{p}} T\big(\Gamma^{-\frac{1-\theta}{p}} Y \Gamma^{-\frac{\theta}{p}}\big)\Gamma^{\frac{\theta}{p}}, \qquad Y \in S_n^p. \tag{3.10}$$

Then

$$\big\|T_{p,\theta}\colon L^p(M_n,\varphi) \longrightarrow L^p(M_n,\varphi)\big\| = \big\|U_{p,\theta}\colon S_n^p \longrightarrow S_n^p\big\|.$$

4. Extension results

This section is devoted to two cases for which Question 2.2 has a positive answer. Let M be a von Neumann algebra equipped with a faithful normal state φ, and let $D \in L^1(M,\varphi)^+$ denote its density.

Theorem 4.1. *Let $T\colon M \to M$ be a 2-positive map such that $\varphi \circ T \le \varphi$. For any $p \ge 2$ and for any $\theta \in [0,1]$, the mapping $T_{p,\theta}\colon \mathcal{A}_{p,\theta} \to \mathcal{A}_{p,\theta}$ defined by equation (1.2) extends to a bounded map $L^p(M,\varphi) \to L^p(M,\varphi)$.*

Proof. Consider a 2-positive map $T\colon M \to M$ such that $\varphi \circ T \le \varphi$. We start with the case $p = 2$. For any $x \in M$, we have

$$T(x)^* T(x) \le \|T\| T(x^* x),$$

by the Kadison–Schwarz inequality [5]. By equation (2.6), we have

$$\|T(x) D^{\frac{1}{2}}\|_2^2 = \varphi\big(T(x)^* T(x)\big) \le \|T\| \varphi\big(T(x^* x)\big) \le \|T\| \varphi(x^* x) = \|T\| \|x D^{\frac{1}{2}}\|_2^2.$$

This shows that $T_{2,1}$ is bounded. The proof that $T_{2,0}$ is bounded is similar.

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Now, let $\theta \in (0,1)$ and let us show that $T_{2,\theta}$ is bounded. Consider the open strip

$$\mathcal{S} = \{z \in \mathbb{C} \,:\, 0 < \mathrm{Re}(z) < 1\}.$$

Let $x, a \in M_a$, and define $F \colon \overline{\mathcal{S}} \to \mathbb{C}$ by

$$F(z) = \mathrm{Tr}\Big(T\Big(\sigma^{\varphi}_{\frac{i}{2}(1-z)}(x)\Big)D^{\frac12}\sigma^{\varphi}_{-\frac{iz}{2}}(a)D^{\frac12}\Big).$$

This is a well-defined function which is actually the restriction to $\overline{\mathcal{S}}$ of an entire function. For all $t \in \mathbb{R}$, we have

$$\begin{aligned} F(it) &= \mathrm{Tr}\Big(D^{\frac12}T\Big(\sigma^{\varphi}_{\frac{i}{2}}(\sigma^{\varphi}_{\frac{t}{2}}(x))\Big)D^{\frac12}\sigma^{\varphi}_{\frac{t}{2}}(a)\Big) \\ &= \mathrm{Tr}\Big(D^{\frac12}T\Big(D^{-\frac12}\sigma^{\varphi}_{\frac{t}{2}}(x)D^{\frac12}\Big)D^{\frac12}\sigma^{\varphi}_{\frac{t}{2}}(a)\Big) \\ &= \mathrm{Tr}\Big(T_{2,0}\Big(\sigma^{\varphi}_{\frac{t}{2}}(x)D^{\frac12}\Big)D^{\frac12}\sigma^{\varphi}_{\frac{t}{2}}(a)\Big), \end{aligned}$$

by equation (2.7). Hence, by equation (2.2),

$$\begin{aligned} |F(it)| &\le \Big\|T_{2,0}\Big(\sigma^{\varphi}_{\frac{t}{2}}(x)D^{\frac12}\Big)\Big\|_2\Big\|D^{\frac12}\sigma^{\varphi}_{\frac{t}{2}}(a)\Big\|_2 \\ &\le \big\|T_{2,0}\big\|\big\|D^{\frac{it}{2}}(xD^{\frac12})D^{-\frac{it}{2}}\big\|_2\big\|D^{\frac{it}{2}}(D^{\frac12}a)D^{-\frac{it}{2}}\big\|_2 \\ &= \big\|T_{2,0}\big\|\big\|xD^{\frac12}\big\|_2\big\|D^{\frac12}a\big\|_2. \end{aligned}$$

Likewise,

$$F(1+it) = \mathrm{Tr}\Big(T_{2,1}\Big(\sigma^{\varphi}_{\frac{t}{2}}(x)D^{\frac12}\Big)D^{\frac12}\sigma^{\varphi}_{\frac{t}{2}}(a)\Big),$$

hence

$$|F(1+it)| \le \big\|T_{2,1}\big\|\big\|xD^{\frac12}\big\|_2\big\|D^{\frac12}a\big\|_2.$$

By the three lines lemma, we deduce that

$$|F(\theta)| \le \big\|T_{2,0}\big\|^{1-\theta}\big\|T_{2,1}\big\|^{\theta}\big\|xD^{\frac12}\big\|_2\big\|D^{\frac12}a\big\|_2.$$

To calculate $F(\theta)$, we apply equation (2.7) again and we obtain

$$\begin{aligned} F(\theta) &= \mathrm{Tr}\Big(T\big(D^{-\frac{1-\theta}{2}}xD^{\frac{1-\theta}{2}}\big)D^{\frac12}D^{\frac{\theta}{2}}aD^{-\frac{\theta}{2}}D^{\frac12}\Big) \\ &= \mathrm{Tr}\Big(D^{\frac{1-\theta}{2}}T\big(D^{-\frac{1-\theta}{2}}xD^{\frac12}D^{-\frac{\theta}{2}}\big)D^{\frac{\theta}{2}}D^{\frac12}a\Big) \\ &= \mathrm{Tr}\Big(T_{2,\theta}\big(xD^{\frac12}\big)D^{\frac12}a\Big). \end{aligned}$$

Thus,

$$\Big|\mathrm{Tr}\Big(T_{2,\theta}\big(xD^{\frac12}\big)D^{\frac12}a\Big)\Big| \le \big\|T_{2,0}\big\|^{1-\theta}\big\|T_{2,1}\big\|^{\theta}\big\|xD^{\frac12}\big\|_2\big\|D^{\frac12}a\big\|_2.$$

Since $M_aD^{\frac12}$ and $D^{\frac12}M_a$ are both dense in $L^2(M,\varphi)$, this estimate shows that $T_{2,\theta}$ is bounded, with $\|T_{2,\theta}\| \le \big\|T_{2,0}\big\|^{1-\theta}\big\|T_{2,1}\big\|^{\theta}$.

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We now let $p \in (2, \infty)$. The proof in this case is a variant of the proof of [9, Theorem 5.1]. We use Kosaki's theorem which is presented after Lemma 3.1; see equations (3.4) and (3.5). Let $\theta \in [0, 1]$. Let $\mathfrak{J}_\theta\colon M \to L^2(M, \varphi)$ be defined by $\mathfrak{J}_\theta(x) = D^{\frac{1-\theta}{2}} x D^{\frac{\theta}{2}}$ for all $x \in M$. Equip $\mathfrak{J}_\theta(M)$ with

$$\big\|D^{\frac{1-\theta}{2}} x D^{\frac{\theta}{2}}\big\|_{\mathfrak{J}_\theta(M)} = \|x\|_M, \qquad x \in M. \tag{4.1}$$

Consider $(\mathfrak{J}_\theta(M), L^2(M, \varphi))$ as an interpolation couple. In analogy with equation (3.4), we set

$$E(p, \theta) = \big[\mathfrak{J}_\theta(M), L^2(M, \varphi)\big]_{\frac{2}{p}},$$

subspace of $L^2(M, \varphi)$ given by the complex interpolation method. Let $q \in (2, \infty)$ such that

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{2}.$$

We introduce one more mapping $U_\theta\colon L^2(M, \varphi) \to L^1(M, \varphi)$ defined by

$$U_\theta(\zeta) = D^{\frac{1-\theta}{2}} \zeta D^{\frac{\theta}{2}}, \qquad \zeta \in L^2(M, \varphi).$$

By equation (3.5), U_θ is an isometric isomorphism from $L^2(M, \varphi)$ onto $C(2, \theta)$. Since U_θ restricts to an isometric isomorphism from $\mathfrak{J}_\theta(M)$ onto $J_\theta(M)$, by equations (3.3) and (4.1), it induces an isometric isomorphism from $E(p, \theta)$ onto $\big[J_\theta(M), C(2, \theta)\big]_{\frac{2}{p}}$. By equation (3.4) and the reiteration theorem for complex interpolation (see [3, Theorem 4.6.1]), the latter is equal to $C(p, \theta)$. Hence, U_θ actually induces an isometric isomorphism

$$E(p, \theta) \overset{U_\theta}{\simeq} C(p, \theta). \tag{4.2}$$

Since $\frac{1}{p'} = \frac{1}{2} + \frac{1}{q}$, we have

$$U_\theta\big(D^{\frac{1-\theta}{q}} y D^{\frac{\theta}{q}}\big) = D^{\frac{1-\theta}{p'}} y D^{\frac{\theta}{p'}}$$

for all $y \in L^p(M, \varphi)$. Applying equations (3.5) and (4.2), we deduce that

$$E(p, \theta) = D^{\frac{1-\theta}{q}} L^p(M, \varphi) D^{\frac{\theta}{q}},$$

with

$$\big\|D^{\frac{1-\theta}{q}} y D^{\frac{\theta}{q}}\big\|_{E(p,\theta)} = \|y\|_{L^p(M,\varphi)}, \qquad y \in L^p(M, \varphi). \tag{4.3}$$

Now, let

$$S = T_{2,\theta}\colon L^2(M, \varphi) \longrightarrow L^2(M, \varphi)$$

be given by the first part of the proof (boundedness of $T_{2,\theta}$). By equation (4.1), S is bounded on $\mathfrak{J}_\theta(M)$. Hence, by the interpolation theorem, S is bounded on $E(p, \theta)$.

Using equation (4.3), we deduce that for all $x \in M$,

$$\begin{aligned}
\big\|D^{\frac{1-\theta}{p}} T(x) D^{\frac{\theta}{p}}\big\|_{L^p(M,\varphi)} &= \big\|D^{\frac{1-\theta}{2}} T(x) D^{\frac{\theta}{2}}\big\|_{E(p,\theta)} \\
&\leq \big\|S\colon E(p, \theta) \to E(p, \theta)\big\| \big\|D^{\frac{1-\theta}{2}} x D^{\frac{\theta}{2}}\big\|_{E(p,\theta)} \\
&= \big\|S\colon E(p, \theta) \to E(p, \theta)\big\| \big\|D^{\frac{1-\theta}{p}} x D^{\frac{\theta}{p}}\big\|_{L^p(M,\varphi)}.
\end{aligned}$$

This proves that $T_{p,\theta}$ is bounded and completes the proof. □

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Remark 4.2. Let $T\colon M \to M$ be a 2-positive map such that $\varphi \circ T \leq C_1 T$ for some $C_1 \geq 0$, and let $C_\infty = \|T\|$. It follows from the above proof and an obvious scaling that for any $p \geq 2$ and any $\theta \in [0,1]$, we have

$$\left\|T_{p,\theta}\colon L^p(M,\varphi) \longrightarrow L^p(M,\varphi)\right\| \leq C_\infty^{1-\frac{1}{p}} C_1^{\frac{1}{p}}.$$

Theorem 4.3. *Let $T\colon M \to M$ be a 2-positive map such that $\varphi \circ T \leq \varphi$, and let $1 \leq p \leq 2$. If*

$$1 - \frac{p}{2} \leq \theta \leq \frac{p}{2}, \tag{4.4}$$

then $T_{p,\theta}\colon \mathcal{A}_{p,\theta} \to \mathcal{A}_{p,\theta}$ extends to a bounded map $L^p(M,\varphi) \to L^p(M,\varphi)$.

Proof. We will use Theorem 4.1 on $L^2(M,\varphi)$, as well as the fact that $T_{1,\frac{1}{2}}$ is bounded; see [9, Lemma 5.3] or Remark 2.3. Let $p \in (1,2)$, let θ satisfying equation (4.4), and let

$$\eta = \frac{\theta - \left(1 - \frac{p}{2}\right)}{p-1}.$$

Then $\eta \in [0,1]$. This interpolation number is chosen in such a way that

$$\frac{\eta}{p'} + \frac{1-\theta}{p} = \frac{\theta}{p} + \frac{1-\eta}{p'} = \frac{1}{2}, \tag{4.5}$$

where p' is the conjugate number of p.

We set

$$S = T_{1,\frac{1}{2}}\colon L^1(M,\varphi) \longrightarrow L^1(M,\varphi).$$

Let $V\colon L^2(M,\varphi) \to L^1(M,\varphi)$ defined by $V(y) = D^{\frac{\eta}{2}} y D^{\frac{1-\eta}{2}}$ for all $y \in L^2(M,\varphi)$. According to equation (3.5), V is an isometric isomorphism from $L^2(M,\varphi)$ onto $C(2,1-\eta)$. Hence, for all $x \in M$, we have

$$\begin{aligned}
\left\|S(D^{\frac{1}{2}} x D^{\frac{1}{2}})\right\|_{C(2,1-\eta)} &= \left\|D^{\frac{\eta}{2}} D^{\frac{1-\eta}{2}} T(x) D^{\frac{\eta}{2}} D^{\frac{1-\eta}{2}}\right\|_{C(2,1-\eta)} \\
&= \left\|D^{\frac{1-\eta}{2}} T(x) D^{\frac{\eta}{2}}\right\|_{L^2(M,\varphi)} \\
&\leq \left\|T_{2,\eta}\right\| \left\|D^{\frac{1-\eta}{2}} x D^{\frac{\eta}{2}}\right\|_{L^2(M,\varphi)} \\
&= \left\|T_{2,\eta}\right\| \left\|D^{\frac{1}{2}} x D^{\frac{1}{2}}\right\|_{C(2,1-\eta)}.
\end{aligned}$$

Here, the boundedness of $T_{2,\eta}$ is provided by Theorem 4.1. This proves that S is bounded on $C(2,1-\eta)$.

By equation (3.4) and the reiteration theorem, we have

$$C(p,1-\eta) = \left[C(2,1-\eta), L^1(M,\varphi)\right]_{\frac{2}{p}-1}.$$

Therefore, S is bounded on $C(p,1-\eta)$. Using equation (3.5) again, as well as equation (4.5), we deduce that for any $x \in M$,

$$\begin{aligned}\big\|D^{\frac{1-\theta}{p}}T(x)D^{\frac{\theta}{p}}\big\|_{L^p(M,\varphi)} &= \big\|D^{\frac{\eta}{p'}}D^{\frac{1-\theta}{p}}T(x)D^{\frac{\theta}{p}}D^{\frac{1-\eta}{p'}}\big\|_{C(p,1-\eta)}\\ &= \big\|D^{\frac{1}{2}}T(x)D^{\frac{1}{2}}\big\|_{C(p,1-\eta)}\\ &\leq \big\|S\colon C(p,1-\eta)\to C(p,1-\eta)\big\|\big\|D^{\frac{1}{2}}xD^{\frac{1}{2}}\big\|_{C(p,1-\eta)}\\ &= \big\|S\colon C(p,1-\eta)\to C(p,1-\eta)\big\|\big\|D^{\frac{1-\theta}{p}}xD^{\frac{\theta}{p}}\big\|_{L^p(M,\varphi)}.\end{aligned}$$

This shows that $T_{p,\theta}$ is bounded. □

5. The use of infinite tensor products

In this section, we show how to reduce the problem of constructing a unital completely positive map $T\colon (M,\varphi)\longrightarrow (M,\varphi)$ such that $\varphi\circ T=\varphi$ and $T_{p,\theta}$ is unbounded, for a certain pair (p,θ), to a finite-dimensional question. In the sequel, by a matrix algebra A, we mean an algebra $A=M_n$ for some $n\geq 1$.

Lemma 5.1. *Let A_1, A_2 be two matrix algebras, and for $i=1,2$, consider a faithful state φ_i on A_i. Let $B=A_1\otimes_{\min} A_2$ and consider the faithful state $\psi=\varphi_1\otimes\varphi_2$ on B. Let $T_i\colon A_i\longrightarrow A_i$ be a linear map, for $i=1,2$, and consider $T=T_1\otimes T_2\colon B\longrightarrow B$. Then for any $1\leq p<\infty$ and any $\theta\in[0,1]$, we have*

$$\begin{aligned}&\big\|T_{p,\theta}\colon L^p(B,\psi)\longrightarrow L^p(B,\psi)\big\|\geq\\ &\qquad\big\|\{T_1\}_{p,\theta}\colon L^p(A_1,\varphi_1)\longrightarrow L^p(A_1,\varphi_1)\big\|\big\|\{T_2\}_{p,\theta}\colon L^p(A_2,\varphi_2)\longrightarrow L^p(A_2,\varphi_2)\big\|.\end{aligned}$$

Proof. Let $n_1,n_2\geq 1$ such that $A_1=M_{n_1}$ and $A_2=M_{n_2}$ and let $n=n_1n_2$. For $i=1,2$, let $\Gamma_i\in M_{n_i}$ such that $\varphi_i(X_i)=\mathrm{tr}(\Gamma_iX_i)$ for all $X_i\in M_{n_i}$. As in Proposition 3.3, consider the mapping $\{U_i\}_{p,\theta}\colon S^p_{n_i}\longrightarrow S^p_{n_i}$ defined by $\{U_i\}_{p,\theta}(Y_i)=\Gamma_i^{\frac{1-\theta}{p}}T_i\big(\Gamma_i^{-\frac{1-\theta}{p}}Y_i\Gamma_i^{-\frac{\theta}{p}}\big)\Gamma_i^{\frac{\theta}{p}}$ for all $Y_i\in S^p_{n_i}$. Using the standard identification

$$B=M_{n_1}\otimes_{\min}M_{n_2}\simeq M_n, \tag{5.1}$$

we observe that $\psi(X)=\mathrm{tr}\big((\Gamma_1\otimes\Gamma_2)X\big)$ for all $X\in M_n$. Hence, using the identification $S^p_n=S^p_{n_1}\otimes S^p_{n_2}$ inherited from equation (5.1), we obtain the the mapping $U_{p,\theta}$ defined by equation (3.10) is actually given by

$$U_{p,\theta}=\{U_1\}_{p,\theta}\otimes\{U_2\}_{p,\theta}.$$

For any $Y_1\in S^p_{n_1}$ and $Y_2\in S^p_{n_2}$, we have $\|Y_1\otimes Y_2\|_p=\|Y_1\|_p\|Y_2\|_p$. Hence, we deduce

$$\begin{aligned}\|\{U_1\}_{p,\theta}(Y_1)\|\|\{U_2\}_{p,\theta}(Y_2)\| &= \|\{U_1\}_{p,\theta}(Y_1)\otimes\{U_2\}_{p,\theta}(Y_2)\|\\ &= \|U_{p,\theta}(Y_1\otimes Y_2)\|\\ &\leq \|U_{p,\theta}\|\|Y_1\|_p\|Y_2\|_p.\end{aligned}$$

This implies that $\|\{U_1\}_{p,\theta}\|\|\{U_2\}_{p,\theta}\|\leq\|U_{p,\theta}\|$ Applying Proposition 3.3, we obtain the requested inequality. □

Throughout the rest of this section, we let $(A_k)_{k\geq 1}$ be a sequence of matrix algebras. For any $k\geq 1$, let φ_k be a faithful state on A_k. Let

$$(M,\varphi)=\overline{\otimes}_{k\geq 1}(A_k,\varphi_k)$$

be the infinite tensor product associated with the (A_k,φ_k). We refer to [21, Section XIV.1] for the construction and the properties of this tensor product. We merely recall that if we regard $(A_1\otimes\cdots\otimes A_n)_{n\geq 1}$ as an increasing sequence of (finite-dimensional) algebras in the natural way, then

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$$\mathcal{B} := \bigcup_{n\geq 1} A_1 \otimes \cdots \otimes A_n \tag{5.2}$$

is w^*-dense in M. Further, φ is a normal faithful state on M such that

$$\varphi_1 \otimes \cdots \otimes \varphi_n = \varphi_{|A_1\otimes\cdots\otimes A_n},$$

for all $n \geq 1$.

Proposition 5.2. *Let $1 \leq p < \infty$ and $\theta \in [0, 1]$. For any $k \geq 1$, let $T_k : A_k \to A_k$ be a unital completely positive map such that $\varphi_k \circ T_k = \varphi_k$. Assume that*

$$\prod_{k=1}^{n} \left\| \{T_k\}_{p,\theta} : L^p(A_k, \varphi_k) \to L^p(A_k, \varphi_k) \right\| \longrightarrow \infty \qquad \text{when } n \to \infty.$$

Then there exists a unital completely positive map $T : M \to M$ such that $\varphi \circ T = \varphi$ and $T_{p,\theta}$ is unbounded.

Proof. For any $n \geq 1$, we introduce $B_n = A_1 \otimes_{\min} \cdots \otimes_{\min} A_n$ and the faithful state

$$\psi_n = \varphi_1 \otimes \cdots \otimes \varphi_n$$

on B_n. According to [21, Proposition XIV.1.11], the modular automorphism group of φ preserves B_n. Consequently (see Remark 2.4), there exists a unique normal conditional expectation $E_n : M \to B_n$ such that $\varphi = \psi_n \circ E_n$, and the preadjoint of E_n yields an isometric embedding

$$L^1(B_n, \psi_n) \hookrightarrow L^1(M, \varphi).$$

Likewise, let $F_n : B_{n+1} \to B_n$ be the conditional expectation defined by

$$F_n(a_1 \otimes \cdots \otimes a_n \otimes a_{n+1}) = \varphi_{n+1}(a_{n+1})\, a_1 \otimes \cdots \otimes a_n, \tag{5.3}$$

for all $a_1 \in A_1, \ldots, a_n \in A_n, a_{n+1} \in A_{n+1}$. Then the preadjoint of F_n yields an isometric embedding

$$J_n : L^1(B_n, \psi_n) \hookrightarrow L^1(B_{n+1}, \psi_{n+1}).$$

We can therefore consider $\left(L^1(B_n, \psi_n)\right)_{n\geq 1}$ as an increasing sequence of subspaces of $L^1(M, \varphi)$. We introduce

$$\mathcal{L} := \bigcup_{n\geq 1} L^1(B_n, \psi_n) \subset L^1(M, \varphi).$$

Let $D \in L^1(M, \varphi)$ be the density of φ. It follows from Remark 2.4 that

$$\mathcal{L} = D^{\frac{1}{2}} \mathcal{B} D^{\frac{1}{2}},$$

where $\mathcal{B}$ is defined by equation (5.2). Since $\mathcal{B}$ is w^*-dense, it is dense in M for the strong operator topology given by the standard representation $M \hookrightarrow B(L^2(M, \varphi))$. Hence, by [12, Lemma 2.2], $\mathcal{B}D^{\frac{1}{2}}$ is dense in $L^2(M, \varphi)$. This implies that $\mathcal{L}$ is dense in $L^1(M, \varphi)$.

For any $n \geq 1$, let

$$V(n) := T_1 \otimes \cdots \otimes T_n : B_n \longrightarrow B_n.$$

This is a unital completely positive map. Hence, its norm is equal to 1. Let

$$S_n = V(n)_* : L^1(B_n, \psi_n) \longrightarrow L^1(B_n, \psi_n)$$

be the preadjoint of $V(n)$. Then $\|S_n\| = 1$. We observe that

$$J_n \circ S_n = S_{n+1} \circ J_n. \tag{5.4}$$

Indeed by duality, this amounts to show that $V(n) \circ F_n = F_n \circ V(n+1)$, where F_n is given by equation (5.3). The latter is true because $\varphi_{n+1} \circ T_{n+1} = \varphi_{n+1}$.

Thanks to equation (5.4), one may define

$$\mathcal{S} \colon \mathcal{L} \longrightarrow \mathcal{L}$$

by letting $\mathcal{S}(\eta) = S_n(\eta)$ if $\eta \in L^1(B_n, \psi_n)$. Then $\mathcal{S}$ is bounded, with $\|\mathcal{S}\| = 1$. Owing to the density of $\mathcal{L}$, there exists a unique bounded $S \colon L^1(M, \varphi) \to L^1(M, \varphi)$ extending $\mathcal{S}$. Using the duality (2.4), we set

$$T = S^* \colon M \longrightarrow M.$$

By construction, T is a contraction. Furthermore, for each $n \geq 1$, $S_n^* = V(n)$ is a unital completely positive map and $\psi_n \circ S_n^* = \psi_n$. We deduce that T is unital and completely positive and that

$$\varphi \circ T = \varphi.$$

Let $1 \leq p < \infty$, and let $\theta \in [0, 1]$. Let us use the isometric embedding

$$L^p(B_n, \psi_n) \hookrightarrow L^p(M, \varphi) \tag{5.5}$$

as explained in Remark 2.4. If D_n denotes the density of ψ_n, then it follows from [9, Proposition 5.5] that the embedding (5.5) maps $D_n^{\frac{1-\theta}{p}} x D_n^{\frac{\theta}{p}}$ to $D^{\frac{1-\theta}{p}} x D^{\frac{\theta}{p}}$ for all $x \in B_n$. Then the restriction of $T_{p,\theta} \colon \mathcal{A}_{p,\theta} \to L^p(M, \varphi)$ coincides with

$$V(n)_{p,\theta} \colon L^p(B_n, \psi_n) \longrightarrow L^p(B_n, \psi_n).$$

Finally we observe that by a simple iteration of Lemma 5.1, we have

$$\|V(n)_{p,\theta}\| \geq \prod_{k=1}^{n} \big\| \{T_k\}_{p,\theta} \colon L^p(A_k, \varphi_k) \to L^p(A_k, \varphi_k) \big\|.$$

The assumption that this product of norms tends to ∞ therefore implies that the operator $T_{p,\theta}$ is unbounded. □

6. Nonextension results

The aim of this section is to show the following.

Theorem 6.1. *Let $1 \leq p < 2$. If either*

$$0 \leq \theta < \frac{1}{2}\big(1 - \sqrt{p-1}\big) \qquad \text{or} \qquad \frac{1}{2}\big(1 + \sqrt{p-1}\big) < \theta \leq 1, \tag{6.1}$$

then there exist a von Neumann algebra M equipped with a normal faithful state φ, as well as a unital completely positive map $T \colon M \to M$ such that $\varphi \circ T = \varphi$ and the mapping $T_{p,\theta} \colon \mathcal{A}_{p,\theta} \to \mathcal{A}_{p,\theta}$ defined by equation (1.2) is unbounded.

This result will be proved at the end of this section, as a simple combination of Proposition 5.2 and the following key result. Recall that M_2 denotes the space of 2×2 matrices.

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Proposition 6.2. *Let $1 \leq p < 2$, and let $\theta \in [0,1]$ be satisfying equation (6.1). Then there exist a unital completely positive map $T\colon M_2 \to M_2$ and a faithful state φ on M_2 such that $\varphi \circ T = \varphi$ and $\|T_{p,\theta}\| > 1$.*

Proof. Let $c \in (0,1)$, and consider

$$\Gamma = \begin{pmatrix} 1-c & 0 \\ 0 & c \end{pmatrix}.$$

This is a positive invertible matrix with trace equal to 1. We let φ denote its associated faithful state on M_2, that is, $\varphi(X) = \mathrm{tr}(\Gamma X) = (1-c)x_{11} + cx_{22}$, for all $X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$ in M_2.

Let $E_{i,j}$, $1 \leq i,j \leq 2$, denote the standard matrix units of M_2. Let $T\colon M_2 \to M_2$ be the linear map defined by

$$T(E_{11}) = (1-c)I_2, \quad T(E_{22}) = cI_2, \quad \text{and} \quad T(E_{21}) = T(E_{12}) = \big(c(1-c)\big)^{\frac{1}{2}}\big(E_{12} + E_{21}\big).$$

Let $A = \big[T(E_{ij})\big]_{1\leq i,j\leq 2} \in M_2(M_2)$. If we regard A as an element of M_4, we have

$$A = \begin{pmatrix} 1-c & 0 & 0 & (c(1-c))^{\frac{1}{2}} \\ 0 & 1-c & (c(1-c))^{\frac{1}{2}} & 0 \\ 0 & (c(1-c))^{\frac{1}{2}} & c & 0 \\ (c(1-c))^{\frac{1}{2}} & 0 & 0 & c \end{pmatrix}.$$

Clearly, A is unitarily equivalent to $B \otimes I_2$, with

$$B = \begin{pmatrix} 1-c & (c(1-c))^{\frac{1}{2}} \\ (c(1-c))^{\frac{1}{2}} & c \end{pmatrix}.$$

It is plain that B is positive. Consequently, A is positive. Hence, T is completely positive, by Choi's theorem (see, for example, [18, Theorem 3.14]). Furthermore, T is unital. We note that $\varphi(T(E_{11})) = \varphi(E_{11}) = 1-c$, $\varphi(T(E_{22})) = \varphi(E_{22}) = c$, $\varphi(T(E_{12})) = \varphi(E_{12}) = 0$ and $\varphi(T(E_{21})) = \varphi(E_{21}) = 0$. Thus,

$$\varphi \circ T = \varphi.$$

Our aim is now to estimate $\|T_{p,\theta}\|$, using Proposition 3.3. We let $U_{p,\theta}\colon S_2^p \to S_2^p$ be defined by equation (3.10). We shall focus on the action of $U_{p,\theta}$ on the antidiagonal part of S_2^p. First, we have

$$\Gamma^{-\frac{1-\theta}{p}} E_{12} \Gamma^{-\frac{\theta}{p}} = (1-c)^{-\frac{1-\theta}{p}} c^{-\frac{\theta}{p}} E_{12}.$$

Hence

$$\begin{aligned} T\big(\Gamma^{-\frac{1-\theta}{p}} E_{12} \Gamma^{-\frac{\theta}{p}}\big) &= (1-c)^{-\frac{1-\theta}{p}} c^{-\frac{\theta}{p}} T(E_{12}) \\ &= (1-c)^{-\frac{1-\theta}{p}} c^{-\frac{\theta}{p}} (c(1-c))^{\frac{1}{2}} \big(E_{12} + E_{21}\big). \end{aligned}$$

Hence,

$$\begin{aligned} U_{p,\theta}(E_{12}) &= (1-c)^{-\frac{1-\theta}{p}} c^{-\frac{\theta}{p}} (c(1-c))^{\frac{1}{2}} \Big(\Gamma^{\frac{1-\theta}{p}} E_{12} \Gamma^{\frac{\theta}{p}} + \Gamma^{\frac{1-\theta}{p}} E_{21} \Gamma^{\frac{\theta}{p}}\Big) \\ &= (1-c)^{-\frac{1-\theta}{p}} c^{-\frac{\theta}{p}} (c(1-c))^{\frac{1}{2}} \Big((1-c)^{\frac{1-\theta}{p}} c^{\frac{\theta}{p}} E_{12} + c^{\frac{1-\theta}{p}} (1-c)^{\frac{\theta}{p}} E_{21}\Big) \\ &= (c(1-c))^{\frac{1}{2}} \left(E_{12} + \left(\frac{1-c}{c}\right)^{\frac{2\theta-1}{p}} E_{21}\right). \end{aligned}$$

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Likewise, we have

$$U_{p,\theta}(E_{21}) = (c(1-c))^{\frac{1}{2}}\Big(\Big(\frac{c}{1-c}\Big)^{\frac{2\theta-1}{p}} E_{12} + E_{21}\Big).$$

Set

$$\delta = \Big(\frac{1-c}{c}\Big)^{\frac{2\theta-1}{p}}. \tag{6.2}$$

Consider

$$Y = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} \qquad \text{with} \quad |a|^p + |b|^p = 1$$

so that $\|Y\|_p = 1$. Then

$$\begin{aligned} U_{p,\theta}(Y) &= (c(1-c))^{\frac{1}{2}}\big(aE_{12} + a\delta E_{21} + b\delta^{-1}E_{12} + bE_{21}\big) \\ &= (c(1-c))^{\frac{1}{2}}\big((a + b\delta^{-1})E_{12} + (a\delta + b)E_{21}\big). \end{aligned}$$

Hence,

$$\|U_{p,\theta}(Y)\|_p^p = (c(1-c))^{\frac{p}{2}}\big((a + b\delta^{-1})^p + (a\delta + b)^p\big). \tag{6.3}$$

To prove Proposition 6.2, it therefore suffices to show that for any $1 \leq p < 2$ and $\theta \in [0,1]$ satisfying equation (6.1), there exist $a, b > 0$ and $c \in (0,1)$ such that

$$a^p + b^p = 1 \qquad \text{and} \qquad (c(1-c))^{\frac{p}{2}}\big((a + b\delta^{-1})^p + (a\delta + b)^p\big) > 1,$$

where δ is given by equation (6.2).

We first assume that $p > 1$. We let $q = \frac{p}{p-1}$ denote its conjugate exponent. Given $c \in (0,1)$ and δ as above, we define

$$a = \Big(\frac{\delta^q}{1+\delta^q}\Big)^{\frac{1}{p}} \qquad \text{and} \qquad b = \Big(\frac{1}{1+\delta^q}\Big)^{\frac{1}{p}}. \tag{6.4}$$

They satisfy $a^p + b^p = 1$ as required. Note that these values of (a, b) are chosen in order to maximize the quantity $(c(1-c))^{\frac{p}{2}}\big((a + b\delta^{-1})^p + (a\delta + b)^p\big)$, according to the Lagrange multiplier method.

We set

$$c_t = \frac{1}{2} + t, \qquad -\frac{1}{2} < t < \frac{1}{2}.$$

Then we denote by δ_t, a_t, b_t the real numbers δ, a, b defined by equations (6.2) and (6.4) when $c = c_t$. Also, we set

$$\gamma_t = (c_t(1-c_t))^{\frac{p}{2}} \qquad \text{and} \qquad \mathfrak{m}_t = \gamma_t\big((a_t + b_t\delta_t^{-1})^p + (a_t\delta_t + b_t)^p\big).$$

It follows from above that it suffices to show that $\mathfrak{m}_t > 1$ for some $t \in \big(0, \frac{1}{2}\big)$. We will prove this property by writing the second-order Taylor expansion of $\mathfrak{m}_t$.

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We have

$$(a_t + b_t\delta_t^{-1})^p + (a_t\delta_t + b_t)^p = (1+\delta_t^{-p})(a_t\delta_t + b_t)^p.$$

Moreover,

$$a_t\delta_t = \frac{\delta_t^{\frac{q}{p}+1}}{(1+\delta_t^q)^{\frac{1}{p}}} = \frac{\delta_t^q}{(1+\delta_t^q)^{\frac{1}{p}}}.$$

Hence,

$$(a_t + b_t\delta_t^{-1})^p + (a_t\delta_t + b_t)^p = (1+\delta_t^{-p})(1+\delta_t^q)^{p-1}.$$

Consequently,

$$\mathfrak{m}_t = \gamma_t(1+\delta_t^{-p})(\delta_t^q+1)^{p-1}.$$

In the sequel, we write

$$f_t \equiv g_t$$

when $f_t = g_t + o(t^2)$ when $t \longrightarrow 0$.

We note that $c_t(1-c_t) = \left(\frac{1}{2}+t\right)\left(\frac{1}{2}-t\right) = \frac{1}{4}\left(1-4t^2\right)$. We deduce that

$$\gamma_t \equiv \frac{1}{2^p}(1-2pt^2). \tag{6.5}$$

We set $\lambda = 2\theta - 1$ for convenience. Then we have

$$\begin{aligned}
\delta_t &= \left(\frac{1-2t}{1+2t}\right)^{\frac{\lambda}{p}} \\
&\equiv \left((1-2t)(1-2t+4t^2)\right)^{\frac{\lambda}{p}} \\
&\equiv (1-4t+8t^2)^{\frac{\lambda}{p}} \\
&\equiv 1 - \frac{4\lambda}{p}t + \frac{8\lambda}{p}t^2 + \frac{1}{2}\,\frac{\lambda}{p}\left(\frac{\lambda}{p}-1\right)(4t)^2 \\
&\equiv 1 - \frac{4\lambda}{p}t + \frac{8\lambda^2}{p^2}t^2.
\end{aligned}$$

This implies that

$$\begin{aligned}
\delta_t^q &\equiv 1 - \frac{4\lambda q}{p}t + \frac{8\lambda^2 q}{p^2}t^2 + \frac{1}{2}\,q(q-1)\left(\frac{4\lambda}{p}\right)^2 t^2 \\
&\equiv 1 - \frac{4\lambda q}{p}t + \frac{8\lambda^2 q^2}{p^2}t^2.
\end{aligned}$$

Likewise,

$$\delta_t^{-p} \equiv 1 + 4\lambda t + 8\lambda^2 t^2. \tag{6.6}$$

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Since $p-1=\frac{p}{q}$, we have

$$\begin{aligned}(1+\delta_t^q)^{p-1} &\equiv 2^{\frac{p}{q}}\Big(1-\frac{2\lambda q}{p}t+\frac{4\lambda^2q^2}{p^2}t^2\Big)^{\frac{p}{q}}\\ &\equiv 2^{\frac{p}{q}}\Big(1-2\lambda t+\frac{4\lambda^2 q}{p}t^2+\frac{1}{2}\,\frac{p}{q}\Big(\frac{p}{q}-1\Big)\Big(\frac{2\lambda q}{p}\Big)^2t^2\Big)\\ &\equiv 2^{\frac{p}{q}}\big(1-2\lambda t+2\lambda^2qt^2\big).\end{aligned}$$

Combining this expansion with equations (6.5) and (6.6), we deduce that

$$\begin{aligned}\mathfrak{m}_t &\equiv \frac{1}{2^p}(1-2pt^2)\cdot 2(1+2\lambda t+4\lambda^2t^2)\cdot 2^{\frac{p}{q}}(1-2\lambda t+2\lambda^2qt^2)\\ &\equiv (1-2pt^2)(1+2\lambda^2qt^2).\end{aligned}$$

Consequently,

$$\mathfrak{m}_t\equiv 1+\alpha t^2 \qquad \text{with} \qquad \alpha=2(\lambda^2q-p). \tag{6.7}$$

The second-order coefficient α can be written as

$$\begin{aligned}\alpha &= 2q\Big((2\theta-1)^2-\frac{p}{q}\Big)\\ &= 8q\Big(\theta^2-\theta+\frac{q-p}{4q}\Big)\\ &= 8q(\theta-\theta_0)(\theta-\theta_1),\end{aligned}$$

with

$$\theta_0=\frac{1}{2}\big(1-\sqrt{p-1}\big) \qquad \text{and} \qquad \theta_1=\frac{1}{2}\big(1+\sqrt{p-1}\big).$$

Now, assume equation (6.1). Then $\alpha>0$. Hence, equation (6.7) ensures the existence of $t>0$ such that $\mathfrak{m}_t>1$, which concludes the proof (in the case $p>1$).

We now consider the case $\underline{p=1.}$ We apply the same method as before, with

$$a=1 \qquad \text{and} \qquad b=0.$$

According to equation (6.3), it will suffice to show that whenever $\theta\neq\frac{1}{2}$, there exists $c\in(0,1)$ such that $(c(1-c))^{\frac{1}{2}}(1+\delta)>1$.

Again, we set $c_t=\frac{1}{2}+t$, for $-\frac{1}{2}<t<\frac{1}{2}$, we define δ_t accordingly, and we set

$$\mathfrak{m}_t=(c_t(1-c_t))^{\frac{1}{2}}(1+\delta_t).$$

It follows from the previous calculations that

$$(c_t(1-c_t))^{\frac{1}{2}}=\frac{1}{2}+o(t) \qquad \text{and} \qquad \delta_t=1-4(2\theta-1)t+o(t).$$

Consequently,

$$\mathfrak{m}_t=1-2(2\theta-1)t+o(t).$$

This order one expansion ensures that if $\theta\neq\frac{1}{2}$, then there exists $t\in\big(-\frac{1}{2},\frac{1}{2}\big)$ such that $\mathfrak{m}(t)>1$, which concludes the proof (in the case $p=1$). □

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Proof of Theorem 6.1. Let (p, θ) satisfying equation (6.1). Thanks to Proposition 6.2, let $T_0 \colon M_2 \to M_2$ and let φ_0 be a faithful state on M_2 such that $\varphi_0 \circ T_0 = \varphi_0$ and $\|\{T_0\}_{p,\theta}\| > 1$. We apply Proposition 5.2 with $(A_k, \varphi_k, T_k) = (M_2, \varphi_0, T_0)$ for all $k \geq 1$. In this case,

$$\prod_{k=1}^{n} \left\|\{T_k\}_{p,\theta}\right\| = \|\{T_0\}_{p,\theta}\|^n,$$

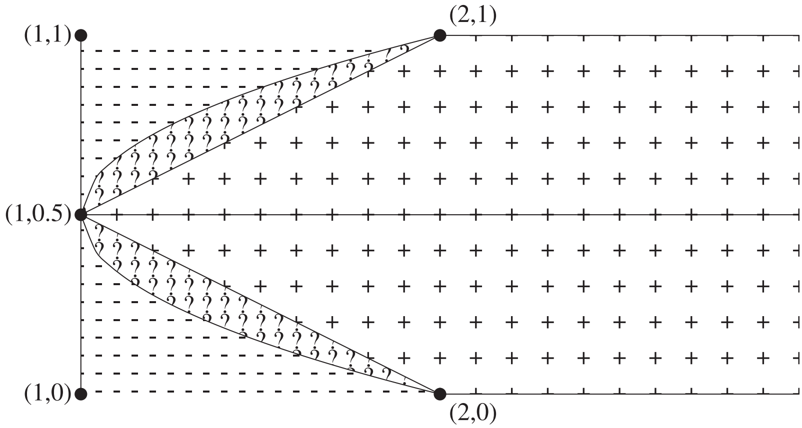
and the latter goes to ∞ when $n \to \infty$. Hence, $T_{p,\theta}$ is unbounded. □

Remark 6.3. With Theorem 4.1, Theorem 4.3 and Theorem 6.1, we have solved Question 2.2 in the following cases: (i) $p \geq 2$ and $\theta \in [0, 1]$; (ii) $1 \leq p < 2$ and $\theta \in \left[1 - p/2, p/2\right]$; (iii) $1 \leq p < 2$ and $\theta \in \left[0, 2^{-1}(1 - \sqrt{p-1})\right)$; (iv) $1 \leq p < 2$ and $\theta \in \left(2^{-1}(1 + \sqrt{p-1}), 1\right]$.

However, we do not know the answer to Question 2.2 when $1 \leq p < 2$ and

$$\theta \in \left[2^{-1}(1 - \sqrt{p-1}), 1 - p/2\right) \qquad \text{or} \qquad \theta \in \left(p/2, 2^{-1}(1 + \sqrt{p-1})\right].$$

Writing a (+) when Question 2.2 has a positive answer, a (−) when it has a negative answer and a (?) when we do not know the answer, we obtain the following diagram:



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Competing interest. The authors have no competing interest to declare.

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