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RIGIDITY IN ORDER-TYPES

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Abstract

A totally ordered set (and corresponding order-type) is said to be rigid if it is not similar to any proper initial segment of itself. The class of rigid order-types is closed under addition and multiplication, satisfies both cancellation laws from the left, and admits a partial ordering that is an extension of the ordering of the ordinals. Under this ordering, limits of increasing sequences of rigid order-types are well defined, rigid and satisfy the usual limit laws concerning addition and multiplication. A decomposition theorem is obtained, and is used to prove a characterization theorem on rigid order-types that are additively prime. Wherever possible, use of the Axiom of Choice is eschewed, and theorems whose proofs depend upon Choice are marked.

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Unless the contrary is either obvious or stated explicitly, the term "set" is throughout this paper taken to mean "ordered set", with the ordering generally being denoted by $<$. Sets and elements of sets are denoted by upper and lower case Latin letters respectively, order-types by lower case Greek letters, and—where necessary—classes of order-type by upper case Greek letters. The first transfinite ordinal is as usual denoted by ω, and natural numbers by k, m, n, p, q. We assume understanding of the concepts of initial (final) segment, and of interval.

An order-preserving map $f\colon A \to B$ having the property that $f''A$ $(=\{f(a) \in B; a \in A\})$ is an initial segment of B is said to be a monomorphism; an isomorphism is a surjective monomorphism. If $f\colon A \to B$ is an isomorphism then A and B are said to be similar ($A \simeq B$). The order-type of a set A is $o(A)$, and ordered unions and ordered products are denoted by $\dot\cup$ and $\dot\times$ respectively; we take the product $A \dot\times B$ of two sets A, B as being ordered antilexicographically; thus $(a,b) < (a',b')$ if either $b < b'$ or $b = b'$ and $a < a'$. The converse of a set A (order-type α) is $A^*(\alpha^*)$.

Whilst it has not proved practicable to do without the Axiom of Choice entirely, we have attempted to keep its use to a minimum. Our reason for this course is two-fold; firstly, a general interest in delineating the role of Choice in the classical theory of order-types; secondly, the class of order-types examined in this present

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paper is a generalization of classes of order-types studied in previous papers (Hickman, 1975; Hickman, submitted), and which are of intrinsic interest only when Choice is rejected.

DEFINITION 1. Let A be a nonempty set.

(1) A is said to be *rigid* if there is no proper initial segment B of A with $B \simeq A$. We say that A is fully rigid if every nonempty initial segment of A is rigid.

(2) A is said to be *regressive* if A is not rigid, and to be fully regressive if every nonempty initial segment of A is regressive.

(3) A is said to be *atomic* if A is rigid and every nonempty proper initial segment of A is regressive.

Naturally the above nomenclature is extended to order-types in the obvious manner. Our reason for excluding 0 from either rigidity or regressiveness is simply one of practical convenience.

Obviously every nonzero ordinal is rigid, in fact fully rigid, whilst η, the order-type of the rationals, is fully regressive. The order-type $\eta+\omega$ is rigid but not fully rigid, whereas $\omega+\eta$ is regressive but not fully regressive. The simplest nontrivial atomic order-type is probably $\eta+2$. Finally, in case anyone is tempted to conjecture that every fully rigid order-type is an ordinal, we present two examples to disillusion him. To obtain the first of these, we let N^- be the set of negative integers (ordered in the usual fashion); the required order-type is then $\sum\{\omega^{-n}; n \in N^-\}$, that is, $\ldots+\omega^{n+1}+\omega^n+\ldots+\omega^2+\omega$, where $n \geqslant 1$. For the second, we let Q be the set of rationals (again with the standard ordering), and let $f\colon Q \to \omega$ be an arbitrary but fixed bijection. Our order-type is then $\Sigma\{\omega^{f(q)}; q \in Q\}$.

It is immediate from the definition that a nonzero order-type α is rigid if and only if there is no $\beta \neq 0$ such that $\alpha+\beta=\alpha$. It was shown in Hickman (1977) that an order-type α is regressive if and only if $\alpha=\beta+\gamma\omega^*$ for some β, γ with $\gamma \neq 0$.

DEFINITION 2. A binary relation $\downarrow$ is defined on the class of all order-types by setting $\alpha \downarrow \beta$ if and only if $\beta = \alpha+\gamma$ for some $\gamma \neq 0$. The relation "$\alpha \downarrow \beta$ or $\alpha=\beta$" is written "$\alpha \stackrel{+}{=} \beta$". The order-type α is said to be a subtype of the order-type β if $\alpha \stackrel{+}{=} \beta$.

DEFINITION 3. $\Gamma=\{\alpha; \alpha \text{ is rigid}\}$.

It is routine to show that the restriction of $\downarrow$ to Γ defines a strict partial order on Γ, that is, $\downarrow$ is irreflexive and transitive on Γ. This is not true if we remove the restriction of $\downarrow$ to Γ; for example, we have $\eta \downarrow \eta+1 \downarrow \eta$, but not $\eta=\eta+1$.

The following result was obtained in Hickman (1977), but as its proof is short and the result itself plays a role of importance in our present work, we re-prove it here. After the above paper was written, the author discovered that this result constitutes one half of a theorem by Tarski (Theorem 1.27 of Tarski, 1956). Despite receiving arguments to the contrary, however, the author is not at all convinced that a direct translation of Tarski's proof into the present context would avoid the use of Choice, and hence he prefers to rely upon the proof set forth below.

THEOREM 1. *For any order-types* α, β, γ, *if* $\alpha = \beta + \alpha + \gamma$, *then* $\beta + \alpha = \alpha = \alpha + \gamma$.

PROOF: Let A be a representative set for α. The hypothesis says that $A = B \cup A^0 \cup C$ for some sets B, A^0, C with respective order-types β, α, γ. Let $f: A \simeq A^0$ be an isomorphism, and define $g: A \to A$ by $g(x) = \min\{x, f(x)\}$ for each $x \in A$. It is routine to show that g is order-preserving and that $g''A = B \cup A^0$. Hence $\alpha = \beta + \alpha$. But now we have $\alpha + \gamma = \beta + \alpha + \gamma = \alpha$.

We wish to show that the class Γ is closed under order-type addition and multiplication, and we achieve this via proofs that the elements of Γ satisfy both left-cancellation laws. With respect to addition, this has already been demonstrated in Hickman (1977), but once again for the sake of convenience we repeat the proof.

THEOREM 2. *Let* α *be any nonzero order-type. Then* $\alpha \in \Gamma$ *if and only if for any order-types* β, γ, *we have* $\alpha + \beta = \alpha + \gamma \Rightarrow \beta = \gamma$.

PROOF. If $\alpha \notin \Gamma$, then there is some $\beta \neq 0$ such that $\alpha + \beta = \alpha = \alpha + 0$. Now suppose that $\alpha + \beta = \alpha + \gamma$ for some β, γ with $\beta \neq \gamma$. Let A, B, C be pairwise disjoint representative sets for α, β, γ respectively, and let $f: A \cup B \simeq A \cup C$ be an isomorphism. As we cannot have $f''B = C$ (since $\beta \neq \gamma$), it must be the case that either $f(a) \in C$ for some $a \in A$ or $f(b) \in A$ for some $b \in B$. In the first instance A is a proper initial segment of $f''A$, and in the second instance $f''A$ is a proper initial segment of A, and so whichever of these two alternatives is the case, we have $\alpha \notin \Gamma$.

THEOREM 3. *For all* $\alpha, \beta \in \Gamma$, *we have* $\alpha + \beta \in \Gamma$.

PROOF. Take any order-types γ, δ, and suppose that $\alpha + \beta + \gamma = \alpha + \beta + \delta$. Since $\alpha \in \Gamma$, Theorem 2 tells us that $\beta + \gamma = \beta + \delta$. Repeating, we obtain $\gamma = \delta$, and one further appeal to Theorem 2 yields $\alpha + \beta \in \Gamma$.

We could show that rigid order-types are multiplicatively left-cancellable by observing that for any rigid order-type α and an arbitrary order-type β, we have $\alpha = \alpha\beta$ if and only if $\beta = 1$, then applying Theorem 3.10 of Morel (1959). However, this procedure would once again involve the Axiom of Choice, and so we prefer to give a direct proof.

THEOREM 4. *For any* $\alpha \in \Gamma$ *and order-types* β, γ, *if* $\alpha\beta = \alpha\gamma$, *then* $\beta = \gamma$.

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PROOF. Let A, B, C be representative sets for α, β, γ respectively, and let $f: A \dot{\times} B \simeq A \dot{\times} C$ be an isomorphism. Take any $b \in B$; we claim that one of the following is true:

(1) there is a unique $c \in C$ such that $f''(A \dot{\times} \{b\}) \subseteq A \dot{\times} \{c\}$;

(2) there are unique elements c_0, c_1 of C such that c_1 immediately succeeds c_0 and

$$f''(A \dot{\times} \{b\}) \cap A \dot{\times} \{c_0\} \neq \emptyset \neq f''(A \dot{\times} \{b\}) \cap A \dot{\times} \{c_1\}.$$

In case (1), if such a c exists, then it is clearly unique. Suppose that with regard to case (2) the existence of $c_0, c_1 \in C$ satisfying the conditions has been established. Then these two elements are uniquely determined. For if there is another such pair, (c_0', c_1'), then as c_1 (c_1') immediately succeeds c_0 (c_0') we may without loss of generality assume that $c_1 \leqslant c_0'$. This means, however, that

$$f''(A \dot{\times} \{b\}) = D \cup A \dot{\times} \{c_1\} \cup D'$$

for some $D, D' \neq \emptyset$; an application of Theorem 1 immediately tells us that $\alpha \notin \Gamma$, contrary to hypothesis. Therefore, in order to establish our claim, it suffices to show, on the assumption that case (1) does not apply, the existence of $c_0, c_1 \in C$ satisfying the conditions of case (2). Now since (1) does not hold, there certainly exist $c_0, c_1 \in C$ with $c_0 < c_1$ and $f''(A \dot{\times} \{b\}) \cap A \dot{\times} \{c_i\} \neq \emptyset$, $i = 0, 1$. Suppose that $c_0 < c < c_1$ for some $c \in C$. Then $f''(A \dot{\times} \{b\}) = D \cup A \dot{\times} \{c\} \cup D'$ for some $D, D' \neq \emptyset$, and we arrive at the same contradiction as before. Thus no such c exists, and our claim is established.

We have just shown that for each $b \in B$ there is a greatest $c \in C$ (with respect to the ordering on C) such that $f''(A \dot{\times} \{b\}) \cap A \dot{\times} \{c\} \neq \emptyset$, and we now define a map $g: B \to C$ by letting $g(b)$ be this c. It is clear from this definition that if we take $b_0, b_1 \in B$ with $b_0 < b_1$, then $g(b_0) \leqslant g(b_1)$; but it is not necessarily true that $g(b_0) < g(b_1)$. Nevertheless, the function g has the following property:

If $b_0, b_1 \in B$ are such that $b_0 < b_1$ and $g(b_0) = g(b_1)$, then

(1) there is no $b \in B$ with $b_0 < b < b_1$;

(2) there is an interval $I(b_0)$ of B such that

($\#$) (i) $o(I(b_0)) = \omega^*$,

(ii) b_0 is the greatest element of $I(b_0)$ (with respect to the ordering on B),

(iii) for any $d_0, d_1 \in I(b_0)$, if $d_0 < d_1$ then $g(d_0) < g(d_1)$.

To demonstrate this, we take $b_0, b_1 \in B$ and suppose that $b_0 < b_1$ and $g(b_0) = g(b_1)$. It follows at once from the definition of g that $f''(A \dot{\times} \{b_1\}) \subseteq A \dot{\times} \{g(b_1)\}$, and so from Theorem 1, the assumption that $g(b_0) = g(b_1)$, and the fact that A is rigid, we conclude that $f''(A \dot{\times} \{b_1\})$ is a (nonempty) proper final segment of $A \dot{\times} \{g(b_1)\}$. But now a second application of Theorem 1 tells us that we cannot have

$$f''(A \dot{\times} \{b_0\}) \subseteq A \dot{\times} \{g(b_1)\},$$

whence it follows that $g(b_1)$ $(= g(b_0))$ has an immediate predecessor $c_1 \in C$ with $f''(A \dot{\times} \{b_0\}) \cap A \dot{\times} \{c_1\} \neq \emptyset \neq f''(A \dot{\times} \{b_0\}) \cap A \dot{\times} \{g(b_1)\}$. If now we had $b \in B$ with

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$b_0 < b < b_1$, then $f''(A \dot{\times} \{b\})$ would be a nonempty, nonfinal segment of $A \dot{\times} \{g(b_1)\}$, which by Theorem 1 is impossible. Thus we have established (1) of ($\#$).

We have seen that $g(b_0)$ has an immediate predecessor $c_1 \in C$ and that $f''(A \dot{\times} \{b_0\}) \cap A \dot{\times} \{c_1\} \neq \varnothing$. But since $f''(A \dot{\times} \{b_0\}) \cap A \dot{\times} \{g(b_0)\} \neq \varnothing$, we know from Theorem 1 that we cannot have $A \dot{\times} \{c_1\} \subseteq f''(A \dot{\times} \{b_0\})$. Since f is an isomorphism, there must exist $d_1 \in B$ with $d_1 < b_0$ and $f''(A \dot{\times} \{d_1\}) \cap A \dot{\times} \{c_1\} \neq \varnothing$. The same argument as before shows that d_1 immediately precedes b_0 in B, and it is clear that $g(d_1) = c_1$.

A routine induction argument now tells us that there exist elements d_n of B and elements c_n of C such that for each $n \geqslant 1$, d_{n+1} immediately precedes d_n, c_{n+1} immediately precedes c_n, and $g(d_n) = c_n$. Setting $I(b_0) = \{b_0\} \cup \{d_n; n \geqslant 1\}$, we see that (2) of ($\#$) is satisfied.

We define a map $h: B \to C$ as follows. Take $b \in B$; there are two possibilities.

(I) We have $b \in I(b_0)$ for some $b_0 \in B$ such that b_0 has an immediate successor $b_1 \in B$ with $g(b_0) = g(b_1)$. Then we have seen that b has an immediate predecessor $b' \in B$, and we put $h(b) = g(b')$.

(II) We have $b \in I(b_0)$ for no such $b_0 \in B$. In this case we put $h(b) = g(b)$.

Obviously h is order-preserving. Thus if h is surjective, then we will have $B \simeq C$, the desired result. Assume for the moment that g is surjective, take $c \in C$, and let $b \in B$ be such that $g(b) = c$. Then either $h(b) = c$ or else b has an immediate successor $b^0 \in B$ and $h(b^0) = g(b) = c$.

It simply remains to show that g is surjective. Take $c \in C$: since f is an isomorphism, there must exist $b \in B$ such that $f''(A \dot{\times} \{b\}) \cap A \dot{\times} \{c\} \neq \varnothing$, and we have seen that there are at most two such b. Let b^0 be the least such b; then $g(b^0) = c$.

Unlike the additive case, the converse to Theorem 4 is not necessarily true. For it is fairly obvious that if σ is any order-type with $\sigma \neq 1$, then $\omega^* \sigma \neq \omega^*$. However, it is shown in Morel (1959) that an order-type ψ is multiplicatively left-cancellable if and only if $\psi\tau \neq \psi$ for every $\tau \neq 1$.

THEOREM 5. *For all* $\alpha, \beta \in \Gamma$ *we have* $\alpha\beta \in \Gamma$.

PROOF. Take $\alpha, \beta \in \Gamma$ and let A, B be representative sets for α, β respectively. Assume that $\alpha\beta \downarrow \alpha\beta$. Thus for some nonempty set C we have an isomorphism $f: A \dot{\times} B \simeq A \dot{\times} B \cup C$. By considering the initial segment $f^{-1}(A \dot{\times} B)$ of $A \dot{\times} B$, we see that we must have $\alpha\beta = \alpha\delta + \mu$ for some order-types δ, μ with $\delta \downarrow \beta$, $\mu \downarrow \alpha$. If $\mu = 0$, then we would have $\alpha\beta = \alpha\delta$, whence by Theorem 4 we would obtain the contradiction $\beta = \delta \downarrow \beta$. Thus we must have $\mu \neq 0$, and we now use this fact to show that B has a final segment of type ω^*, once again contradicting the hypothesis $\beta \in \Gamma$.

Since $\alpha\beta = \alpha\delta + \mu$, it follows that $A \dot{\times} B$ has a final segment M of type μ. But $0 \downarrow \mu \downarrow \alpha$, and so Theorem 1 tells us that B has a final element b_0, for otherwise we would have $M = D_0 \cup D_1 \cup D_2$ for some sets D_i with $D_1 \simeq A$ and $D_2 \neq \emptyset$.

Suppose now that we have shown that $B = B_n \cup C_n$ for some sets B_n, C_n with $o(C_n) = n$, where n is some positive integer. Put $\theta = o(B_n)$; then we have $\alpha\theta + \alpha n = \alpha\beta = \alpha\delta + \mu$. From the rigidity of α and Theorem 1 we conclude that $\delta = \tau + (n-1)$ for some τ. Thus $\alpha\theta + \alpha n = \alpha\tau + (\alpha(n-1) + \mu)$, and since $\alpha(n-1) + \mu \downarrow \alpha n$, we can repeat the above argument and show that B_n has a final element b_n.

By induction therefore, it follows that for each $n \geqslant 1$, we have $B = B_n \cup C_n$ for some sets B_n, C_n with $o(C_n) = n$, and from this it is easy to see that B has a final segment of type ω^*.

We wish now to turn to what we consider to be one of the most interesting features of the class Γ; that it is possible to give a natural definition of a limit operation and show that Γ is closed under this operation.

DEFINITION 4. Let $\{\alpha_i\}_{i \in I}$ be a family of order-types, where the index set I is ordered. We say that $\{\alpha_i\}_{i \in I}$ is *increasing* if $\alpha_i \stackrel{+}{=} \alpha_j$ for all $i, j \in I$ with $i \leqslant j$, and in this case we define α to be the limit of $\{\alpha_i\}_{i \in I}$ if α is the unique order-type satisfying the following:

(1) for each $i \in I$ we have $a_i \stackrel{+}{=} \alpha$;

(2) for any order-type β, if $\beta \downarrow \alpha$ then $\beta \stackrel{+}{=} \alpha_i$ for some $i \in I$.

If the limit of $\{\alpha_i\}_{i \in I}$ exists, then it is denoted by $\lim_{i \in I} \alpha_i$.

It is the uniqueness restriction in the above definition that causes the trouble. As will become apparent in the proof of the following theorem, for any increasing family of order-types there will be order-types satisfying conditions (1) and (2) above; it is not usually the case, however, that there is only one such order-type. For example, if we take I such that $o(I) = \omega$ and $\alpha_i = \eta$ (the order-type of the rationals) for each $i \in I$, then it is readily seen that both η and $\eta + 1$ satisfy (1) and (2).

With the aid of the Axiom of Choice, however, we can show that limits of increasing families of rigid order-types exist and are themselves rigid.

* THEOREM 6. *Let $\{\alpha_i\}_{i \in I}$ be an increasing family of rigid order-types. Then $\lim_{i \in I} \alpha_i$ exists and is rigid.*

PROOF. Using Choice, we choose for each $i \in I$ a set A_i with $o(A_i) = \alpha_i$, and for each pair $(i, j) \in I \times I$ with $i \leqslant j$ we let $f_{ij}: A_i \to A_j$ be a monomorphism. Without loss of generality we may assume the A_i to be pairwise disjoint, and by using Choice again we may also assume that $f_{jk} f_{ij} = f_{ik}$ for all $i, j, k \in I$ with $i \leqslant j \leqslant k$.

We now put $A^0 = \bigcup\{A_i;\ i \in I\}$, and define an equivalence relation $\sim$ on A^0 by setting $x \sim y$ if (1) $x = y$, (2) $x \in A_i$, $y \in A_j$ for some $i, j \in I$ with $i < j$ and $f_{ij}(x) = y$, or (3) $x \in A_i$, $y \in A_j$ for some $i, j \in I$ with $j < i$ and $f_{ji}(y) = x$. It is easily checked that $\sim$ is an equivalence relation, and we let A be the quotient structure $A^0/\sim$. Take $[x], [y] \in A$; then there exists $i \in I$ and $x', y' \in A_i$ such that $x' \in [x]$, $y' \in [y]$, and we set $[x] \leqslant [y]$ if $x' \leqslant y'$. It is routine to show that $\leqslant$ is a well-defined order-relation on A, and with respect to this order-relation we set $\alpha = o(A)$ and claim that α is the required order-type.

Before demonstrating this, we observe that the correspondence $x \mapsto [x]$ defines for each $i \in I$ a monomorphism $f_i: A_i \to A$, and that for each $[y] \in A$ there exists $i \in I$ and $y \in A_i$ such that $[y] = f_i(y)$. Therefore we may assume without loss of generality that each A_i is an initial segment of A and that $A = \bigcup\{A_i;\ i \in I\}$. We are of course now relinquishing our previous assumption that the A_i were pairwise disjoint.

We show firstly that α is rigid. Suppose not; then there is a proper initial segment B of A such that $B \simeq A$. Since B is proper in A, we must have $B \subseteq A_i$ for some $i \in I$. But because $B \simeq A$,there exists an initial segment C of B with $C \simeq A_i$. It is easily seen that C must be proper in A_i ,which contradicts the rigidity of A_i. Thus α must be rigid.

From the above observation and assumption, it is clear that α satisfies (1) and (2) of Definition 4, and so it remains to show that α is unique in this respect.

Suppose there is an order-type $\alpha' \neq \alpha$ satisfying (1) and (2), and let A' be a representative set for α'. For each $i \in I$ we have a monomorphism $A_i \to A'$, and so the Axiom of Choice gives us a monomorphism $A \to A'$. Thus $\alpha \downarrow \alpha'$, and in a similar manner we can show that $\alpha' \downarrow \alpha$. Hence $\alpha \downarrow \alpha$, which is a contradiction.

The above limit operation has some of the usual properties with respect to addition and multiplication.

THEOREM 7. *Let $\{\alpha_i\}_{i \in I}$ be an increasing family of rigid order-types, put $\alpha = \lim_{i \in I} \alpha_i$, and take any $\beta \in \Gamma$. Then $\{\beta + \alpha_i\}_{i \in I}$ and $\{\beta\alpha_i\}_{i \in I}$ are increasing and have respective limits $\beta + \alpha$ and $\beta\alpha$.*

PROOF. It is clear that $\{\beta + \alpha_i\}_{i \in I}$ and $\{\beta\alpha_i\}_{i \in I}$ are increasing families of rigid order-types, and so we concentrate on determining their limits. First of all we have $\beta + \alpha_i \mathrel{\dot{\pm}} \beta + \alpha$ for each $i \in I$ because $\alpha_i \mathrel{\dot{\pm}} \alpha$. Now take $\gamma \downarrow \beta + \alpha$. If $\gamma \mathrel{\dot{\pm}} \beta$, then obviously $\gamma \mathrel{\dot{\pm}} \beta + \alpha_i$ for some $i \in I$, and so we may assume that $\beta \downarrow \gamma \downarrow \beta + \alpha$, whence $\gamma = \beta + \delta$ for some $\delta \downarrow \alpha$, and once again it is clear that $\gamma \mathrel{\dot{\pm}} \beta + \alpha_i$ for some $i \in I$. Thus $\beta + \alpha = \lim_{i \in I} \beta + \alpha_i$.

We turn to multiplication, and since $\beta\alpha$ clearly satisfies (1) of Definition 4, we take $\gamma \downarrow \beta\alpha$. Then $\gamma = \beta\delta + \mu$ for some $\delta \downarrow \alpha$ and some $\mu \downarrow \beta$. If $\mu = 0$, then it is

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immediate that $\gamma \stackrel{+}{=} \beta\alpha_i$ for some $i \in I$, and so we may assume that $\mu \neq 0$, whence $\delta+1 \stackrel{+}{=} \alpha$. If $\delta+1 \downarrow \alpha$, then again the result is clear, and so there only remains the case $\delta+1=\alpha$. Now since $\delta \stackrel{+}{=} \alpha_i$ for some $i \in I$ and $\alpha = \lim_{i \in I} \alpha_i$, we must have $\alpha = \alpha_j$ for some $j \in I$, and we have once more attained our goal. Thus $\beta\alpha = \lim_{i \in I} \beta\alpha_i$.

We remark that a definition of the limit of a decreasing family of order-types can also be formulated, although it does not correspond exactly to Definition 4. The situation here, however, is not as satisfactory as the one discussed above, as far as the class Γ is concerned anyway. For in the first place limits of decreasing families of rigid order-types do not always exist (again it is the uniqueness condition that gives trouble), and in the second place even when they do exist, they are not necessarily rigid.

In the theory of ordinal numbers the prime components play an interesting and often very useful role: we recall that an ordinal α is called a "prime component" if $\alpha > 0$ and $\beta + \alpha = \alpha$ for every $\beta < \alpha$. Since Γ can be regarded as a fairly natural extension of the class of ordinals, it is plausible to ask whether the concept of prime component can be extended to Γ in any manner that yields interesting results. The remainder of this paper is mainly devoted to this question.

DEFINITION 5. A rigid order-type α is said to be *primitive* if $\beta + \alpha = \alpha$ for every rigid order-type β such that $\beta \downarrow \alpha$.

Since every prime component is obviously primitive, the concept of a primitive order-type is indeed an extension of that of a prime component. Moreover, every atomic order-type is primitive, and so the extension is nontrivial. Our aim is to show that every primitive order-type can be expressed as a product $\mu\delta$, where either (i) μ is atomic and δ is a prime component, or (ii) μ is fully regressive and δ is continuous, fully rigid, and such that if $\gamma \downarrow \delta$ and $1 \stackrel{+}{=} \gamma^*$, then $\delta = \gamma\rho$ for some prime component ρ. We do not really believe that an order-type having the properties attributed to δ in (ii) can exist, but despite fairly strenuous efforts, we have been unable to back up our belief with a formal proof.

We approach the result given above via a decomposition theorem that is in the style of Theorem 3.1 of Morel (1959), but which does not seem to be derivable from it.

THEOREM 8. *Let A be a nonempty set. There is an ordered family $\{A_i\}_{i \in I}$ of pairwise disjoint intervals of A such that*

(1) $A = \cup\{A_i;\ i \in I\}$;

(2) *for each $i \in I$, A_i is either atomic or fully regressive*;

(3) *for any interval B of A, if B is either atomic or fully regressive, then $B \subseteq A_i$ for some $i \in I$.*

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If $(A_j^0)_{j\in J}$ *is a second such decomposition of* A, *then there is an isomorphism* $f\colon I\simeq J$ *such that* $A_i = A^0_{f(i)}$ *for each* $i\in I$. *Finally, if* J *is some nonempty initial segment of* I *such that either* J *has no last element or else* $J=\{i\in I;\ i<i^0\}$ for some $i^0\in I$, then $\cup\{A_j;\ j\in J\}$ *is rigid.*

PROOF. Take any $x\in A$. We define subsets $S(x)$, $U(x)$, $T(x)$ of A by:

$$S(x)=\{y\in A;\ y<x\};$$
$$U(x)=\bigcup\{S\subseteq A;\ S \text{ is a rigid initial segment of } S(x)\};$$
$$T(x)=\{y\in A;\ U(y)=U(x)\}.$$

It is clear that $S(x)$ and $U(x)$ are both initial segments of A; moreover, from * Theorem 6 we see that $U(x)$ is either empty or rigid (an analysis of the proof of * Theorem 6 shows that the Axiom of Choice is not used here). We claim that $T(x)$ is an interval of A, that $T(x)$ is either atomic or fully regressive, and that for any $a,b\in A$, either $T(a)=T(b)$ or $T(a)\cap T(b)=\emptyset$.

Firstly, take $a,b,c\in A$ with $a\leqslant b\leqslant c$, and suppose that $a,c\in T(x)$. Obviously $U(a)\subseteq U(b)\subseteq U(c)$, and as $U(a)=U(x)=U(c)$, we conclude that $U(b)=U(x)$, that is, $b\in T(x)$.

Let S be any nonempty proper initial segment of $T(x)$: we will show that S is regressive, thereby proving that $T(x)$ is either atomic or fully regressive. Now $U(x)\cap T(x)=\emptyset$ and $U(x)\cup T(x)$ is an initial segment of A. For if $y\in T(x)$, then $U(y)=U(x)$, whence $U(x)\subseteq S(y)$, and so $y\notin U(x)$: hence $U(x)\cap T(x)=\emptyset$. Take y, $z\in A$ with $y\in T(x)$ and $z\leqslant y$. Then either $z\in T(x)$ or $U(z)\subsetneq U(y)=U(x)$, which yields $z\in U(x)$. Since it is clear that for any $y\in T(x)$ and $z\in U(x)$ we have $z\leqslant y$, this shows that $U(x)\cup T(x)$ is an initial segment of A. Now if S were rigid, then $V=U(x)\cup S$ would be a rigid initial segment of A having $U(x)$ as a proper initial segment. However, since S is proper in $T(x)$, there must exist $y\in T(x)$ such that $S\subseteq S(y)$. Therefore V is a rigid initial segment of $S(y)$, and so we conclude that $U(x)\subsetneq V\subseteq U(y)=U(x)$, a contradiction. Thus S is regressive.

Finally, take $a,b\in A$, and suppose that $y\in T(a)\cap T(b)$ for some $y\in A$. Then $U(a)=U(y)=U(b)$, whence it follows easily that $T(a)=T(b)$.

Put $I=\{T(x);\ x\in A\}$; since the $T(x)$ are pairwise disjoint intervals of A, the ordering on A induces an ordering on I. If we now put $A_i=i$ for each $i\in I$, we obtain an ordered family $\{A_i\}_{i\in I}$ of pairwise disjoint intervals of A satisfying (1) and (2).

Let B be a nonempty interval of A, and suppose that B is either atomic or fully regressive; we wish to show that $B\subseteq A_i$ for some $i\in I$. There is of course $a\in A$ such that $B\cap T(a)\neq\emptyset$, and we claim that $T(a)$ is the appropriate A_i.

Suppose that it is not the case that $B\subseteq T(a)$. Then there exists $b\in B$ such that either $x<b$ for all $x\in T(a)$, or $x>b$ for all $x\in T(a)$. Consider the first possibility. We must have $U(x)\subsetneq U(b)$ for all $x\in T(a)$, and so $T(a)\subseteq U(b)\subseteq S(b)$, whence it

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follows that $B \cap U(b)$ is a proper rigid initial segment of B, contradicting our assumption on B.

We must therefore have $x > b$ for all $x \in T(a)$; since it has been shown that $U(a) \cup T(a)$ is an initial segment of A, we conclude that $b \in U(a)$. Thus $B \cap U(a)$ is a proper rigid initial segment of B, once again contradicting our assumption on B. Thus $B \subseteq T(a)$. This shows that $\{A_i\}_{i \in I}$ satisfies (3).

Let $\{A_j^0\}_{j \in J}$ be a second decomposition of A satisfying (1)–(3). Then (3) tells us that for each $i \in I$ there exists $f(i) \in J$ such that $A_i = A_{f(i)}^0$, and clearly this defines an isomorphism $f\colon I \simeq J$.

Finally, let J be a nonempty initial segment of I such that either J has no last element, or else $J = \{i \in I;\ i < i^0\}$ for some $i^0 \in I$. Put $B = \bigcup \{A_i;\ i \in J\}$, and suppose firstly that J has no last element. It follows that for each $i \in J$ we have $A_i \subseteq U(b)$ for some $b \in B$, whence * Theorem 6 tells us that B is rigid (again no use of Choice is made here).

We may thus assume that J has a final element j^0, whence it follows from our assumption on J that j^0 has an immediate successor i^0 in I. We claim that A_{j^0} is atomic. For if A_{j^0} is fully regressive, then for any $y \in A_{i^0}$ the interval C_y defined by $C_y = A_{j^0} \cup \{x \in A_{i^0};\ x \leqslant y\}$ is either atomic or fully regressive, and so by (3) we have $C_y \subseteq A_i$ for some $i \in I$. This of course is absurd, and so A_{j^0} is atomic. But now $\mathrm{B} = U(x)$ for any $x \in A_{i^0}$, and so B is rigid.

We remark at this stage that if A and $\{A_i\}_{i \in I}$ are as in Theorem 8 and B is any nonempty interval or segment of A, then $\{B \cap A_j\}_{j \in J}$ is the corresponding decomposition of B, where $J = \{i \in I;\ B \cap A_i \neq \varnothing\}$. The proof of this is routine.

THEOREM 9. *Let α be a primitive order-type, and let $\beta \stackrel{+}{=} \alpha$ be rigid. Then $\alpha = \beta\rho$ for some prime component ρ.*

PROOF. Put $\Upsilon = \{\tau;\ \tau \text{ is a prime component and } \beta\tau \stackrel{+}{=} \alpha\}$. Then $1 \in \Upsilon$ and so $\Upsilon \neq \varnothing$. At the same time it is clear that Υ is a set of ordinals, and so $\rho = \sup \Upsilon$ is a well-defined ordinal, which is itself easily seen to be a prime component. By * Theorem 7 we have $\beta\rho = \lim_{\tau \in \Upsilon} \beta\tau$, and since $\beta\tau \stackrel{+}{=} \alpha$ for every $\tau \in \Upsilon$, it follows that $\beta\rho \stackrel{+}{=} \alpha$. Now if $\beta\rho \downarrow \alpha$, then we must have $\beta\rho + \alpha = \alpha$, and a simple induction argument shows that $\beta\rho n + \alpha = \alpha$, whence $\beta\rho n \downarrow \alpha$, for every $n < \omega$. Using limits again, we obtain $\beta\rho\omega \stackrel{+}{=} \alpha$, which means that $\rho\omega \in \Upsilon$, contradicting the definition of ρ. Hence $\beta\rho = \alpha$.

* THEOREM 10. *Let α be a rigid order-type. Then α is primitive if and only if $\alpha = \mu\delta$ for some order-types μ, δ such that*

either (1) *μ is atomic and δ is a prime component,*

or (2) *μ is fully regressive and*

(a) *δ is continuous,*

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(b) δ *is fully rigid,*

(c) *if* $\gamma \downarrow \delta$ *is nonzero and such that* $\sim(1 \stackrel{+}{=} \gamma^*)$, *then* $\delta = \gamma\rho$ *for some prime component* ρ.

PROOF. Suppose that $\alpha = \mu\delta$ has the decomposition given in (1), and let $\beta \downarrow \alpha$ be rigid. Then $\beta = \mu\psi + \varepsilon$ for some $\psi < \delta$ and some $\varepsilon \downarrow \mu$. Now since μ is atomic, we cannot have $\varepsilon \neq 0$, for otherwise ε and hence β would be regressive. Thus $\beta = \mu\psi$, and so we have $\beta + \alpha = \mu\psi + \mu\delta = \mu(\psi + \delta) = \mu\delta = \alpha$.

Now suppose that $\alpha = \mu\delta$ has the decomposition given in (2), and let $\beta \downarrow \delta$ be rigid. Once again we have $\beta = \mu\psi + \varepsilon$ for some $\psi \downarrow \delta$ and some $\varepsilon \downarrow \mu$, and since μ is this time fully regressive, it follows as above that $\varepsilon = 0$. Moreover, we cannot have $1 \stackrel{+}{=} \psi^*$, since otherwise there would be $\zeta \downarrow \psi$ such that $\beta = \mu\zeta + \mu$, which implies that β is regressive. This means that $\delta = \psi\rho$ for some prime component ρ, whence $\beta + \alpha = \beta + \beta\rho = \beta\rho = \alpha$, since $\rho > 1$.

We turn now to the converse, assume that α is primitive, and let $\{\alpha_i\}_{i \in I}$ be the decomposition of α corresponding to Theorem 8. Our first aim is to show that I has a first element and that $\alpha_i = \alpha_j$ for all $i, j \in I$. To this end we may of course assume that $|I| > 1$.

Choose any $i^0 \in I$ such that $i < i^0$ for some $i \in I$, and put $\beta = \sum\{\alpha_i;\ i < i^0\}$; then Theorem 8 tells us that β is rigid, and so by * Theorem 9 we have $\alpha = \beta\rho$ for some prime component ρ. Since $\beta \neq \alpha$, we must have $\rho > 1$. Put $I^0 = \{i \in I;\ i \geqslant i^0\}$, and $\gamma = \sum\{\alpha_i;\ i \in I^0\}$. From the remark following Theorem 8 we see that $\{\alpha_i\}_{i \in I^0}$ is the decomposition of γ corresponding to Theorem 8. But $\beta + \alpha = \alpha = \beta + \gamma$, and so $\alpha = \gamma$. Thus by Theorem 8 there is an isomorphism $I \simeq I^0$; since I^0 has a first element, it follows that I has one too, and we let this be i'.

We now show that $\alpha_{i^0} = \alpha_{i'}$; since i^0 was chosen arbitrarily in $I - \{i'\}$, this will establish our claim. Let A, C be representative sets for α, γ respectively, and let $\{A_i\}_{i \in I}$, $\{C_i\}_{i \in I^0}$ be the respective decompositions of A, C given by Theorem 8. We know that there is an isomorphism $f\colon A \simeq C$, and so by (3) of Theorem 8 we see that $f''A_{i'} \subseteq C_{i^0}$. Similarly we obtain $f^{-1}C_{i^0} \subseteq A_{i'}$, and so $A_{i'} \simeq C_{i^0}$, whence $\alpha_{i'} = \alpha_{i^0}$.

We thus see (using Choice) that $\alpha = \mu\delta$ for some order-types μ, δ, where μ is either atomic or fully regressive. If μ is atomic, then by * Theorem 9 there is some prime component ρ such that $\alpha = \mu\rho$, and from Theorem 4 we conclude that $\delta = \rho$. Hence we have obtained the decomposition given by (1).

Assume that μ is fully regressive, let M, D be representative sets for μ, δ respectively, and consider $M \dot{\times} D$. Without loss of generality we may assume that $A = M \dot{\times} D$.

(1) D is dense. Take $d_0, d_1 \in D$ and suppose that d_0 immediately precedes d_1. Then $N = M \dot{\times} \{d_0, d_1\}$ is a fully regressive interval of A, and so by (3) of Theorem 8 we must have $N \subseteq M \times \{d\}$ for some $d \in D$. But this is absurd.

(2) D is continuous. Let E be a nonempty proper initial segment of D, and suppose that E has no last element. By Theorem 8, $M \dot{\times} E$ is a rigid initial segment of A, and of course $A \neq M \dot{\times} E$. Thus by * Theorem 9 we have $\alpha = o(M \dot{\times} E)\rho$ for some prime component $\rho > 1$. The remark following Theorem 8 tells us that $M \dot{\times} (D-E)$ is the decomposition of $A-(M \dot{\times} E)$ given by Theorem 8; furthermore, from $\alpha = o(M \dot{\times} E)\rho$ and $\rho > 1$ it follows that $A \simeq A-(M \dot{\times} E)$. Therefore $D-E \simeq D$. However, $D \simeq I$, and we have seen that I has a first element. Thus $D-E$ has a first element.

(3) D is fully rigid. Let E be a nonempty initial segment of D. Since D is continuous, either E has no last element or else $E = E^0 \cup \{d\}$ for some $d \in D$ and some initial segment E^0 of D having no last element. Clearly in this latter case E is rigid if E^0 is, and so it suffices to prove E rigid under the assumption that E has no last element. But under this assumption $M \dot{\times} E$ is rigid, by Theorem 8, and it follows at once that E is also rigid.

Now let E be a nonempty initial segment of D without last element. As usual $M \dot{\times} E$ is rigid and $A \simeq (M \dot{\times} E) \dot{\times} R$ for some set R such that $o(R)$ is a prime component. It is easy to see, however, that $M \dot{\times} (E \dot{\times} R)$ is a decomposition of A satisfying the conditions of Theorem 8, and so we must have $D \simeq L \dot{\times} R$.

This concludes the proof of our theorem.

Corollary. *Let α be a fully rigid order-type. Then α is a primitive if and only if α is a prime component.*

Proof. Suppose that α is primitive, and let $\alpha = \mu\delta$ be the decomposition given by * Theorem 10. We know that μ is either atomic or fully regressive, and so if $\mu \neq 1$, there exist ε, ϕ with ε regressive and $\varepsilon + \phi = \mu$. Since $\delta \neq 0$, there exist β, γ such that $\delta = \beta + 1 + \gamma$, and we have $\alpha = \mu\beta + \varepsilon + \phi + \mu\gamma$. But of course $\mu\beta + \varepsilon$ is regressive, contradicting the fact that α is fully rigid. Hence $\mu = 1$, whence δ is a prime component and $\alpha = \delta$.

We observe that in the first part of the proof of * Theorem 10, only property (c) was used to show that if α had the decomposition (2), then α was primitive. The trivial case in which μ is any fully regressive order-type and $\delta = 1$ shows that the assumption that α is rigid is essential in the proof of primitivity. This contrasts with the situation in (1), for clearly if μ is atomic and δ is a prime component, then $\mu\delta$ is rigid and hence primitive.

With the exception of the case $\delta = 1$, we have thus far been unable to produce any order-type satisfying (a), (b), (c); on the other hand, all our attempts to prove that such order-types do not exist have ended in equal failure. There are of course rigid continuous order-types (other than 1), the simplest being perhaps $(1+\lambda)\omega_1$, where λ is the order-type of the reals. However, the construction of a nontrivial

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fully rigid continuous order-type in the presence of the Axiom of Choice does not appear to be a particularly simple task. (Without Choice it is a different matter, for clearly every ordered medial set is fully rigid—see Hickman, 1975.)

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