ON VECTOR LATTICE-VALUED MEASURES

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1. Introduction. E. Hewitt [1] used the Daniell approach to define a real-valued measure function on a σ-algebra of the real line. He began by defining an arbitrary non-negative linear functional I on $L_{\infty\infty}(R)$, (the space of all complex-valued continuous functions on the real line R which vanish off some compact subset of R).

In this paper, it is shown how this method can be generalized to obtain a vector lattice-valued measure function on a σ-algebra of the real line.

This is done by first defining an arbitrary order preserving linear transformation I from the normed linear space L_r (of continuous real-valued functions on the real line which vanish off a compact subset) into an arbitrary boundedly complete Banach lattice B. A Banach lattice B is a Banach space which is also a lattice in which:

1) $\vee$ and $\wedge$ (least upper bound and greatest lower bound, respectively, of any two members of B) are continuous functions of both their variables and,

2) $|x|_B \leq |y|_B$ implies that $\|x\|_B \leq \|y\|_B$ for $x, y \in B$, where $\| \ \|_B$ is the norm of the Banach space B and, for any $z \in B$, $|z|_B = (x \vee 0) - (x \wedge 0)$. The additive identity of B is 0 and $\leq$ is the partial ordering of B.

A boundedly complete Banach lattice B is a Banach lattice in which every non-void bounded set has a least upper bound and a greatest lower bound in B.

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The space of real or complex numbers is a special case of B with respect to the partial ordering and the bounded completeness property.

In the following, let L_+ denote the set of all $x \in L_r$ such that $x(t) \geq 0$ for all $t \in R$ and let M_+ denote the set of all real-valued, non-negative, lower semi-continuous functions on the real line. For each $x \in M_+$ let $Y_x = \{y \mid y \in L_+, y \leq x\}$.

2. When the vector lattice-valued operator I is extended from L_r to M_+ by $\bar{I} = \sup\{I(y) \mid y \in Y_x\}$, the extended Banach lattice $\bar{B}$ is considered where B is a boundedly complete Banach lattice and $\bar{B} = B \cup \{U, D\}$ is a complete lattice. (U is such that, for any $x \in B$, $x < U$ and D is such that, $D < x$.) $\bar{I}(x) \in \bar{B}$, but not necessarily in B.

Next, we consider the space of all real-valued functions defined on R and extend $\bar{I}$ to a transformation $\bar{\bar{I}}$ on this space by $\bar{\bar{I}}(z) = \inf\{\bar{I}(y) \mid y \in M_+, y \geq z\}$. All the characteristic functions are in this space of functions and a set function may be defined on the subsets of R by $\mu(A) = \bar{\bar{I}}(\chi_A)$, $A \subset R$. This set function μ is a vector lattice-valued outer measure on the subsets of R.

We need the following result later for the proof of Theorem 1. For any $A \subset R$, $\mu(A) = \inf\{U(G) \mid G$ is open, $G \supset A\}$. The proof of this is analogous to that given in E. Hewitt [1, Th. 4.1.35, p. 217].

The main problem now is to obtain a σ-algebra of the real line on which μ becomes a countably additive vector lattice-valued measure function.

DEFINITION 1. Let L_1 be the set of all equivalence classes ξ of real-valued functions x defined almost everywhere on R, with respect to μ, such that ξ contains some function x defined everywhere on R for which $\bar{\bar{I}}(|x|) \in B$.

L_1 is a complete normed linear space with its norm defined by $\|\xi\|_1 = \|\bar{\bar{I}}(|x|)\|_B$. We write $\|x\|_1$ instead of $\|\xi\|_1$, where x is as in the definition above.

DEFINITION 2. Let E_1 be the space of all the functions $x \in L_1$ such that for some sequence $\{y_n\}_{n=1}^{\infty} \subset L_r$, $\lim_{n\to\infty} \|y_n - x\|_1 = 0$. The elements of E_1 are called summable functions.

E_1 is a real Banach space with the linear operations and norm that it inherits from L_1, and L_r is a dense linear subspace of E_1. The operator I admits an extension, again called I, over E_1 such that I is linear on E_1 and $\|I(x)\|_B \leq \|x\|_1$ for all $x \in E_1$. The extension I is unique under the restrictions that it be linear and satisfy $\|I(x)\|_B \leq A\|x\|_1$ for some $A > 0$ and all $x \in E_1$.

The σ-algebra on the real line is obtained by first defining a subset A of R to be summable if $\chi_A \in E_1$. Note that for summable sets $\mu(A) = \bar{\bar{I}}(\chi_A) = I(\chi_A)$. A subset P of R is said to be measurable if $P \cap F$ is summable for all compact sets F. The family of measurable sets is denoted by M and is a σ-algebra of sets which contain the Borel sets.

The following theorems show that the vector lattice-valued outer measure μ becomes a measure function over M.

THEOREM 1. If $A \subset R$ is summable, then for every $\epsilon > 0$ there exists an open set G and a compact set F such that $F \subset A \subset G$ and $\|\mu(G) - \mu(F)\|_B < \epsilon$.

Proof. If $A \subset R$ is summable, then $\chi_A \in E_1$. Let G_1 be any open set such that $\mu(G_1) > 0$ and $G_1 \supset A$. Since $\mu(A) = \inf\{\mu(G) \mid G \text{ is open}, G \supset A\}$, for $\epsilon > 0$ there exists an open set $G \supset A$ such that $\mu(G) < \mu(A) + \dfrac{\epsilon\,\mu(G_1)}{2\|\mu(G_1)\|_B}$. From

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this it follows that $\|\mu(G)\|_B < \|\mu(A)\|_B + \epsilon/2$ or, also, $\|\mu(G) - \mu(A)\|_B < \frac{\epsilon}{2}$.

Next, there exists a non-negative upper semi-continuous function z in E_1, having a compact support, such that $z \leq \chi_A$ and $\|I(\chi_A - z)\|_B < \frac{\epsilon}{4}$. If $\|\mu(A)\|_B = 0$, let $F = \emptyset$ to get $\|\mu(G) - \mu(F)\|_B \leq \|\mu(G)\|_B + \|\mu(F)\|_B = \|\mu(G)\|_B < \frac{\epsilon}{2} < \epsilon$. Otherwise, let $\delta = \min(\epsilon/8\ \|\mu(A)\|_B,\ 1)$ and let $F = \{t \mid z(t) \geq \delta\}$. F is compact, $F \subset A$, and $A - F$ is summable. Now $z \leq \chi_F + \delta\chi_{A-F}$ so that $I(z) \leq I(\chi_F) + \delta I(\chi_{A-F}) \leq \mu(F) + \epsilon\,\mu(A)/8\,\|\mu(A)\|_B$. Therefore

$$\|\mu(A) - \mu(F)\|_B \leq \|\mu(A) - (I(z) - \epsilon\,\mu(A)/8\,\|\mu(A)\|_B)\|_B$$

$$\leq \|I(\chi_A - z)\|_B + \frac{\epsilon}{8} < \frac{\epsilon}{4} + \frac{\epsilon}{8} < \frac{\epsilon}{2}.$$

Then $\|\mu(G) - \mu(F)\|_B = \|\mu(G) - \mu(A) + \mu(A) - \mu(F)\|_B$

$$\leq \|\mu(G) - \mu(A)\|_B + \|\mu(A) - \mu(F)\|_B < \epsilon.$$

THEOREM 2. Let $A \subset R$ be summable. Then there exists a sequence $G_1 \supset G_2 \supset \ldots \supset G_n \supset \ldots$ of open sets where $G_n \supset A$ and $\mu((\bigcap_{n=1}^{\infty} G_n) - A) = 0$. Furthermore, there exists a sequence of pairwise disjoint compact sets $\{F_n\}_{n=1}^{\infty}$ such that $F_n \subset A$ and $\mu(A - (\bigcup_{n=1}^{\infty} F_n)) = 0$.

THEOREM 3. Let $A \subset R$ and $\mu(A) \epsilon B$. Then A is contained in the union of a null set and the union of a countable family of pairwise disjoint compact sets.

THEOREM 4. If $A \epsilon M$ and $\mu(A) \epsilon B$, then A is summable.

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The proofs of Theorems 2, 3 and 4 are similar to those for the special case of the reals as presented by Hewitt [1, pp. 230-232].

It now follows that μ is countably additive on M and is therefore a vector lattice-valued measure on R.

Proof. Let $\{A_n\}_{n=1}^{\infty}$ be a pairwise disjoint family of sets in M. If $\mu(A_n) \notin B$ for some n, then $\mu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$.

If $\mu(A_n) \in B$ for all n, then by Theorem 4 and by the result that if $\{B_n\}_{n=1}^{\infty}$ is a pairwise disjoint collection of summable sets, then $\mu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$; the result follows.

Example: Let B be the boundedly complete Banach lattice of sequences $s = \{s_i\}_{i=-\infty}^{\infty}$ such that each s_i is a real number. Let the partial ordering be defined by $s \geq r$ if and only if $s_i \geq r_i$ for each integer i, and let the norm be defined by $\|s\|_B = (\sum_{i=-\infty}^{\infty} |s_i|^2)^{1/2}$. A sequence s is not in B if $\|s\|_B = \infty$. If a non-negative linear transformation I is defined on L_r with values in B, then a measure on a σ-algebra of the real line with values in B is obtained.

In particular, if I is defined by $I(x) = \{x(n)\}_{n=-\infty}^{\infty}$ for $x \in L_r$, then $\bar{\bar{I}}(x) = 0$ if and only if x is zero at the integers. Also for any $A \subset R$, $\{\ldots, 0, \ldots\} \leq \mu(A) \leq \{\ldots, 1, \ldots\}$.

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