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The Polynomial Behavior of Weight Multiplicities for the Affine Kac–Moody Algebras $A_r^{(1)}$

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Abstract. We prove that the multiplicity of an arbitrary dominant weight for an irreducible highest weight representation of the affine Kac–Moody algebra $A_r^{(1)}$ is a polynomial in the rank r. In the process we show that the degree of this polynomial is less than or equal to the depth of the weight with respect to the highest weight. These results allow weight multiplicity information for small ranks to be transferred to arbitrary ranks.

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Introduction

The irreducible highest weight representations of affine Kac–Moody algebras have played an increasingly important role in diverse areas of mathematics and physics. When its level is positive, such a representation is infinite-dimensional. It is parameterized by a dominant integral highest weight and has finite-dimensional weight spaces. The formal character of such a representation records the multiplicity of each weight, and the well-known *Weyl–Kac character formula* ([K2, p. 173]) provides a precise expression for the character. The character formula involves a sum over the Weyl group in both its numerator and denominator which makes it impractical for explicitly computing multiplicities. However, when the character formula is applied to the one-dimensional trivial representation, it gives the *denominator identity*, and from the denominator identity Peterson [P] has derived Freudenthal- type recursive formulas for calculating root and weight multiplicities. These formulas enabled Kass, Moody, Patera, and Slansky [KMPS] to develop tables of weight multiplicities for certain weights of low level irreducible highest weight representations for affine Kac–Moody algebras having rank less than 8.

In 1987 while analyzing the weight multiplicities of the irreducible highest weight representations of the untwisted classical affine Kac–Moody algebras, Benkart and Kass (see [BK]) conjectured certain polynomial behavior for the weight multiplicities of these representations and introduced the notion of a 'rank-zero string function'. The conjectures were confirmed in [BKM2] for any irreducible highest weight representation of the affine Kac–Moody algebras $A_r^{(1)}$ for weights having depth $\leqslant 2$. In fact, in [BKM2] the multiplicities of such weights were given by explicit polynomials whose coefficients involve Kostka numbers. However, it seems to be very difficult to extend the methods of [BKM2], which were based on the root multiplicity formula for Kac–Moody algebras obtained in [Ka2] and the representation theory of $\mathfrak{sl}(r+1, \mathbf{C})$, to prove the conjecture for arbitrary depths.

In this paper, we adopt a completely different approach to prove that the multiplicity of an arbitrary dominant weight for an irreducible highest weight representation of the affine Kac–Moody algebra $A_r^{(1)}$ is a polynomial in the rank r. Although the precise degree of these polynomials is not determined in this work, an upper bound is obtained for the degree, and this upper bound coincides with the degree conjectured by Benkart and Kass (see [BKM2], Conjecture A).

Briefly, our argument proceeds as follows: Let $L(\lambda)$ denote the irreducible highest weight $A_r^{(1)}$-module with highest weight $\lambda = \sum_{i=0}^{r} a_i \Lambda_i - m\delta$, where $\Lambda_0, \Lambda_1, \ldots, \Lambda_r$ are the fundamental weights and δ is the null root. We consider the minimal graded Lie algebra L with local part $L(\lambda) \oplus A_r^{(1)} \oplus L^*(\lambda)$, where $L^*(\lambda)$ is the finite dual space of $L(\lambda)$ (see Section 2). Then L is isomorphic to the indefinite Kac–Moody algebra $\hat{\mathfrak{g}}$ associated to the Cartan matrix $\hat{\mathfrak{A}} = (a_{i,j})_{i,j=-1,0,1,\ldots,r}$, whose first column consists of the entries 2, $-a_0$, $-a_1$, $\cdots$, $-a_r$. When the first row and the first column of $\hat{\mathfrak{A}}$ are deleted, the result is the Cartan matrix of the affine Kac–Moody algebra $A_r^{(1)}$. Now any weight μ of $L(\lambda)$ can be viewed as a root in $\hat{\mathfrak{g}}$, and its multiplicity as a root of $\hat{\mathfrak{g}}$ is the same as its multiplicity as a weight of $L(\lambda)$. We use Peterson's recursive root multiplicity formula in conjunction with a tricky inductive argument to establish the polynomial behavior of the dominant weights of $L(\lambda)$.

In a sequel to this paper [BKLS], weights for the other classical affine algebras are shown to exhibit polynomial behavior. The $A_r^{(1)}$ case requires special treatment because its Dynkin diagram is a cycle, so we address that case separately in this work. The proof of the polynomial conjecture permits much of the information in [KMPS] to be extended to arbitrary ranks, and it provides a means of relating string functions for various algebras. A better understanding of the polynomial nature of the multiplicities for more general sequences of Kac–Moody algebras (beyond the affine cases treated here and in [BKLS]) would allow results about multiplicities for affine algebras to be transferred to hyperbolic and indefinite Kac–Moody algebras, where only very limited information is currently known.

1. Affine Weight Lattice and the Conjecture

Suppose that $I = \{0, 1, \ldots, r\}$, and let $\mathfrak{A} = (a_{i,j})_{i,j \in I}$ be the affine Cartan matrix of type $A_r^{(1)}$:

$$\mathfrak{A} = \begin{pmatrix} 2 & -1 & 0 & 0 & \cdots & 0 & -1 \\ -1 & 2 & -1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 2 & -1 \\ -1 & 0 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix}. \tag{1.1}$$

Let $\mathfrak{h}$ be a vector space over $\mathbf{C}$ with a basis $\{h_0, h_1, \ldots, h_r, d\}$. Define linear functionals $\alpha_i \in \mathfrak{h}^*$ $(i \in I)$ by

$$\alpha_i(h_j) = a_{j,i} \quad \text{for } j \in I, \quad \alpha_i(d) = \delta_{i,0}. \tag{1.2}$$

Then the triple $(\mathfrak{h}, \Pi = \{\alpha_i |\ i \in I\}, \Pi^\vee = \{h_i |\ i \in I\})$ provides a realization of the matrix $\mathfrak{A}$ in the sense of [K2, Chap. 1]. The Kac–Moody algebra $\mathfrak{g}$ associated with the affine matrix $\mathfrak{A}$ is the *affine Kac–Moody algebra of type* $A_r^{(1)}$. We denote by e_i, f_i, h_i $(i \in I)$ and d the generators of the algebra $\mathfrak{g}$. The subalgebra $\mathfrak{g}_0$ of $\mathfrak{g}$ generated by e_i, f_i, h_i $(i = 1, \ldots, r)$ is a finite-dimensional simple Lie algebra of type A_r which is isomorphic to the Lie algebra $\mathfrak{sl}(r+1, \mathbf{C})$ of $(r+1) \times (r+1)$ complex matrices of trace zero.

Let $c = h_0 + h_1 + \cdots + h_r$. Then $[c, x] = 0$ for all $x \in \mathfrak{g}$, and c is the *canonical central element* of $\mathfrak{g}$. Note that $\{h_1, \ldots, h_r, c, d\}$ forms another basis of $\mathfrak{h}$. Since the matrix $\mathfrak{A}$ is symmetric, there is a nondegenerate symmetric bilinear form on $\mathfrak{h}$ which satisfies

$$\begin{aligned} &(h_i|h_j) = a_{i,j} \quad \text{for } i, j = 1, \ldots, r, \\ &(h_i|c) = (h_i|d) = 0 \quad \text{for } i = 1, \ldots, r, \\ &(c|c) = (d|d) = 0, \quad (c|d) = 1. \end{aligned} \tag{1.3}$$

Define linear functionals $\Lambda_i \in \mathfrak{h}^*$ $(i \in I)$ and $\delta \in \mathfrak{h}^*$ by

$$\begin{aligned} &\Lambda_i(h_j) = \delta_{i,j}, \quad \Lambda_i(d) = 0, \\ &\delta(h_j) = 0, \quad \delta(d) = 1 \quad \text{for } j \in I. \end{aligned} \tag{1.4}$$

Then δ can be expressed as $\delta = \alpha_0 + \alpha_1 + \cdots + \alpha_r$. It is easy to see that $\{\Lambda_0, \Lambda_1, \ldots, \Lambda_r, \delta\}$ and $\{\Lambda_0, \alpha_0, \alpha_1, \ldots, \alpha_r\}$ are both bases of the complex vector space $\mathfrak{h}^*$, and

$$\alpha_i = -\Lambda_{i-1} + 2\Lambda_i - \Lambda_{i+1} + \delta_{i,0}\delta \quad (i \bmod r+1). \tag{1.5}$$

The *affine weight lattice* P of *integral weights* is defined to be $P = \mathbf{Z}\Lambda_0 \oplus \mathbf{Z}\Lambda_1 \oplus \cdots \oplus \mathbf{Z}\Lambda_r \oplus \mathbf{Z}\delta$, and the elements of $P^+ = \{\lambda \in P |\ \lambda(h_i) \geqslant 0$

for all $i \in I\}$ are the *dominant integral weights* for the algebra $\mathfrak{g}$. For $\lambda, \mu \in P$, we say that μ is *related* to λ, which we denote by $\mu \sim \lambda$, if $\lambda - \mu \in Q$, where $Q = \bigoplus_{i=0}^{r} \mathbf{Z}\alpha_i$ is the root lattice. For example, if $\mu \in P$ is a weight of the irreducible highest weight module $L(\lambda)$ over $\mathfrak{g}$ with highest weight $\lambda \in P$, then $\lambda - \mu \in Q_+ = \bigoplus_{i=0}^{r} \mathbf{Z}_{\geqslant 0}\alpha_i$, and hence μ is related to λ.

Let l be a positive integer. A dominant integral weight $\lambda \in P^+$ is said to have *level* l if $\lambda(c) = l$. The weight λ can be uniquely expressed in the form

$$\lambda = a_0\Lambda_0 + a_1\Lambda_1 + \cdots + a_r\Lambda_r - m\delta, \tag{1.6}$$

where $m \in \mathbf{Z}$ and $a_i \in \mathbf{Z}_{\geqslant 0}$ for $i = 0, 1, \ldots, r$. Since $c = h_0 + h_1 + \cdots + h_r$, we have

$$\lambda(c) = a_0 + a_1 + \cdots + a_r = l. \tag{1.7}$$

Let $\mu \in P$ be an integral weight and suppose μ is related to λ. We write

$$\mu = b_0\Lambda_0 + b_1\Lambda_1 + \cdots + b_r\Lambda_r - n\delta, \tag{1.8}$$

where $n \in \mathbf{Z}$ and $b_i \in \mathbf{Z}$ for $i = 0, 1, \ldots, r$, and we set $d_i = b_i - a_i$ $(i \in I)$. Since μ is related to λ, we can write $\mu = \lambda - \sum_{i=0}^{r} k_i\alpha_i$ for some $k_i \in \mathbf{Z}$ $(i = 0, 1, \ldots, r)$. Therefore, since $\alpha_i(c) = 0$ for all i, we must have

$$\mu(c) = b_0 + b_1 + \cdots + b_r = l,$$

which implies

$$d_0 + d_1 + \cdots + d_r = 0. \tag{1.9}$$

Using the linear system

$$\begin{aligned} &\mu(h_j) = a_j - \sum_{i=0}^{r} k_i a_{j,i} = b_j \quad \text{for } j = 0, 1, \ldots, r, \\ &\mu(d) = -m - k_0 = -n, \end{aligned} \tag{1.10}$$

we can solve for the k_i's to obtain

$$\begin{aligned} &k_0 = n - m, \\ &k_i = n - m + d_{i+1} + 2d_{i+2} + \cdots + (r-i)d_r - (r-i+1)\frac{N}{r+1} \\ &\qquad\qquad \text{for } i = 1, \ldots, r, \end{aligned} \tag{1.11}$$

where $N = d_1 + 2d_2 + \cdots + rd_r$. In particular,

$$k_r = n - m - \frac{N}{r+1} \in \mathbf{Z}.$$

Thus

$$N = d_1 + 2d_2 + \cdots + rd_r \equiv 0 \mod r+1. \tag{1.12}$$

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Conversely, suppose $\mu = \sum_{i=0}^{r} b_i\Lambda_i - n\delta \in P$ is an integral weight with $b_i \in \mathbf{Z}$ $(i \in I)$ and $n \in \mathbf{Z}$ which satisfies (1.9) and (1.12) for $d_i = b_i - a_i \ \ (i \in I)$. Then we can write $\mu = \lambda - \sum_{i=0}^{r} k_i\alpha_i$, where the k_i are given by (1.11). Hence μ is related to λ. Therefore, we obtain:

PROPOSITION 1.13. *Let* $\lambda = \sum_{i=0}^{r} a_i\Lambda_i - m\delta \ (a_i \in \mathbf{Z}_{\geqslant 0}, m \in \mathbf{Z})$ *be a dominant integral weight of level* l*, and let* $\mu = \sum_{i=0}^{r} b_i\Lambda_i - n\delta \in P \ (b_i \in \mathbf{Z}, n \in \mathbf{Z})$ *be an integral weight. Then* μ *is related to* λ *if and only if*

$$\begin{aligned} &d_0 + d_1 + \cdots + d_r = 0, \\ &d_1 + 2d_2 + \cdots + rd_r \equiv 0 \mod r+1, \end{aligned} \tag{1.14}$$

where $d_i = b_i - a_i \ \ (i \in I)$.

Now fix a positive integer l and a dominant integral weight λ of level l given by (1.6). Since $\lambda(c) = a_0 + a_1 + \cdots + a_r = l$, there must be a *gap* in the expression (1.6) for λ if $r \geqslant l$. That is, there exist nonnegative integers s and t with $s + t \leqslant r$ such that

$$a_{s-1} \neq 0, \quad a_{r-t+1} \neq 0, \quad a_s = a_{s+1} = \cdots = a_{r-t} = 0. \tag{1.15}$$

The viewpoint we adopt here is that the weight λ is completely determined by the following data: (i) an s-tuple of nonnegative integers $\underline{a} = (a_0, a_1, \ldots, a_{s-1})$ with $a_{s-1} \neq 0$, (ii) a t-tuple of nonnegative integers $\underline{a}' = (a_{r-t+1}, a_{r-t+2}, \ldots, a_r)$ with $a_{r-t+1} \neq 0$, and (iii) an integer m. Note that this determining data is independent of r. Thus, dominant integral weights will be regarded as the same for all $r \geqslant l$ provided they have the same determining data. It is important to observe that a different choice of gap in expression (1.6) yields a different weight. For example, consider $\lambda = 2\Lambda_2 + \Lambda_4 - \delta$ when $r = 5$. If we take $s = 5$ and $t = 0$, then the determining data for λ is $\underline{a} = (0, 0, 2, 0, 1)$, $\underline{a}' = \emptyset$, $m = 1$, and λ can be written as $\lambda = 2\Lambda_2 + \Lambda_4 - \delta$ for all $r \geqslant 5$. On the other hand, we can choose a different gap by taking $s = 3$, $t = 2$. In this case, the determining data for λ is $\underline{a} = (0, 0, 2)$, $\underline{a}' = (1, 0)$, $m = 1$, and λ can be expressed as $\lambda = 2\Lambda_2 + \Lambda_{r-1} - \delta$ for all $r \geqslant 5$.

Suppose the dominant integral weight λ is given by the determining data $\underline{a} = (a_0, a_1, \ldots, a_{s-1})$, $\underline{a}' = (a_{r-t+1}, a_{r-t+2}, \ldots, a_r)$, and m. Let $\mu = \sum_{i=0}^{r} b_i\Lambda_i - n\delta$ $(b_i \in \mathbf{Z}_{\geqslant 0}, n \in \mathbf{Z})$ be a dominant integral weight of level l. Then $\mu(c) = b_0 + b_1 + \cdots + b_r = l$, and if $r \geqslant l$, there must be a gap in the expression (1.8) for μ. Moreover, if $r \geqslant l + s + t$, then the gap of λ is sufficiently large that there exists a gap of μ which overlaps the gap of λ. As a result, we can associate determining data $\underline{b} = (b_0, b_1, \ldots, b_{s'-1})$, $\underline{b}' = (b_{r-t'+1}, b_{r-t'+2}, \ldots, b_r)$, and $n \in \mathbf{Z}$ to μ, where $b_{s'-1} \neq 0$, $b_{r-t'+1} \neq 0$, and s', t' are nonnegative integers satisfying $s' + t' \leqslant r$, $s + t' \leqslant r$, and $s' + t \leqslant r$. Hence, if we let $p = \max(s, s')$ and

$q = \max(t, t')$, the weights λ and μ share a *common gap*:

$$\begin{aligned} a_p &= a_{p+1} = \cdots = a_{r-q} = 0, \\ b_p &= b_{p+1} = \cdots = b_{r-q} = 0. \end{aligned} \tag{1.16}$$

(Note that $p \leqslant r - t$ and $s \leqslant r - q$.)

From now on we assume that $r \geqslant l + s + t$. If μ is related to λ for infinitely many values of $r \geqslant l + s + t$, then the congruence equation (1.14) holds for all those values of r. Hence,

$$\begin{aligned} N &= d_1 + 2d_2 + \cdots + rd_r \\ &= \big(d_1 + 2d_2 + \cdots + (p-1)d_{p-1}\big) - \\ &\quad - \big(qd_{r-q+1} + (q-1)d_{r-q+2} + \cdots + 2d_{r-1} + d_r\big) + \\ &\quad + (r+1)\big(d_{r-q+1} + d_{r-q+2} + \cdots + d_r\big) \end{aligned} \tag{1.17}$$

is divisible by $r + 1$ for all such values of r, which implies

$$d_1 + 2d_2 + \cdots + (p-1)d_{p-1} = qd_{r-q+1} + \cdots + 2d_{r-1} + d_r. \tag{1.18}$$

We now define

$$\begin{aligned} d_\lambda(\mu) &= n - m - (d_1 + 2d_2 + \cdots + (p-1)d_{p-1}) \\ &= n - m - (qd_{r-q+1} + \cdots + 2d_{r-1} + d_r), \end{aligned} \tag{1.19}$$

and refer to $d_\lambda(\mu)$ as the *depth of* μ *with respect to* λ. It follows from (1.11) and (1.19) that

$$k_i = \begin{cases} d_{i+1} + 2d_{i+2} + \cdots + (p-i-1)d_{p-1} + d_\lambda(\mu) \\ \qquad \text{for } i = 0, 1, \ldots, p-2, \\ d_\lambda(\mu) \ \text{ for } i = p-1, p, \ldots, r-q, r-q+1, \\ (i - (r-q+1))d_{r-q+1} + \cdots + 2d_{i-2} + d_{i-1} + d_\lambda(\mu) \\ \qquad \text{for } i = r-q+2, \ldots, r-1, r. \end{cases} \tag{1.20}$$

For $i = 0, 1, \ldots, r$, let $m_i = k_i - d_\lambda(\mu)$, and define $\mu_0 = \lambda - \sum_{i=0}^{r} m_i\alpha_i$. Then $\mu = \mu_0 - d_\lambda(\mu)\delta$ and $d_\lambda(\mu_0) = 0$. Note that $\mu(h_i) = \mu_0(h_i) = b_i$ for all $i = 0, 1, \ldots, r$. By (1.20), we obtain

$$m_i = \begin{cases} d_{i+1} + 2d_{i+2} + \cdots + (p-i-1)d_{p-1} \\ \qquad \text{for } i = 0, 1, \ldots, p-2, \\ 0 \ \text{ for } i = p-1, p, \ldots, r-q, r-q+1, \\ (i - (r-q+1))d_{r-q+1} + \cdots + 2d_{i-2} + d_{i-1} \\ \qquad \text{for } i = r-q+2, \ldots, r-1, r. \end{cases} \tag{1.21}$$

In particular, the values of m_i's do not depend on r.

To summarize the above discussion, we have

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PROPOSITION 1.22. *Let $\lambda \in P^+$ be a dominant integral weight of level $l > 0$ with determining data $\underline{a} = (a_0, a_1, \ldots, a_{s-1})$, $\underline{a}' = (a_{r-t+1}, a_{r-t+2}, \ldots, a_r)$, and $m \in \mathbf{Z}$. Assume that $r \geqslant l + s + t$ and that μ is a dominant integral weight of level l with determining data $\underline{b} = (b_0, b_1, \ldots, b_{s'-1})$, $\underline{b}' = (b_{r-t'+1}, b_{r-t'+2}, \ldots, b_r)$, and $n \in \mathbf{Z}$ such that $s' + t' \leqslant r$, $s + t' \leqslant r$, and $s' + t \leqslant r$. If μ is related to λ for infinitely many values of $r \geqslant l + s + t$, then μ can be uniquely written as $\mu = \mu_0 - d_\lambda(\mu)\delta$, where $d_\lambda(\mu)$ and $\mu_0 = \lambda - \sum_{i=0}^{r} m_i\alpha_i$ are determined by (1.19) and (1.21) for $d_i = b_i - a_i$ ($i \in I$), and $d_\lambda(\mu_0) = 0$.*

The following lemma plays an important role in proving our main theorem (Theorem 3.4).

LEMMA 1.23. *Let $\lambda \in P^+$ be a dominant integral weight of level $l > 0$ with determining data*

$$\underline{a} = (a_0, a_1, \ldots, a_{s-1}), \qquad \underline{a}' = (a_{r-t+1}, a_{r-t+2}, \ldots, a_r), \quad \text{and} \quad m \in \mathbf{Z}.$$

Assume that $r \geqslant l + s + t$. Let $\mu \in P^+$ be a dominant integral weight of level l with determining data $\underline{b} = (b_0, b_1, \ldots, b_{s'-1})$, $\underline{b}' = (b_{r-t'+1}, b_{r-t'+2}, \ldots, b_r)$, and $n \in \mathbf{Z}$ such that $s' + t' \leqslant r$, $s + t' \leqslant r$, and $s' + t \leqslant r$. Let $\tau \in P^+$ be a dominant integral weight of level l with determining data

$$\underline{c} = (c_0, c_1, \ldots, c_{s''-1}), \qquad \underline{c}' = (c_{r-t''+1}, c_{r-t''+2}, \ldots, c_r), \qquad n' \in \mathbf{Z}$$

satisfying $s'' + t'' \leqslant r$, $s + t'' \leqslant r$, and $s'' + t \leqslant r$. If $\mu \leqslant \tau \leqslant \lambda$, then $d_\lambda(\tau) \leqslant d_\lambda(\mu)$.

Proof. By Proposition 1.22, μ and τ can be expressed as follows:

$$\mu = \lambda - \sum_{i=0}^{r} m_i\alpha_i - d_\lambda(\mu)\delta,$$

$$\tau = \lambda - \sum_{i=0}^{r} m'_i\alpha_i - d_\lambda(\tau)\delta.$$

The condition $\tau \geqslant \mu$ implies

$$\tau - \mu = \sum_{i=0}^{r}(m_i - m'_i)\alpha_i + (d_\lambda(\mu) - d_\lambda(\tau))\delta \in Q_+. \tag{1.24}$$

Since $a_j = 0$ for all $j = s, s+1, \ldots, r-t$, we have

$$d'_j = c_j - a_j = c_j \geqslant 0 \quad \text{for all } j = s, s+1, \ldots, r-t. \tag{1.25}$$

Let $p = \max(s, s')$, $q = \max(t, t')$, $x = \max(s, s'')$, and $y = \max(t, t'')$. Recall that $x \leqslant r - t$ and $s \leqslant r - y$, and assume that $i \in \{s-1, s, \ldots, r-t, r-t+1\}$. It suffices to consider the following three cases:

$$\text{(i) } s-1 \leqslant i \leqslant x-2, \quad \text{(ii) } x-1 \leqslant i \leqslant r-y+1, \quad \text{(iii) } r-y+2 \leqslant i \leqslant r-t+1.$$

If $s-1 \leqslant i \leqslant x-2$, then since $x \leqslant r-t$, it follows from (1.21) and (1.25) that

$$m_i' = d_{i+1}' + 2d_{i+2}' + \cdots + (x-i-1)d_{x-1}' \geqslant 0.$$

If $x-1 \leqslant i \leqslant r-y+1$, then by (1.21) $m_i' = 0$. Finally, if $r-y+2 \leqslant i \leqslant r-t+1$, then because $s \leqslant r-y$, (1.21) and (1.25) yield

$$m_i' = (i-(r-y+1))d_{r-y+1}' + \cdots + 2d_{i-2}' + d_{i-1}' \geqslant 0.$$

Therefore, $m_i' \geqslant 0$ for all $i = s-1, s, \ldots, r-t, r-t+1$. In particular, since $p \geqslant s$, $q \geqslant t$, we conclude that $m_i' \geqslant 0$ for all $i = p-1, p, \ldots, r-q, r-q+1$.

Observe that in (1.24) the coefficient of α_i in $\tau - \mu$ for $i = p-1, p, \ldots, r-q$, $r-q+1$ is

$$-m_i' + d_\lambda(\mu) - d_\lambda(\tau) \geqslant 0. \tag{1.26}$$

Since $m_i' \geqslant 0$ for $i = p-1, p, \ldots, r-q, r-q+1$, it must be that $d_\lambda(\tau) \leqslant d_\lambda(\mu)$. □

The following is a more detailed formulation of a conjecture presented in [BK].

CONJECTURE. Let $\lambda \in P^+$ *be a dominant integral weight of level* $l > 0$ *with determining data* $\underline{a} = (a_0, a_1, \ldots, a_{s-1})$, $\underline{a}' = (a_{r-t+1}, a_{r-t+2}, \ldots, a_r)$, *and* $m \in \mathbf{Z}$. *Assume that* $r \geqslant l+s+t+2$, *and let* $L(\lambda)$ *be the irreducible highest weight module over the affine Kac–Moody algebra* $\mathfrak{g}$ *of type* $A_r^{(1)}$ *with highest weight* λ. *Let* $\mu \in P^+$ *be a dominant integral weight of level* l *with determining data* $\underline{b} = (b_0, b_1, \ldots, b_{s'-1})$, $\underline{b}' = (b_{r-t'+1}, b_{r-t'+2}, \ldots, b_r)$, *and* $n \in \mathbf{Z}$ *such that* $s'+t' \leqslant r$, $s+t' \leqslant r$, *and* $s'+t \leqslant r$. *Suppose that* μ *is related to* λ *for infinitely many values of* $r \geqslant l+s+t+2$. *If* μ *is a weight of the* $\mathfrak{g}$*-module* $L(\lambda)$ *for some* $r_0 \geqslant l+s+t+2$, *then it is a weight of* $L(\lambda)$ *for all* $r \geqslant r_0$, *and the multiplicity of* μ *in* $L(\lambda)$ *is given by a polynomial in* r *of degree* $d_\lambda(\mu)$. *If* $d_\lambda(\mu) < 0$, *then the multiplicity of* μ *in* $L(\lambda)$ *is zero.*

In Section 3, we will prove a weaker version of the above conjecture. We will show that the multiplicity of μ in $L(\lambda)$ is given by a polynomial in r of degree $\leqslant d_\lambda(\mu)$. Our approach is to apply Peterson's formula to a certain indefinite Kac–Moody algebra L which will be constructed in the next section.

2. The Indefinite Kac–Moody Algebra L

Recall that the Cartan subalgebra $\mathfrak{h}$ of the affine Kac–Moody algebra $\mathfrak{g}$ of type $A_r^{(1)}$ has the basis $\{h_1, \ldots, h_r, c, d\}$, where $c = h_0 + h_1 + \cdots + h_r$ is the canonical central element, and there is a nondegenerate symmetric bilinear form $(\ |\ )$ on $\mathfrak{g}$ whose values on $\mathfrak{h}$ are given by (1.3). Since the form $(\,|\,)$ is nondegenerate on $\mathfrak{h}$, for every $\mu \in \mathfrak{h}^*$ there is a unique element t_μ in $\mathfrak{h}$ such that $\mu(h) = (h|t_\mu)$ for all $h \in \mathfrak{h}$. Thus the form $(\,|\,)$ induces a nondegenerate symmetric bilinear form on $\mathfrak{h}^*$, also denoted by $(\,|\,)$, defined by $(\mu|\nu) = (t_\mu|t_\nu)$ for all $\mu, \nu \in \mathfrak{h}^*$. In particular, $t_\delta = c$. We take a basis for $\mathfrak{h}$ and extend it to a basis $\{x_i|\ i \in \Omega\}$ of $\mathfrak{g}$ by adding basis elements for each of the root spaces $\mathfrak{g}_\alpha$. Since $(\mathfrak{g}_\alpha|\mathfrak{g}_\beta) = 0$ unless $\beta = -\alpha$, the dual basis $\{y_i|\ i \in \Omega\}$ of $\mathfrak{g}$ with respect to the form $(\,|\,)$ also consists of vectors in $\mathfrak{h}$ and root vectors.

Assume $\lambda = \sum_{i=0}^{r} a_i\Lambda_i - m\delta$ is a dominant integral weight of level $l > 0$ for $\mathfrak{g}$ as in Section 1. Let $L(\lambda)$ be the irreducible highest weight $\mathfrak{g}$-module with highest weight λ. The finite dual space $L^*(\lambda)$ is the irreducible lowest weight $\mathfrak{g}$-module with lowest weight $-\lambda$, where the $\mathfrak{g}$-module action is given by

$$\langle g \cdot v^*, w\rangle = -\langle v^*, g \cdot w\rangle \tag{2.1}$$

for $g \in \mathfrak{g}$, $v^* \in L^*(\lambda)$, $w \in L(\lambda)$ (see [K2, p. 149]). Define a linear map $\psi\colon L^*(\lambda) \otimes L(\lambda) \to \mathfrak{g}$ by

$$\psi(v^* \otimes w) = -\frac{2}{(\lambda|\lambda)} \sum_{i \in \Omega} \langle v^*, x_i \cdot w\rangle y_i, \tag{2.2}$$

where $\{x_i|\ i \in \Omega\}$ and $\{y_i|\ i \in \Omega\}$ are dual bases of $\mathfrak{g}$ as above. Then ψ is a well-defined $\mathfrak{g}$-module homomorphism, (compare with [FF], [Ka1], [BKM1]), and hence the space $L(\lambda) \oplus \mathfrak{g} \oplus L^*(\lambda)$ has the structure of a local Lie algebra with the bracket defined by

$$\begin{aligned} &[v^*, w] = \psi(v^* \otimes w), \\ &[g, w] = g \cdot w, \quad [g, v^*] = g \cdot v^* \end{aligned} \tag{2.3}$$

for $g \in \mathfrak{g}$, $v^* \in L^*(\lambda)$, $w \in L(\lambda)$ (see [K1]).

Let F_+ (resp. F_-) be the free Lie algebra generated by $L^*(\lambda)$ (resp. $L(\lambda)$), and for $k \geqslant 1$, let F_k (resp. F_{-k}) be the subspace of F_+ (resp. F_-) spanned by the vectors of the form $[u_1[u_2[\ldots[u_{k-1}, u_k]\ldots]]]$ with $u_j \in L^*(\lambda)$ (resp. $L(\lambda)$). In particular, $F_1 = L^*(\lambda)$ and $F_{-1} = L(\lambda)$. Let $F_0 = \mathfrak{g}$ and define $F = F_- \oplus F_0 \oplus F_+ = \bigoplus_{k \in \mathbf{Z}} F_k$. Then F is the maximal graded Lie algebra with local part $L(\lambda) \oplus \mathfrak{g} \oplus L^*(\lambda)$.

For $k \geqslant 2$, define the subspaces $J_{\pm k}$ of $F_{\pm k}$ by

$$J_{\pm k} = \{v \in F_{\pm k}|\ [u_1[u_2[\ldots[u_{k-1}, v]\ldots]]] = 0 \quad \text{forall } u_i \in F_{\mp 1}\}, \tag{2.4}$$

and let $J_\pm = \bigoplus_{k \geqslant 2} J_{\pm k}$. Then $J_\pm$ is a graded ideal of $F_\pm$, and $J = J_- \oplus J_+$ is the maximal graded ideal of F which intersects the local part $L(\lambda) \oplus \mathfrak{g} \oplus L^*(\lambda)$ trivially ([K1], [FF], [Ka1], [BKM1]). The Lie algebra $L = F/J = \bigoplus_{k \in \mathbf{Z}} L_k$ is the minimal graded Lie algebra with local part $L(\lambda) \oplus \mathfrak{g} \oplus L^*(\lambda)$, where $L_k = F_k/J_k$ for $k \in \mathbf{Z}$. In particular, $L_{-1} = F_{-1} = L(\lambda)$, $L_0 = \mathfrak{g}$, and $L_1 = F_1 = L^*(\lambda)$.

Alternately, let $\alpha_{-1} = -\lambda$ and consider the Cartan matrix $\hat{\mathfrak{A}} = (a_{i,j})$ $(i, j = -1, 0, 1, \ldots, r)$ given by

$$a_{i,j} = \frac{2(\alpha_i|\alpha_j)}{(\alpha_i|\alpha_i)} \quad \text{for } i, j = -1, 0, 1, \ldots, r. \tag{2.5}$$

The first column of the matrix $\hat{\mathfrak{A}}$ consists of the entries 2, $-a_0$, $-a_1$, ..., $-a_r$, and deleting the first row and the first column of $\hat{\mathfrak{A}}$ gives the affine Cartan matrix of type $A_r^{(1)}$. If we let $h_{-1} = -2t_\lambda/(\lambda|\lambda)$, where $t_\lambda \in \mathfrak{h}$ is such that $\lambda(h) = (h|t_\lambda)$ for all $h \in \mathfrak{h}$, then the triple $(\mathfrak{h}, \Pi = \{\alpha_{-1}, \alpha_0, \alpha_1, \ldots, \alpha_r\}, \Pi^\vee = \{h_{-1}, h_0, h_1, \ldots, h_r\})$ provides a realization of the matrix $\hat{\mathfrak{A}}$. Let $\hat{\mathfrak{g}}$ be the indefinite Kac–Moody algebra associated with the Cartan matrix $\hat{\mathfrak{A}}$. It is a direct consequence of the Gabber-Kac theorem (see [GK]) that the following holds:

PROPOSITION 2.6 ([FF], [Ka1], [BKM1]). Let v_0 (resp. v_0^*) be the highest (resp. lowest) weight vector of $L(\lambda)$ (resp. $L^*(\lambda)$) such that $\langle v_0^*, v_0\rangle = 1$. If $\hat{\mathfrak{g}}$ is the indefinite Kac–Moody algebra with the Cartan matrix $\hat{\mathfrak{A}}$ given by (2.5), then there is an isomorphism of Lie algebras $\hat{\mathfrak{g}} \cong L$ defined by

$$\begin{aligned} &e_i \longmapsto e_i, \quad f_i \longmapsto f_i, \quad h_i \longmapsto h_i \quad \text{for } i = 0, 1, \ldots, r, \\ &e_{-1} \longmapsto v_0^*, \quad f_{-1} \longmapsto v_0, \quad h_{-1} \longmapsto -\frac{2t_\lambda}{(\lambda|\lambda)}. \end{aligned} \tag{2.7}$$

It follows from Proposition 2.6 that the subspace $L_{\pm k}$ is the sum of all the root spaces $\hat{\mathfrak{g}}_{\pm\alpha}$, where α is of the form $\pm(k\alpha_{-1} + \sum_{i=0}^r k_i\alpha_i)$ with $k, k_i \in \mathbf{Z}_{\geqslant 0}$. In particular, the roots of the form $\pm(\sum_{i=0}^r k_i\alpha_i)$ are roots of the affine Kac–Moody algebra $\mathfrak{g}$, and the roots of the form $-\alpha_{-1} - \sum_{i=0}^r k_i\alpha_i$ are weights of the irreducible highest weight $\mathfrak{g}$-module $L(\lambda)$. Thus to compute the weight multiplicity of $\lambda - \sum_{i=0}^r k_i\alpha_i$ in $L(\lambda)$, it suffices to compute the root multiplicity of $-\alpha_{-1} - \sum_{i=0}^r k_i\alpha_i$ in the indefinite Kac–Moody algebra $L \cong \hat{\mathfrak{g}}$.

3. The Weight Multiplicity Polynomials

In this section, we will prove our main result. Fix a positive integer l and a dominant integral weight λ of level l with determining data $\underline{a} = (a_0, a_1, \ldots, a_{s-1})$, $\underline{a}' = (a_{r-t+1}, a_{r-t+2}, \ldots, a_r)$, and $m \in \mathbf{Z}$. Assume that $r \geqslant l + s + t + 2$, and let $L(\lambda)$ be the irreducible highest weight module over the affine Kac–Moody algebra $\mathfrak{g} = A_r^{(1)}$ with highest weight λ. Suppose μ is a dominant integral weight of level l with determining data $\underline{b} = (b_0, b_1, \ldots, b_{s'-1})$, $\underline{b}' = (b_{r-t'+1}, b_{r-t'+2}, \ldots, b_r)$, and $n \in \mathbf{Z}$, where s', t' are nonnegative integers satisfying $s' + t' \leqslant r$, $s + t' \leqslant r$, and $s' + t \leqslant r$. Since the determining data associated to λ and μ is fixed, the integers s, t, s', t' are all fixed. In particular, the integers $p = \max(s, s')$ and $q = \max(t, t')$ are fixed also. Suppose further that μ is related to λ for infinitely many values of $r \geqslant l + s + t + 2$. Our aim is to prove that if μ is a weight of $L(\lambda)$ for some $r_0 \geqslant l + s + t + 2$, then it is a weight of $L(\lambda)$ for all $r \geqslant r_0$, and the multiplicity of μ in $L(\lambda)$ is a polynomial in r of degree $\leqslant d_\lambda(\mu)$, the depth of μ with respect to λ.

Let $L \cong \hat{\mathfrak{g}}$ be the minimal graded Lie algebra with local part $L(\lambda) \oplus \mathfrak{g} \oplus L^*(\lambda)$ constructed in Section 2. Let $\hat{Q} = \bigoplus_{i=-1}^r \mathbf{Z}\alpha_i$ denote the root lattice of $\hat{\mathfrak{g}}$ with respect to the Cartan subalgebra $\mathfrak{h}$. The roots of $\hat{\mathfrak{g}}$ belong to $\hat{Q}_+ \cup \hat{Q}_-$, where $\hat{Q}_+ = \bigoplus_{i=-1}^{r} \mathbf{Z}_{\geqslant 0}\alpha_i = -\hat{Q}_-$. Furthermore, $\mu = \lambda - \sum_{i=0}^{r} k_i\alpha_i = -\alpha_{-1} - \sum_{i=0}^{r} k_i\alpha_i \in \hat{Q}_-$, where the coefficients k_i are as in (1.20). The weight multiplicity of μ in $L(\lambda)$ is the same as the root multiplicity $\mathrm{mult}(\mu)$ of μ in L, which can be computed using the following Freudenthal-type recursive formula due to Peterson.

PROPOSITION 3.1 ([P], cf. [K2, Exercise 11.12]). *For $\beta \in \hat{Q}_-$, define*

$$c_\beta = \sum_{n \geqslant 1} \frac{1}{n}\mathrm{mult}\left(\frac{\beta}{n}\right).$$

Then

$$(\beta|\beta + 2\rho)c_\beta = \sum_{\substack{\beta',\beta'' \in \hat{Q}_- \\ \beta = \beta' + \beta''}} (\beta'|\beta'')c_{\beta'}c_{\beta''}, \tag{3.2}$$

where $\rho \in \mathfrak{h}^*$ is such that $\rho(h_i) = 1$ for $i = -1, 0, \ldots, r$.

We write $\mu = \mu_0 - d_\lambda(\mu)\delta$, where $d_\lambda(\mu)$ and $\mu_0 = -\alpha_{-1} - \sum_{i=0}^{r} m_i\alpha_i$ are given by (1.19) and (1.21). Since the coefficient of α_{-1} in μ is -1, any decomposition is of the form $\mu = \beta' + \beta''$, where

$$\beta' = -\alpha_{-1} - \sum_{i=0}^{r} s_i\alpha_i \quad \text{and} \quad \beta'' = -\sum_{i=0}^{r} t_i\alpha_i$$

with $s_i, t_i \in \mathbf{Z}_{\geqslant 0}$, or it has this form with the roles of β' and β'' switched. Note that $c_\mu = \mathrm{mult}(\mu)$ and $c_{\beta'} = \mathrm{mult}(\beta')$, where $\mathrm{mult}(\cdot)$ is the multiplicity in L, which, for μ and β', is the same as the multiplicity in $L(\lambda)$. Thus, in order to have a nontrivial contribution to $c_{\beta'}$ and $c_{\beta''}$, β' must be a weight of $L(\lambda)$ and $\beta'' = -k\alpha$ for some $k \geqslant 1$, where α is a positive root of $\mathfrak{g}$.

LEMMA 3.3. *Suppose $d_\lambda(\mu) > 0$. Then $(\mu|\mu + 2\rho)$ is a polynomial in r of degree 1.*

Proof. Since $\mu = -\alpha_{-1} - \sum_{i=0}^{r} m_i\alpha_i - d_\lambda(\mu)\delta$, we have

$$\begin{aligned}(\mu|\mu + 2\rho) &= \left(-\alpha_{-1} - \sum_{i=0}^{r} m_i\alpha_i - d_\lambda(\mu)\delta \,\middle|\, -\alpha_{-1} - \sum_{i=0}^{r} m_i\alpha_i - d_\lambda(\mu)\delta + 2\rho\right) \\ &= -2d_\lambda(\mu)(r+1) - 2\sum_{i=0}^{r} m_i a_i - 2\sum_{i=0}^{r} m_i + \sum_{i,j=0}^{r} m_i m_j a_{i,j} - 2l d_\lambda(\mu) + \\ &\quad + (\alpha_{-1}|\alpha_{-1}) - 2(\rho|\alpha_{-1}).\end{aligned}$$

By (1.21), the terms $\sum_{i=0}^{r} m_i a_i$, $\sum_{i=0}^{r} m_i$, $\sum_{i,j=0}^{r} m_i m_j a_{i,j}$ are all constants. Therefore $(\mu|\mu + 2\rho)$ is a polynomial in r of degree 1. □

We now state and prove our main result.

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THEOREM 3.4. *Let $\mathfrak{g}$ be the affine Kac–Moody algebra of type $A_r^{(1)}$, and let $\lambda \in P^+$ be a dominant integral weight of level $l > 0$ for $\mathfrak{g}$ with determining data $\underline{a} = (a_0, a_1, \ldots, a_{s-1})$, $\underline{a}' = (a_{r-t+1}, a_{r-t+2}, \ldots, a_r)$, and $m \in \mathbf{Z}$. Assume that $r \geqslant l + s + t + 2$, and let $\mu \in P^+$ be a dominant integral weight of level l with determining data $\underline{b} = (b_0, b_1, \ldots, b_{s'-1})$, $\underline{b}' = (b_{r-t'+1}, b_{r-t'+2}, \ldots, b_r)$, and $n \in \mathbf{Z}$ such that $s' + t' \leqslant r$, $s + t' \leqslant r$, and $s' + t \leqslant r$. Suppose that μ is related to λ for infinitely many values of $r \geqslant l + s + t + 2$. If μ is a weight of $L(\lambda)$ for some $r_0 \geqslant l + s + t + 2$, then it is a weight of $L(\lambda)$ for all $r \geqslant r_0$, and the multiplicity of μ in $L(\lambda)$ is given by a polynomial in r of degree $\leqslant d_\lambda(\mu)$.*

Proof. We will prove our assertion by induction on $d_\lambda(\mu)$ and on the partial ordering on the affine weight lattice. Write $\mu = -\alpha_{-1} - \sum_{i=0}^{r} m_i\alpha_i - d_\lambda(\mu)\delta$, where $d_\lambda(\mu)$ and the m_i's are given by (1.19) and (1.21).

If $d_\lambda(\mu) < 0$, then for $i = p-1, p, \ldots, r-q, r-q+1$, the coefficient of α_i in μ is positive. Hence μ cannot be a weight of $L(\lambda)$, and therefore its multiplicity in $L(\lambda)$ is zero.

Let $p = \max(s, s')$, $q = \max(t, t')$. Suppose that $d_\lambda(\mu) = 0$. Then

$$\mu = \lambda - \sum_{i=0}^{p-2} m_i\alpha_i - \sum_{i=r-q+2}^{r} m_i\alpha_i.$$

The multiplicity of μ in $L(\lambda)$ is the number of linearly independent vectors of the form $f_{i_1}f_{i_2}\cdots f_{i_k}\cdot v_0$, where v_0 is the highest weight vector of $L(\lambda)$ and f_j appears m_j times in the expression for each $j \in \{0, 1, \ldots, p-2, r-q+2, r-q+3, \ldots, r\}$. Clearly, this number is independent of r (it may be 0). In particular, if this number is nonzero for some $r_0 \geqslant l + s + t + 2$, then it is nonzero and constant for all $r \geqslant r_0$. Therefore, if μ is a weight of $L(\lambda)$ for some $r_0 \geqslant l + s + t + 2$, then it is a weight of $L(\lambda)$ for all $r \geqslant r_0$, and the multiplicity of μ is a constant. The same argument shows that any $\tau \in P$ such that $d_\lambda(\tau) = 0$ has a constant multiplicity (which may be 0) in $L(\lambda)$.

Suppose $d_\lambda(\mu) \geqslant 1$. Consider a dominant integral weight τ of level l with determining data $\underline{c} = (c_0, c_1, \ldots, c_{s''-1})$, $\underline{c}' = (c_{r-t''+1}, c_{r-t''+2}, \ldots, c_r)$, and $n' \in \mathbf{Z}$ such that $s'' + t'' \leqslant r$, $s + t'' \leqslant r$, and $s'' + t \leqslant r$ which is related to λ for infinitely many values of $r \geqslant l + s + t + 2$, and write $\tau = \tau_0 - d_\lambda(\tau)\delta$, where $d_\lambda(\tau_0) = 0$. Assume that if $d_\lambda(\tau) < d_\lambda(\mu)$ or if $d_\lambda(\tau) = d_\lambda(\mu)$ and $\tau_0 > \mu_0$, our assertion holds for τ. That is, we assume that if τ is a weight of $L(\lambda)$ for some $r_0 \geqslant l + s + t + 2$, then it is a weight of $L(\lambda)$ for all $r \geqslant r_0$, and the multiplicity of τ in $L(\lambda)$ is a polynomial in r of degree $\leqslant d_\lambda(\tau)$.

Consider a decomposition $\mu = \beta' + \beta''$, where $\beta' \in -\alpha_{-1} - Q_- = \lambda - Q_-$ and β'' is a multiple of a negative root of $\mathfrak{g}$. Thus we may assume that β'' is one of the following:

$$\text{(i) } -k\delta, \quad \text{(ii) } -k\gamma, \quad \text{(iii) } -k(k'\delta + \gamma), \quad \text{(iv) } -k(k'\delta - \gamma), \tag{3.5}$$

where $k, k' \geqslant 1$ and γ is a positive root of $\mathfrak{g}_0 = A_r$.

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Note that $\beta'' = -k\delta$ is an imaginary root of $\mathfrak{g}$ for all $k \geqslant 1$, and its multiplicity in $\mathfrak{g}$ (and hence in L) is r (see [K1, Cor. 7.4]). It follows that

$$\begin{aligned} c_{-k\delta} &= \sum_{m \geqslant 1} \frac{\operatorname{mult}(-k\delta/m)}{m} = \sum_{m|k} \frac{\operatorname{mult}(-(k/m)\delta)}{m} \\ &= \sum_{m|k} \frac{r}{m} = \frac{\xi(k)}{k} r, \end{aligned} \tag{3.6}$$

where $\xi(k)$ denotes the sum of all factors of k. On the other hand, the roots γ, $k'\delta \pm \gamma$ are real, and their multiplicities in $\mathfrak{g}$ (and hence in L) are all 1. Moreover, if $k \geqslant 2$, the multiplicities of $k\gamma$ and $k(k'\delta \pm \gamma)$ in $\mathfrak{g}$ are all 0. Therefore we have

$$c_{\beta''} = \frac{1}{k} \quad \text{if } \beta'' = -k\gamma \text{ or } \beta'' = -k(k'\delta \pm \gamma). \tag{3.7}$$

We now treat the four cases separately.

Case 1. Suppose first that $\beta'' = -k\delta$ for $k \geqslant 1$. In this case,

$$\beta' = -\alpha_{-1} - \sum_{i=0}^{r} m_i\alpha_i - (d_\lambda(\mu) - k)\delta \in \hat{Q}_-.$$

Since $m_i = 0$ for $i = p-1, p, \dots, r-q, r-q+1$, we must have $d_\lambda(\mu) - k \geqslant 0$ in order for β' to belong to $\hat{Q}_-$, which implies that k runs from 1 to $d_\lambda(\mu)$. Now

$$\begin{aligned} (\beta'|\beta'') &= \left(-\alpha_{-1} - \sum_{i=0}^{r} m_i\alpha_i - (d_\lambda(\mu) - k)\delta \middle| -k\delta\right) \\ &= k(\alpha_{-1}|\delta) = -kl, \end{aligned}$$

where l denotes the level. As we have seen before, if β' is not a weight of $L(\lambda)$, then $c_{\beta'} = 0$ and there is no contribution to the right-hand side of (3.2). So we may assume that β' is a weight of $L(\lambda)$ for some $r_0 \geqslant l+s+t+2$. Observe that β' is dominant since $\beta'(h_j) = \mu(h_j) \geqslant 0$ for all $j \in I$. Since β' is related to λ for infinitely many values of $r \geqslant l+s+t+2$, and since $d_\lambda(\mu) - k < d_\lambda(\mu)$, it follows from the induction hypothesis that β' is a weight of $L(\lambda)$ for all $r \geqslant r_0$, and $\operatorname{mult}(\beta')$ is a polynomial in r of degree $\leqslant d_\lambda(\mu) - k \leqslant d_\lambda(\mu) - 1$. We have seen in (3.6) that $c_{\beta''} = r\xi(k)/k$, which is a polynomial in r of degree 1. Therefore, the total contribution of the various decompositions of this kind to the right-hand side of (3.2) is a polynomial in r of degree $\leqslant d_\lambda(\mu)$.

Case 2. Suppose $\beta'' = -k\gamma$ for $k \geqslant 1$, where γ is a positive root of $\mathfrak{g}_0 = A_r$. Thus $\gamma = \alpha_u + \alpha_{u+1} + \cdots + \alpha_v$ with $1 \leqslant u \leqslant v \leqslant r$. In this case, we have

$$\beta' = -\alpha_{-1} - \sum_{i=0}^{r} m_i\alpha_i + k\gamma - d_\lambda(\mu)\delta \in \hat{Q}_-.$$

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Note that for all $i = u, u+1, \ldots, v$, the coefficient of α_i in β' must be $\leqslant 0$. That is, $-m_i + k - d_\lambda(\mu) \leqslant 0$ for $i = u, u+1, \ldots, v$. Let $M = \max\{m_i | \ 0 \leqslant i \leqslant r\}$. Then $k \leqslant M + d_\lambda(\mu)$, and hence k ranges from 1 to $M + d_\lambda(\mu)$. Note that M is independent of the value of r. We also have

$$\begin{aligned}(\beta'|\beta'') &= (-\alpha_{-1} - \sum_{i=0}^{r} m_i\alpha_i + k\gamma - d_\lambda(\mu)\delta| - k\gamma)\\ &= k(\alpha_{-1}|\gamma) + k\sum_{i=0}^{r} m_i(\alpha_i|\gamma) - k^2(\gamma|\gamma).\end{aligned}$$

Hence $(\beta'|\beta'')$ is a constant for each k, because $(\lambda|\alpha_i) = a_i = 0$ and $m_i = 0$ for $i = p, p+1, \ldots, r-q$,

Now by (1.5), $\gamma = \alpha_u + \alpha_{u+1} + \cdots + \alpha_v = -\Lambda_{u-1} + \Lambda_u + \Lambda_v - \Lambda_{v+1}$. Observe that

$$\mathfrak{r}_{u-1}(-\Lambda_{u-1} + \Lambda_u + \Lambda_v - \Lambda_{v+1}) = -\Lambda_{u-2} + \Lambda_{u-1} + \Lambda_v - \Lambda_{v+1},$$

and

$$\mathfrak{r}_{v+1}(-\Lambda_{u-1} + \Lambda_u + \Lambda_v - \Lambda_{v+1}) = -\Lambda_{u-1} + \Lambda_u + \Lambda_{v+1} - \Lambda_{v+2},$$

where $\mathfrak{r}_i$ denotes the simple reflection corresponding to the root α_i. Therefore for each $r \geqslant l + s + t + 2$, we apply the simple reflections $\mathfrak{r}_{u-1}, \mathfrak{r}_{u-2}, \ldots, \mathfrak{r}_1, \mathfrak{r}_0, \mathfrak{r}_r, \ldots$ and then $\mathfrak{r}_{v+1}, \mathfrak{r}_{v+2}, \ldots, \mathfrak{r}_r, \mathfrak{r}_0, \mathfrak{r}_1, \ldots$ in succession to get a dominant integral weight. (It may take several rounds of applying the simple reflections in this order to produce a dominant integral weight.) Let w_r denote the corresponding Weyl group element of $A_r^{(1)}$. We can verify that $w_r\beta'$ has the form

$$w_r\beta' = \sum_{i=0}^{r} b_i'(r)\Lambda_i - n'(r)\delta,$$

where $b_i'(r) = 0$ for $i = p+1, \ldots, r-q-1$. Moreover, it is tedious but straightforward to show that the sequences of integers $\underline{c}(r) = (b_0'(r), b_1'(r), \ldots, b_p'(r); b_{r-q}'(r), b_{r-q+1}'(r), \ldots, b_r'(r); n'(r))$ are the same for all $r \geqslant l + s + t + 2$. That is, the determining data of $w_r\beta'$ is given by $\underline{c} = (b_0', b_1', \ldots, b_x')$, $\underline{c}' = (b_{r-y}', b_{r-y+1}', \ldots, b_r')$, and $n' \in \mathbf{Z}$, where $x \leqslant p$, $y \leqslant q$ and $b_i' = b_i'(r)$, $n' = n'(r)$ (for all $r \geqslant l + s + t + 2$). Rather than writing $w_r\beta'$ in what follows, we denote the dominant weight determined by this data as τ.

Since τ is related to λ, τ also has level l, and if we let $d_i' = b_i' - a_i$ for $i = 0, 1, \ldots, r$, then $a_i = b_i' = d_i' = 0$ for $i = p+1, p+2, \ldots, r-q-1$. Hence we may write

$$\tau = -\alpha_{-1} - \sum_{i=0}^{r} m_i'\alpha_i - d_\lambda(\tau)\delta,$$

where $-\alpha_{-1} - \sum_{i=0}^{r} m_i'\alpha_i$ has depth 0 with respect to λ. In addition, since τ is the highest element among the Weyl group conjugates of β', the inequality

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$\mu < \beta' \leqslant \tau$ must hold, and hence

$$\tau - \mu = \sum_{i=0}^{r}(m_i - m_i')\alpha_i + (d_\lambda(\mu) - d_\lambda(\tau))\delta \in \hat{Q}_+, \tag{3.8}$$

and

$$\tau - \beta' = \sum_{i=0}^{r}(m_i - m_i')\alpha_i - k\gamma + (d_\lambda(\mu) - d_\lambda(\tau))\delta \in \hat{Q}_+. \tag{3.9}$$

By the same argument as in Lemma 1.23, we can show that $m_i' \geqslant 0$ for all $i = p-1, p, \ldots, r-q, r-q+1$ and, hence, $d_\lambda(\tau) \leqslant d_\lambda(\mu)$.

If $d_\lambda(\tau) = d_\lambda(\mu)$, then $m_i' \leqslant 0$ by (1.26), and hence $m_i' = 0$ for $i = p-1, p, \ldots, r-q, r-q+1$. Since $m_i = 0$ for $i = p-1, p, \ldots, r-q, r-q+1$, (3.9) can be written as

$$\tau - \beta' = \sum_{i=0}^{p-2}(m_i - m_i')\alpha_i + \sum_{i=r-q+2}^{r}(m_i - m_i')\alpha_i - k\gamma \in \hat{Q}_+. \tag{3.10}$$

Thus, in order for $\tau - \beta'$ to be an element of $\hat{Q}_+$, γ must be a linear combination of the simple roots $\alpha_1, \alpha_2, \ldots, \alpha_{p-2}$, or of $\alpha_{r-q+2}, \ldots, \alpha_{r-1}, \alpha_r$. The number of such γ is at most

$$\frac{(p-2)(p-1)}{2} + \frac{(q-1)q}{2},$$

which is independent of r. Since $\tau > \mu$ and $d_\lambda(\tau) = d_\lambda(\mu)$, we have $\tau_0 > \mu_0$. Hence by the induction hypothesis, if $w_{r_0}\beta'$ is a weight of $L(\lambda)$ for some $r_0 \geqslant l + s + t + 2$, then $\tau = w_r\beta'$ is a weight of $L(\lambda)$ for all $r \geqslant r_0$, and $\text{mult}(\beta') = \text{mult}(\tau)$ is a polynomial in r of degree $\leqslant d_\lambda(\mu)$. Note that $c_{\beta''} = 1/k$ for all $k = 1, \ldots, M + d_\lambda(\mu)$. Therefore, the contribution of these partitions to the right-hand side of (3.2) is a polynomial in r of degree $\leqslant d_\lambda(\mu)$.

Suppose that $d_\lambda(\tau) < d_\lambda(\mu)$. By the induction hypothesis, if $w_{r_0}\beta'$ is a weight of $L(\lambda)$ for some $r_0 \geqslant l + s + t + 2$, then $\tau = w_r\beta'$ is a weight of $L(\lambda)$ for all $r \geqslant r_0$, and $\text{mult}(\beta') = \text{mult}(\tau)$ is a polynomial of degree $\leqslant d_\lambda(\tau) \leqslant d_\lambda(\mu) - 1$. Since there are $(r(r+1)/2)$ positive roots in $\mathfrak{g}_0 = A_r$, a polynomial in r of degree 2, and since $c_{\beta''} = 1/k$ for all $k = 1, \ldots, M + d_\lambda(\mu)$, the contribution of these decompositions to the right-hand side of (3.2) is a polynomial in r of degree $\leqslant d_\lambda(\mu) + 1$.

Therefore, the total contribution of the partitions in Case 2 to the right-hand side of (3.2) is a polynomial in r of degree $\leqslant d_\lambda(\mu) + 1$.

Case 3. Suppose that $\beta'' = -k(k'\delta + \gamma)$, where $k, k' \geqslant 1$ and γ is a positive root of $\mathfrak{g}_0 = A_r$. In this case,

$$\begin{aligned}\beta' &= -\alpha_{-1} - \sum_{i=0}^{r} m_i\alpha_i + k(k'\delta + \gamma) - d_\lambda(\mu)\delta \\ &= -\alpha_{-1} - \sum_{i=0}^{r} m_i\alpha_i + k\gamma - (d_\lambda(\mu) - kk')\delta \in \hat{Q}_-.\end{aligned}$$

Observe that the coefficient of α_0 in β' is $-m_0 - d_\lambda(\mu) + kk'$, which must be $\leqslant 0$. Thus $kk' \leqslant m_0 + d_\lambda(\mu)$, and hence k, k' range from 1 to $m_0 + d_\lambda(\mu)$. We have

$$\begin{aligned}(\beta'|\beta'') &= \left(-\alpha_{-1} - \sum_{i=0}^{r} m_i\alpha_i + k\gamma - (d_\lambda(\mu) - kk')\delta| - k(k'\delta + \gamma)\right) \\ &= -kk'l + k(\alpha_{-1}|\gamma) + k\sum_{i=0}^{r} m_i(\alpha_i|\gamma) - k^2(\gamma|\gamma),\end{aligned}$$

which can be seen to be a constant as in Case 2.

Moreover, by the same argument as in Case 2, for each $r \geqslant l + s + t + 2$, we can verify that β' is Weyl group conjugate to a dominant integral weight $\tau = w_r\beta'$ that has the form $\tau = \sum_{i=0}^{r} b_i'\Lambda_i - n'\delta$, where $b_i' = 0$ for $i = p+1, \ldots, r-q-1$. So if we let $d_i' = b_i' - a_i$ $(i = 0, 1, \ldots, r)$, we may write

$$\tau = -\alpha_{-1} - \sum_{i=0}^{r} m_i'\alpha_i - d_\lambda(\tau)\delta,$$

where $-\alpha_{-1} - \sum_{i=0}^{r} m_i'\alpha_i$ has depth 0 with respect to λ. In addition, we have

$$\tau - \mu = \sum_{i=0}^{r}(m_i - m_i')\alpha_i + (d_\lambda(\mu) - d_\lambda(\tau))\delta \in \hat{Q}_+, \tag{3.11}$$

and

$$\begin{aligned}\tau - \beta' = \sum_{i=0}^{r}(m_i - m_i')\alpha_i - k\gamma + \\ + (d_\lambda(\mu) - d_\lambda(\tau) - kk')\delta \in \hat{Q}_+.\end{aligned} \tag{3.12}$$

By the same argument as in Lemma 1.23, we can show that $m_i' \geqslant 0$ for all $i = p-1, p, \ldots, r-q, r-q+1$ and, hence, $d_\lambda(\mu) - d_\lambda(\tau) - kk' \geqslant 0$. Therefore,

$$d_\lambda(\tau) \leqslant d_\lambda(\mu) - kk' \leqslant d_\lambda(\mu) - 1.$$

By the induction hypothesis, if $w_{r_0}\beta'$ is a weight of $L(\lambda)$ for some $r_0 \geqslant l + s + t + 2$, then $\tau = w_r\beta'$ is a weight of $L(\lambda)$ for all $r \geqslant r_0$, and $\mathrm{mult}(\beta') = \mathrm{mult}(\tau)$ is a polynomial in r of degree $\leqslant d_\lambda(\tau) \leqslant d_\lambda(\mu) - 1$. Since there are $(r(r+1)/2)$ positive roots in $\mathfrak{g}_0$ and

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since $c_{\beta''} = 1/k$ for all $k = 1, \ldots, m_0 + d_\lambda(\mu)$, the total contribution of the partitions in this case to the right side of (3.2) is a polynomial in r of degree $\leqslant d_\lambda(\mu) + 1$.

Case 4. Suppose $\beta'' = -k(k'\delta - \gamma)$, where $k, k' \geqslant 1$ and γ is a positive root of $\mathfrak{g}_0 = A_r$. In this case,

$$\beta' = -\alpha_{-1} - \sum_{i=0}^{r} m_i\alpha_i - k\gamma - (d_\lambda(\mu) - kk')\delta \in \hat{Q}_-.$$

As in Case 3, by looking at the coefficient of α_0 in the above expression, we can show that k and k' range from 1 to $m_0 + d_\lambda(\mu)$. We have

$$\begin{aligned}(\beta'|\beta'') &= \left(-\alpha_{-1} - \sum_{i=0}^{r} m_i\alpha_i - k\gamma - (d_\lambda(\mu) - kk')\delta | - k(k'\delta - \gamma)\right) \\ &= -kk'l - k(\alpha_{-1}|\gamma) - \sum_{i=0}^{r} m_i(\alpha_i|\gamma) - k^2(\gamma|\gamma),\end{aligned}$$

which can be seen to be a constant as in Case 2.

By the identical argument as in Case 2 we can verify for each $r \geqslant l + s + t + 2$ that β' is Weyl group conjugate to a dominant integral weight $\tau = w_r\beta'$ that has the form $\tau = \sum_{i=0}^{r} b'_i\Lambda_i - n'\delta$, where $b'_i = 0$ for $i = p+1, \ldots, r-q-1$. So if we suppose as before $d'_i = b'_i - a_i$ $(i = 0, 1, \ldots, r)$, then

$$\tau = -\alpha_{-1} - \sum_{i=0}^{r} m'_i\alpha_i - d_\lambda(\tau)\delta,$$

where $-\alpha_{-1} - \sum_{i=0}^{r} m'_i\alpha_i$ has depth 0 with respect to λ. Moreover,

$$\tau - \mu = \sum_{i=0}^{r}(m_i - m'_i)\alpha_i + (d_\lambda(\mu) - d_\lambda(\tau))\delta \in \hat{Q}_+, \tag{3.13}$$

and

$$\tau - \beta' = \sum_{i=0}^{r}(m_i - m'_i)\alpha_i + k\gamma + (d_\lambda(\mu) - d_\lambda(\tau) - kk')\delta \in \hat{Q}_+. \tag{3.14}$$

Let us write $\gamma = \alpha_u + \alpha_{u+1} + \cdots + \alpha_v$ with $1 \leqslant u \leqslant v \leqslant r$. Then (3.14) becomes

$$\begin{aligned}\tau - \beta' &= \sum_{i=0}^{r}(m_i - m'_i)\alpha_i - k(\alpha_0 + \alpha_1 + \cdots + \alpha_{u-1}) - \\ &\quad - k(\alpha_{v+1} + \cdots + \alpha_r) + (d_\lambda(\mu) - d_\lambda(\tau) + k - kk')\delta \in \hat{Q}_+.\end{aligned} \tag{3.15}$$

The argument in Lemma 1.23 proves that $m'_i \geqslant 0$ for all $i = p-1, p, \ldots, r-q$, $r-q+1$ and that $d_\lambda(\mu) - d_\lambda(\tau) + k - kk' \geqslant 0$, which yields

$$d_\lambda(\tau) \leqslant d_\lambda(\mu) + k - kk' \leqslant d_\lambda(\mu).$$

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If $d_\lambda(\tau) = d_\lambda(\mu)$, then we must have $k' = 1$ and (3.15) can be written as

$$\begin{aligned}\tau - \beta' = \sum_{i=0}^{r}(m_i - m_i')\alpha_i - \\ - k(\alpha_0 + \alpha_1 + \cdots + \alpha_{u-1}) - k(\alpha_{v+1} + \cdots + \alpha_r) \in \hat{Q}_+.\end{aligned} \tag{3.16}$$

Recall that $m_i = 0$ and $m_i' \geqslant 0$ for $i = p-1, p, \ldots, r-q, r-q+1$. Thus in order for $\tau - \beta'$ to belong to $\hat{Q}_+$, it must be that $m_i' = 0$ for all $i = p-1, p, \ldots, r-q, r-q+1$, and $u \leqslant p-2$, $v \geqslant r-q+1$. Hence, the number of such γ is at most $(p-2)(q-1)$, a constant. As $\tau > \mu$ and $d_\lambda(\tau) = d_\lambda(\mu)$ hold, we have $\tau_0 > \mu_0$. Hence, it follows from the induction hypothesis that if $w_{r_0}\beta'$ is a weight of $L(\lambda)$ for some $r_0 \geqslant l+s+t+2$, $\tau = w_r\beta'$ is a weight of $L(\lambda)$ for all $r \geqslant r_0$, and $\text{mult}(\beta') = \text{mult}(\tau)$ is a polynomial in r of degree $\leqslant d_\lambda(\tau) = d_\lambda(\mu)$. Note that $c_{\beta''} = 1/k$ for all $i = 1, \ldots, m_0 + d_\lambda(\mu)$. Therefore, the contribution of these decompositions to the right-hand side of (3.2) is a polynomial in r of degree $\leqslant d_\lambda(\mu)$.

Suppose that $d_\lambda(\tau) < d_\lambda(\mu)$. Then by the induction hypothesis, if $w_{r_0}\beta'$ is a weight of $L(\lambda)$ for some $r_0 \geqslant l+s+t+2$, then τ is a weight of $L(\lambda)$ for all $r \geqslant r_0$, and $\text{mult}(\beta') = \text{mult}(\tau)$ is a polynomial in r of degree $\leqslant d_\lambda(\tau) \leqslant d_\lambda(\mu) - 1$. Since there are $(r(r+1)/2)$ positive roots in $\mathfrak{g}_0 = A_r$ and since $c_{\beta''} = 1/k$ for all $k = 1, \ldots, m_0 + d_\lambda(\mu)$, the contribution of these decompositions to the right side of (3.2) is a polynomial in r of degree $\leqslant d_\lambda(\mu) + 1$.

Therefore, what the partitions in Case 4 contribute to the right-hand side of (3.2) is a polynomial in r of degree $\leqslant d_\lambda(\mu) + 1$.

Consequently, the sum of all the contributions from Case 1 to Case 4, which is the right side of (3.2), is a polynomial in r of degree $\leqslant d_\lambda(\mu) + 1$. By Lemma 3.3 and (3.2), we have $\text{mult}(\mu) = f/g$, where f is a polynomial in r of degree $\leqslant d_\lambda(\mu) + 1$ and g is a polynomial in r of degree 1. Since $\text{mult}(\mu)$ takes positive integral values for infinitely many values of $r \geqslant l+s+t+2$, it must be a polynomial in r (see [PS], p. 130), and

$$\deg(\text{mult}(\mu)) \leqslant (d_\lambda(\mu) + 1) - 1 = d_\lambda(\mu).$$

This completes the proof of the theorem. □

EXAMPLE 3.17. Tables I–IV illustrate the polynomial behavior of the multiplicity of the weight $\mu - k\delta$ in the irreducible highest weight module $L(\lambda)$ over the affine Kac–Moody algebra $A_r^{(1)}$. The numerical data in these tables was taken from [KMPS]. From now on, let $\underline{a} = (a_0, a_1, \ldots, a_{s-1})$, $\underline{a}' = (a_{r-t+1}, a_{r-t+2}, \ldots, a_r)$, and $m \in \mathbf{Z}$ be the determining data for λ, and let

$$\underline{b} = (b_0, b_1, \ldots, b_{s'-1}), \qquad \underline{b}' = (b_{r-t'+1}, b_{r-t'+2}, \ldots, b_r) \quad \text{and } n \in \mathbf{Z}$$

be the determining data for μ.

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Table I.
$\lambda = \mu = \Lambda_0 + \Lambda_r$;
$\underline{a} = (1)$, $\underline{a}' = (1)$, $m = 0$, $\underline{b} = (1)$, $\underline{b}' = (1)$, $n = 0$;
$d_\lambda(\mu - k\delta) = (k - m) - (d_1 + 2d_2 + \cdots + (p-1)d_{p-1}) = k$.

$k \backslash r$	1	2	3	4	5	6	7	8	Polynomial
0	1	1	1	1	1	1	1	1	1
1	2	4	6	8	10	12	14	16	$2r$
2	4	13	27	46	70	99	133	172	$\frac{1}{2}(5r^2 + 3r)$
3	8	36	98	208	380	628	966	1408	$\frac{1}{3}(7r^3 + 9r^2 + 8r)$
4	14	89	310	804	1740	3329	5824	9520	$\frac{1}{12}(21r^4 + 44r^3 + 87r^2 + 16r)$
5	24	204	888	2768	7012	15396	30436	55520	$\frac{1}{30}(33r^5 + 100r^4 + 315r^3 + 200r^2 + 72r)$

Table II.
$\lambda = 3\Lambda_1$, $\mu = \Lambda_0 + \Lambda_1 + \Lambda_2$;
$\underline{a} = (0, 3)$, $\underline{a}' = \emptyset$, $m = 0$, $\underline{b} = (1, 1, 1)$, $\underline{b}' = \emptyset$, $n = 0$;
$d_\lambda(\mu - k\delta) = (k - m) - (d_1 + 2d_2 + \cdots + (p-1)d_{p-1}) = k$.

$k \backslash r$	2	3	4	5	6	7	Polynomial
0	1	1	1	1	1	1	1
1	4	6	8	10	12	14	$2r$
2	15	31	53	81	115	155	$3r^2 + r + 1$
3	44	126	278	523	884	1384	$\frac{1}{6}(23r^3 + 3r^2 + 40r - 12)$
4	121	456	1267	2901	5808	10541	$\frac{1}{24}(103r^4 - 54r^3 + 533r^2 - 294r + 144)$
5	300	1477	5120	14166	33444	70188	$\frac{1}{120}(513r^5 - 800r^4 + 5815r^3 - 6580r^2 + 7412r - 22)$

Table III.
$\lambda = 2\Lambda_0$, $\mu = \Lambda_1 + \Lambda_r$;
$\underline{a} = (2)$, $\underline{a}' = \emptyset$, $m = 0$, $\underline{b} = (1)$, $\underline{b}' = (1)$, $n = 0$;
$d_\lambda(\mu - k\delta) = (k - m) - (d_1 + 2d_2 + \cdots + (p-1)d_{p-1}) = k - 1$.

$k \backslash r$	2	3	4	5	6	7	8	Polynomial
0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1
2	4	6	8	10	12	14	16	$2r$
3	12	25	43	66	94	127	165	$\frac{1}{2}(5r^2 + r + 2)$
4	32	87	186	343	572	887	1302	$\frac{1}{3}(7r^3 + 3r^2 + 17r - 6)$
5	77	266	693	1513	2923	5162	8511	$\frac{1}{12}(21r^4 + 16r^3 + 129r^2 - 46r + 36)$

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Table IV.
$\lambda = \Lambda_0 + \Lambda_1, \quad \mu = \Lambda_2 + \Lambda_r;$
$\underline{a} = (1, 1), \underline{a}' = \emptyset, m = 0, \quad \underline{b} = (0, 0, 1), \underline{b}' = (1), n = 0;$
$d_\lambda(\mu - k\delta) = (k - m) - (d_1 + 2d_2 + \cdots + (p-1)d_{p-1}) = k - 1.$

$k \setminus r$	3	4	5	6	7	8	Polynomial
0	0	0	0	0	0	0	0
1	2	2	2	2	2	2	2
2	12	17	22	27	32	37	$5r - 3$
3	50	92	148	218	302	400	$7r^2 - 7r + 8$
4	172	396	770	1336	2136	3212	$7r^3 - 9r^2 + 28r - 20$
5	522	1466	3382	6816	12446	21082	$\frac{1}{2}(11r^4 - 16r^3 + 97r^2 - -124r + 84)$

Table V.
$\lambda = 2\Lambda_0 + \Lambda_1, \ \mu = 2\Lambda_1 + \Lambda_r;$
$\underline{a} = (2, 1), \underline{a}' = \emptyset, m = 0, \quad \underline{b} = (0, 2), \underline{b}' = (1), n = 0;$
$d_\lambda(\mu - k\delta) = (k - m) - (d_1 + 2d_2 + \cdots + (p-1)d_{p-1}) = k - 1.$

$k \setminus r$	2	3	4	5	6	7	Polynomial
0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1
2	6	9	12	15	18	21	$3r$
3	22	49	87	136	196	267	$\frac{1}{2}(11r^2 - r + 2)$
4	70	214	492	951	1638	2600	$\frac{1}{6}(47r^3 - 21r^2 + 76r - 24)$
5	193	795	2328	5515	11304	20868	$\frac{1}{8}(75r^4 - 86r^3 + 373r^2 - -290r + 120)$

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