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L^p-CONVERGENCE OF A CERTAIN CLASS OF PRODUCT MARTINGALES

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We establish the Kakutani dichotomy property for two generalized Rademacher–Riesz product measures μ, ν; that *either* μ, ν are equivalent measures *or* they are mutually singular according as a certain series converges or diverges. We further give sufficient conditions so that in the equivalence case the Radon–Nikodym derivative $d\mu/d\nu$ belongs to $L^p(\nu)$ for all positive real numbers p, by proving that a certain product martingale converges in $L^p(\nu)$ for $p \geqq 1$.

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1. Introduction

Let us consider the sequence $(\varepsilon_1, \varepsilon_2, \ldots)$ of digits of the expansion of $x \in [0,1)$ in the base r ($r \geqq 2$), as a formal stochastic process with finite state space $S = \{0, 1, \ldots, r-1\}$ defined on the usual probability space of the unit interval $[0,1]$. Let μ be the Borel probability measure such that $\{\varepsilon_n\}$ is independent under μ and

$$\mu(\{x : \varepsilon_n(x) = i\}) = p_n^{(i)}, \quad i = 0, 1, \ldots, r-1,$$

where $\{p_n^{(0)}, p_n^{(1)}, \ldots, p_n^{(r-1)}\}$ is a set of positive real numbers such that $\sum_{i=0}^{r-1} p_n^{(i)} = 1$ for all n.

In the case $r = 2$, G. Brown and W. Moran proved in **[2]**, among other things, that either μ is absolutely continuous, with respect to Lebesgue measure, and its Radon–Nikodym derivative belongs to $L^p[0,1]$ for all positive real numbers p or μ is a singular measure, according as a certain series converges or diverges. This result is a dichotomy property for this kind of measure, and its proof relies on a convergence theorem for products of random variables, which is essentially a generalization of Kakutani's famous dichotomy criterion for infinite product measures **[4]**. As G. Brown and W. Moran **[2]** pointed out, the measure μ is, in this case, a Riesz product-type measure of the form

$$d\mu = \lim_{n \to \infty} \prod_{k=1}^{n} (1 + a_k r_k)\, dx,$$

where r_k denotes the kth Rademacher function.

In the present article we shall see that for each positive integer r ($r \geqq 2$), the measure μ above is also a Riesz product-type measure based on a certain sequence of independent random variables each of mean zero. So, this gives reason to investigate analogous dichotomy theorems in this general situation as both Kakutani's criterion [4] and Brown and Moran's convergence theorem (see [2, Theorem 1]) seem to be effective.

In Section 2 of this paper we shall show that such a measure μ is a generalized Rademacher–Riesz product as it has been introduced in [5]. Next, by the application of its properties we are able to generalize Brown and Moran's result stated above.

In Section 3 we prove that two generalized Rademacher–Riesz products are either equivalent or mutually singular in terms of the convergence of a certain series. To prove this result we use a formally different dichotomy criterion from those in [2] and [4]. Namely, we apply Kakutani's dichotomy theorem on product martingales as it is stated in [11]. An advantage of our martingale approach on the subject is that we can extract additional information in the equivalence case. Indeed, by the application of the classical Doob's Theorem [3] for the L^p-convergence of martingales we prove sufficient conditions so that in the case where μ, ν are equivalent generalized Rademacher–Riesz products the Radon–Nikodym derivative $d\mu/d\nu \in L^p(\nu)$ for all positive real numbers p.

Some central results concerning equivalence or mutual singularity for the classical Riesz products have been derived by J. Peyrière [7], G. Brown and W. Moran [1] and G. Ritter [8]. In addition, S. J. Kilmer and S. Saeki [6] have given further criteria for mutual absolute continuity and singularity of Riesz products as well as sufficient conditions so that in the case where a Riesz product μ is absolutely continuous with respect to another Riesz product ν, $d\mu/d\nu \in L^p(\nu)$ for appropriate p. It should be noted that G. Ritter proved in [9] dichotomy results for more general infinite products of functions than those in [1], [6], [7] and [8] and he also used martingales in the context of Riesz products.

In our work here we deal with measures different from the classical trigonometric Riesz products and our methods based upon the application of the basic martingale theory.

In the next section we give some basic definitions and results on generalized Rademacher–Riesz products.

2. Preliminaries

Let r be a positive integer ($r \geqq 2$) and let ε_n be the nth digit of the expansion of $x \in [0, 1)$ in the base r, i.e. $x = \sum_{n=1}^{\infty} \varepsilon_n / r^n$ and $\varepsilon_n \in \{0, 1, \dots, r-1\}$.

We define the sequences of functions $(R_n^i)_{n=1}^{\infty}$, for $i = 0, 1, \dots, r-1$ on $[0, 1]$ as follows:

$$R_n^i(x) = 1 - r\delta_{\varepsilon_n, i} \quad \text{for} \quad n = 1, 2, \dots,$$

where $\delta_{\varepsilon_n, i}$ is the usual Kronecker's delta. We also define R_n^i to be zero on each r-adic rational. We call the sequence $(R_n^i)_{n=1}^{\infty}$ the system of r-adic Rademacher functions associated with the digit i.

Let $(a_n^{(0)}), (a_n^{(1)}), \ldots, (a_n^{(r-1)})$ be sequences of positive real numbers satisfying $\sum_{i=0}^{r-1} a_n^{(i)} = 1$, for all n.

We define the sequence of functions $(X_n)_{n=1}^{\infty}$ on $[0,1]$ as follows:

$$X_n = \sum_{i=0}^{r-1} a_n^{(i)} R_n^i \quad \text{for} \quad n = 1, 2, \ldots.$$

Since the functions X_n are obviously Borel measurable, it is more convenient in the sequel to employ the terminology of the probability theory. As it is easily seen $(X_n)_{n=1}^{\infty}$ is a sequence of independent random variables, each of mean 0, on the probability space $([0,1], \mathscr{B}, \lambda)$, where $\mathscr{B}$ is the σ-algebra of the Borel subsets of $[0,1]$ and λ is the Lebesgue measure on $\mathscr{B}$.

We shall call a cylinder of order n and r-adic interval of the form:

$$E_{n,j} = \left(\frac{j-1}{r^n}, \frac{j}{r^n} \right) \quad \text{for} \quad j = 1, 2, \ldots, r^n.$$

Let $E_n(x)$ be the cylinder of order n which contains x, for $n = 1, 2, \ldots$ and let $E_0(x) = [0, 1)$.

We define

$$\mu(E_n(x)) = \frac{1}{r^n} \prod_{k=1}^{n} (1 - X_k(x)).$$

Let $E_{n,j} = \bigcup_{i=1}^{r} E_{n+1,(j-1)r+i}$. Clearly, $E_{n+1,(j-1)r+i}$, $i = 1, 2, \ldots, r$ are the cylinders of order $n+1$ that $E_{n,j}$ is divided into. Then it is easily checked that the set function μ satisfies the following conditions:

(1) $\mu(E_{n,j}) = \sum_{i=1}^{r} \mu(E_{n+1,(j-1)r+i})$ for every n and $j = 1, 2, \ldots, r^n$.
(2) $\sum_{j=1}^{r^n} \mu(E_{n,j}) = 1$ for all n.

Therefore, μ may be extended to a Borel probability measure on $[0,1]$ in a unique manner.

We denote by $\mathscr{B}_n$ the σ-algebra generated by $\{E_{n,j} : 1 \leqq j \leqq r^n\}$. It is plain that the restriction of the measure μ to $\mathscr{B}_n$ is the measure μ_n defined by

$$d\mu_n = \prod_{k=1}^{n} (1 - X_k) \, d\lambda.$$

It is also clear that the measure μ is the limit of the sequence of measures $(\mu_n)_{n=1}^{\infty}$ in the weak* topology of $M([0,1])$, where $M([0,1])$ is the Banach space of all regular Borel measures on $[0,1]$. We call such a measure μ a *generalized Rademacher–Riesz product* associated with the sequences $(a_n^{(i)})$, and in what follows we shall employ the notation

$$d\mu = \prod_{n=1}^{\infty} (1 - X_n)\, d\lambda. \tag{2.1}$$

Some basic properties of a measure μ defined above are given by the following proposition.

Proposition. *Let μ be a generalized Rademacher–Riesz product as in* (2.1). *Then, we have the following conclusions:*

(i) *The random variables $(R_n^i)_{n=1}^{\infty}$ are independent on the probability space $([0,1], \mathscr{B}, \mu)$ such that*

$$\int_0^1 R_n^i\, d\mu = 1 - r a_n^{(i)}$$

for $i = 0, 1, \ldots, r-1$ and $n = 1, 2, \ldots$.

(ii) *$(\varepsilon_n)_{n=1}^{\infty}$ is a sequence of independent random variables on the probability space $([0,1], \mathscr{B}, \mu)$, so that, if*

$$U_{n,i} = \{x \in [0,1) : \varepsilon_n(x) = i\}$$

then

$$\mu(U_{n,i}) = a_n^{(i)}, \quad i = 0, 1, \ldots, r-1.$$

(iii) *Let*

$$\xi_n = c_n + \sum_{i=0}^{r-1} c_n^{(i)} R_n^i,$$

where (c_n), $(c_n^{(i)})$ are sequences of real numbers such that at least one $c_n^{(i)} \neq 0$ and one $c_n^{(i)} \neq 1$ for every n. Then the random variables ξ_n are independent on the probability space $([0,1], \mathscr{B}, \mu)$.

Proof. Let $A_{n,i} = \{x \in [0,1) : R_n^i(x) = 1 - r\}$. It can be easily checked that $\mu(A_{n,i}) = a_n^{(i)}$. From this it follows that $\int_0^1 R_n^i\, d\mu = 1 - r a_n^{(i)}$. It is not hard to see that

$$\mu(A_{n,i} \cap A_{m,i}) = \mu(A_{n,i}) \cdot \mu(A_{m,i})$$

for every pair of positive integers n, m, which establishes (i). To prove assertion (ii) it is sufficient to observe that

$$A_{n,i} = U_{n,i} \quad \text{for all} \quad i = 0, 1, \ldots, r-1 \quad \text{and} \quad n = 1, 2, \ldots.$$

The proof of (iii) directly follows from (i) and (ii). □

Remark. In the case $r=2$ we have $R_n^1 = r_n$ and $R_n^0 = -r_n$, where $(r_n)_{n=1}^{\infty}$ are the usual Rademacher functions on $[0,1]$, defined by $r_n(x) = 1 - 2\varepsilon_n$. Taking

$$a_n^{(0)} = \frac{1+a_n}{2} \quad \text{and} \quad a_n^{(1)} = \frac{1-a_n}{2},$$

where $|a_n| \leqq 1$, the measure (2.1) has the form

$$d\mu = \prod_{n=1}^{\infty} (1 + a_n r_n)\, d\lambda.$$

In what follows, we shall maintain all the notation and the terminology of this section.

3. The main results

The following theorem establishes the dichotomy property for two generalized Rademacher–Riesz product measures.

Theorem 1. *Let*

$$d\mu = \prod_{n=1}^{\infty} (1 - X_n)\, d\lambda \quad \textit{and} \quad d\nu = \prod_{n=1}^{\infty} (1 - Y_n)\, d\lambda,$$

where

$$X_n = \sum_{i=0}^{r-1} a_n^{(i)} R_n^i \quad \textit{and} \quad Y_n = \sum_{i=0}^{r-1} b_n^{(i)} R_n^i.$$

Then μ, ν are equivalent measures (i.e., $\mu \ll \nu$ and $\nu \ll \mu$) if and only if

$$\sum_{n=1}^{\infty} \sum_{i=0}^{r-1} \left(\sqrt{a_n^{(i)}} - \sqrt{b_n^{(i)}}\right)^2 < \infty. \tag{3.1}$$

Otherwise, the measures μ, ν are mutually singular.

Before proving this theorem it is convenient to gather together some basic results and notation, that we shall use throughout this section, in the following lemma.

Lemma 1. *We suppose that X_n, Y_n are as in Theorem* 1. *We further define*

$$Z_n = \frac{1-X_n}{1-Y_n} \quad \textit{for} \quad n=1,2,\dots$$

and for $p>0$

$$I_n(p) = \int_0^1 (Z_n)^p \, dv,$$

where the measure v *is as in the theorem above. Then*

(i) *for all positive real numbers* p

$$(Z_n)^p = \frac{1}{r}\sum_{i=0}^{r-1}\left(\frac{a_n^{(i)}}{b_n^{(i)}}\right)^p - \frac{1}{r}\sum_{i=0}^{r-1}\left(\frac{a_n^{(i)}}{b_n^{(i)}}\right)^p R_n^i.$$

(ii) $\{(Z_n)^p\}_{n=1}^{\infty}$ *is a sequence of independent random variables on the probability space* $([0,1],\mathscr{B},v)$, *for any* $p\colon 0<p<\infty$.

(iii) $$I_n(p) = \sum_{i=0}^{r-1} \frac{[a_n^{(i)}]^p}{[b_n^{(i)}]^{p-1}},$$

for any $p\colon 0<p<\infty$.

(iv) *In particular, for the sequence of independent random variables* $(Z_n)_{n=1}^{\infty}$ *we have*

$$\int_0^1 Z_n \, dv = 1.$$

Proof. (i) follows easily from the definition of X_n, Y_n and Z_n. Combining (i) and part (iii) of the Proposition in Section 2 we have (ii). Clearly (i) and part (i) of the Proposition mentioned above imply (iii). Finally, (iv) is an immediate consequence of (ii) and (iii). □

Proof of Theorem 1. We define

$$F_n = \prod_{k=1}^{n} Z_k, \quad \text{for} \quad n=1,2,\dots, \tag{3.2}$$

where Z_k, $k=1,2,\dots,n$, are as in Lemma 1. It is clear that the stochastic sequence $(F_n,\mathscr{B}_n)_{n\geq 1}$ is a martingale on the probability space $([0,1],\mathscr{B},v)$, and Doob's Theorem (cf. [3, p. 319]) ensures that $\lim_{n\to\infty} F_n = F$ exists v-almost everywhere.

Now, let μ_n, ν_n be the restrictions of the measures μ, ν to $\mathscr{B}_n$ respectively. It is easy to see that $\mu_n \ll \nu_n$ and $d\mu_n/d\nu_n = F_n$ for $n=1,2,\ldots$. It is well-known that for every $A \in \mathscr{B}$

$$\mu(A) = \int_A F\,d\nu + \sigma(A), \tag{3.3}$$

where the measures σ, ν are mutually singular (see [10, p. 493]).

It follows from Lemma 1 that

$$I_n\left(\frac{1}{2}\right) = \sum_{i=0}^{r-1} \sqrt{a_n^{(i)} \cdot b_n^{(i)}}$$

whence

$$1 - I_n\left(\frac{1}{2}\right) = \frac{1}{2} \sum_{i=0}^{r-1} \left(\sqrt{a_n^{(i)}} - \sqrt{b_n^{(i)}}\right)^2.$$

We suppose that the series in (3.1) converges. Then by Kakutani's Theorem on product martingales (cf. [**11**, p. 144]) we infer that the martingale $(F_n, \mathscr{B}_n)_{n \geq 1}$ is uniformly integrable, F_n converges to F in $L^1(\nu)$ and

$$\int_0^1 F\,d\nu = 1. \tag{3.4}$$

By combining (3.3) and (3.4) we conclude that μ is absolutely continuous with respect to ν.

When the series in (3.1) diverges by applying again Kakutani's Theorem mentioned above we deduce that

$$F = 0 \ \nu\text{-almost everywhere,}$$

which in combination with (3.3) implies that μ, ν are mutually singular measures. This completes the proof of the theorem. □

Theorem 2. *Let the measures* μ, ν *be as in Theorem* 1. *We define*

$$M_n = \max_{0 \leq i \leq r-1} \frac{a_n^{(i)}}{b_n^{(i)}}, \quad n = 1, 2, \ldots.$$

We assume that the series in (3.1) *converges and*

$$\sup_n M_n < \infty. \tag{3.5}$$

Then $d\mu/d\nu$ belongs to $L^p(\nu)$ for all positive real numbers p.

For the proof of this theorem we need the following elementary lemma.

Lemma 2. *Let*

$$a_i > 0, \quad b_i > 0 \quad \textit{for} \quad i = 1, 2, \dots, n.$$

We set

$$M = \max_{1 \leqq i \leqq n} \frac{a_i}{b_i}.$$

Then the inequality

$$\sum_{i=1}^{n} \frac{(a_i - b_i)^2}{b_i} \leqq 4 \max(1, M) \sum_{i=1}^{n} (\sqrt{a_i} - \sqrt{b_i})^2$$

holds.

Proof. For each pair of positive real numbers x, y we have

$$\frac{(x-y)^2}{y} \leqq 4 \max\left(1, \frac{x}{y}\right) \cdot (\sqrt{x} - \sqrt{y})^2. \tag{3.6}$$

This inequality can be easily proved by the mean value theorem and the observation that

$$\max\left(1, \frac{x}{y}\right) = \frac{\max(x, y)}{y}.$$

From (3.6) we have

$$\begin{aligned} \frac{(a_i - b_i)^2}{b_i} &\leqq 4 \max\left(1, \frac{a_i}{b_i}\right)(\sqrt{a_i} - \sqrt{b_i})^2 \\ &\leqq 4 \max(1, M)(\sqrt{a_i} - \sqrt{b_i})^2 \quad \text{for} \quad i = 1, 2, \dots, n \end{aligned} \tag{3.7}$$

(since $a < b$ implies $\max(c, a) \leqq \max(c, b)$). From (3.7) the inequality of the lemma follows. □

Proof of Theorem 2. We consider the martingale $(F_n, \mathscr{B}_n)_{n \geq 1}$, where F_n is defined by (3.2). According to Theorem 1, F_n converges ν-almost everywhere and in $L^1(\nu)$ towards a function F, where $F=d\mu/d\nu$. So, it suffices to examine the convergence of F_n in $L^p(\nu)$, for $p>1$.

It follows from Lemma 1(ii) that for any p

$$\int_0^1 F_n^p\,d\nu=\prod_{k=1}^n\int_0^1 Z_k^p\,d\nu=\prod_{k=1}^n I_k(p). \tag{3.8}$$

Taking into consideration part (iii) of Lemma 1 one can easily see that

$$I_k(2)=1+\sum_{i=0}^{r-1}\frac{(a_k^{(i)}-b_k^{(i)})^2}{b_k^{(i)}}.$$

By Lemma 2 we evidently have

$$\sum_{i=0}^{r-1}\frac{(a_k^{(i)}-b_k^{(i)})^2}{b_k^{(i)}}\leqq 4\max(1,M_k)\sum_{i=0}^{r-1}\left(\sqrt{a_k^{(i)}}-\sqrt{b_k^{(i)}}\right)^2. \tag{3.9}$$

From this and the assumptions of the theorem we infer that

$$\sum_{n=1}^{\infty}\sum_{i=0}^{r-1}\frac{(a_n^{(i)}-b_n^{(i)})^2}{b_n^{(i)}}<\infty.$$

From (3.8), for $p=2$, and the above we have

$$\sup_n\int_0^1 F_n^2\,d\nu<\infty.$$

By applying Doob's Theorem (see [3, p. 319]) we deduce that $F\in L^2(\nu)$ and F_n converges to F in $L^2(\nu)$.

We observe that

$$I_k(p)=\sum_{i=0}^{r-1}\frac{[a_k^{(i)}]^p}{[b_k^{(i)}]^{p-1}}=\sum_{i=0}^{r-1}b_k^{(i)}\left(1+\frac{a_k^{(i)}-b_k^{(i)}}{b_k^{(i)}}\right)^p.$$

For each positive real number p the following inequality holds true:

$$(1+t)^p\leqq 1+pt+c_p(|t|^p+t^2)\quad\text{for any}\quad t\geqq -1,$$

where

$$c_p=2^p+\sum_{k=2}^{\infty}\left|\binom{p}{k}\right|,$$

(cf. [6, Lemma 1.1]). By applying the above inequality we obtain

$$b_k^{(i)}\left(1+\frac{a_k^{(i)}-b_k^{(i)}}{b_k^{(i)}}\right)^p \leqq b_k^{(i)}+p(a_k^{(i)}-b_k^{(i)})+c_p\left[\frac{|a_k^{(i)}-b_k^{(i)}|^p}{[b_k^{(i)}]^{p-1}}+\frac{(a_k^{(i)}-b_k^{(i)})^2}{b_k^{(i)}}\right],$$

for $i=0,1,\dots,r-1$. Hence

$$I_k(p)\leqq 1+c_p\left[\sum_{i=0}^{r-1}\frac{|a_k^{(i)}-b_k^{(i)}|^p}{[b_k^{(i)}]^{p-1}}+\sum_{i=0}^{r-1}\frac{(a_k^{(i)}-b_k^{(i)})^2}{b_k^{(i)}}\right]. \tag{3.10}$$

For $p>2$ we have

$$\frac{|a_k^{(i)}-b_k^{(i)}|^p}{[b_k^{(i)}]^{p-1}}=\left(\frac{|a_k^{(i)}-b_k^{(i)}|}{b_k^{(i)}}\right)^{p-2}\frac{(a_k^{(i)}-b_k^{(i)})^2}{b_k^{(i)}}\leqq(1+M_k)^{p-2}\frac{(a_k^{(i)}-b_k^{(i)})^2}{b_k^{(i)}}.$$

A combination of this with (3.9) gives

$$\sum_{i=0}^{r-1}\frac{|a_k^{(i)}-b_k^{(i)}|^p}{[b_k^{(i)}]^{p-1}}+\sum_{i=0}^{r-1}\frac{(a_k^{(i)}-b_k^{(i)})^2}{b_k^{(i)}}\leqq[1+(1+M_k)^{p-2}]\sum_{i=0}^{r-1}\frac{(a_k^{(i)}-b_k^{(i)})^2}{b_k^{(i)}}$$

$$\leqq 4[1+(1+M_k)^{p-2}]\max(1,M_k)\sum_{i=0}^{r-1}(\sqrt{a_k^{(i)}}-\sqrt{b_k^{(i)}})^2.$$

From (3.10) and the above we obtain

$$I_k(p)\leqq 1+c_p\cdot N_k(p)\cdot\sum_{i=0}^{r-1}(\sqrt{a_k^{(i)}}-\sqrt{b_k^{(i)}})^2, \tag{3.11}$$

where

$$N_k(p)=4[1+(1+M_k)^{p-2}]\max(1,M_k).$$

Clearly (3.8) and (3.11) imply that

$$\int_0^1 F_n^p\,dv\leqq\prod_{k=1}^{n}(1+c_p\cdot D_k(p)), \tag{3.12}$$

where

$$D_k(p)=N_k(p)\cdot\sum_{i=0}^{r-1}(\sqrt{a_k^{(i)}}-\sqrt{b_k^{(i)}})^2.$$

It follows from our assumptions that

$$\sum_{n=1}^{\infty} D_n(p) < \infty. \tag{3.13}$$

Finally, from (3.12) and (3.13) we have

$$\sup_n \int_0^1 F_n^p \, dv \leqq \prod_{n=1}^{\infty} (1 + c_p \cdot D_n(p)) < \infty,$$

and by the application of Doob's Theorem (see [3, p. 319]) we conclude that F_n converges to F in $L^p(v)$. Consequently, $F = d\mu/dv$ belongs to $L^p(v)$ for every positive real number p. Furthermore,

$$\|F\|_p \leqq \exp(c_p \cdot D(p)) \quad \text{for all} \quad p,$$

where $D(p) = \sum_{n=1}^{\infty} D_n(p)$ and $\| \ \|_p$ denotes the norm of $L^p(v)$. The proof is complete. □

As a consequence of Theorem 2 we have the following remarkable fact:

Corollary 1. *Let the measures μ, v be as in Theorem* 1 *and let $B_n = \min_{0 \leqq i \leqq r-1} b_n^{(i)}$. We suppose that*

$$\liminf_{n \to \infty} B_n > 0 \tag{3.14}$$

and

$$\sum_{n=1}^{\infty} \sum_{i=0}^{r-1} (a_n^{(i)} - b_n^{(i)})^2 < \infty. \tag{3.15}$$

Then μ, v are equivalent measures and $d\mu/dv$ belongs to $L^p(v)$ for all positive real numbers p.

Proof. Plainly (3.14) and (3.15) imply the convergence of the series in (3.1). On the other hand, (3.14) shows that condition (3.5) is also satisfied.

In the case where $b_n^{(i)} = 1/r$ for each $i = 0, 1, \ldots, r-1$, the measure v above coincides with the Lebesgue measure λ; so by applying Corollary 1, the following corollary can easily be shown.

Corollary 2. *Let μ be a generalized Rademacher–Riesz product as in Theorem* 1. *We set*

$$d_n^2 = \sum_{i=0}^{r-1} (1 - r a_n^{(i)})^2.$$

If $\sum_{n=1}^{\infty} d_n^2 < \infty$ then $\mu \ll \lambda$ *and* $d\mu/d\lambda$ *belongs to* $L^p([0,1])$ *for any* p: $0 < p < \infty$. *If* $\sum_{n=1}^{\infty} d_n^2 = \infty$ *then* μ *is a singular measure.*

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