ON THE RADII OF STARLIKENESS AND CONVEXITY OF CERTAIN CLASSES OF REGULAR FUNCTIONS

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1. Introduction

Let R_n denote the class of functions $f(z) = z + a_n z^n + \cdots$ $(n \geqq 2)$ which are regular in the open disc $|z| < 1$ (hereafter called E) and satisfy

$$\text{Re}\left(\frac{f(z)}{z}\right) > 0, \tag{1.1}$$

for all z in E. R_n is a subclass of the class of close-to-star functions in E [9, p. 61]. MacGregor showed that the radius of univalence and starlikeness of R_n is $[1-n+(n^2-2n+2)^{\frac{1}{2}}]^{1/(n-1)}$, see [4,5]. The radius of convexity of $R = R_2$ is $r_0 = 0.179\cdots$, where r_0 is the smallest positive root of the equation $1-5r-3r^2-r^3 = 0$, see [8].

In this paper we consider a subclass $R_n(\alpha)$ of the class R_n, the members of $R_n(\alpha)$ being those members of R_n which satisfy

$$\left|\frac{f(z)}{z} - \alpha\right| < \alpha \quad (\alpha > \tfrac{1}{2}), \tag{1.2}$$

for all z in E. The main purpose of this paper is to find the radius of convexity of $R(\alpha) = R_2(\alpha)$. To obtain the result in more general form we further assume that $f(z)$ is k-fold symmetric, that is, it has power series expansion of the form

$$f(z) = z + \sum_{m=1}^{\infty} a_{mk+1} z^{mk+1}.$$

We also obtain the radius of univalence and starlikeness of $R_n(\alpha)$. Corresponding result for the class $R(1)$ is known to be $\frac{1}{2}$, see [6]. By making α tend to infinity in the above results for the class $R_n(\alpha)$, we can obtain the corresponding results for the class R_n.

For the above classes the identity function z plays a key role. It would naturally be interesting to see how the radii of univalence and convexity vary when the identity function is replaced by some other function $g(z)$ such that $g(0) = 0$;

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that is, to investigate similar problems for the class of functions $f(z)$ which are regular in E and satisfy $f(0) = 0, f'(0) = 1$, and

$$\left|\frac{f(z)}{g(z)} - \alpha\right| < \alpha \quad (\alpha > \tfrac{1}{2}), \tag{1.3}$$

for all z in E. We take $g(z) = z + az^2 (|a| \leqq 1)$ and find the radius of univalence and starlikeness of the above class. We also find the radius of convexity of the above class when $\alpha = 1$, or ∞. It is found that these radii decrease monotonically as $|a|$ increases from 0 to 1.

The estimates used to obtain the above results are further used to obtain the radius of univalence and starlikeness of a subclass of the class of typically real functions.

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We shall need the following lemmas.

Throughout this paper $P(z)$ denotes a function which is regular in E and satisfies $P(0) = 1$, $\operatorname{Re} P(z) > 0$ for all z in E.

LEMMA 1. *If $P(z)$ has a power series expansion of the form*

$$P(z) = 1 + c_n z^n + c_{n+1} z^{n+1} + \cdots \ (n \geqq 1), \tag{2.1}$$

then for $|z| = r, 0 \leqq r < 1$,

$$|P'(z)| \leqq \frac{2nr^{n-1}}{(1-r^{2n})} \operatorname{Re} P(z), \tag{2.2}$$

and

$$\left|\frac{P'(z)}{P(z)+\mu}\right| \leqq \frac{2nr^{n-1}}{(1-r^n)[(1+r^n)+(1-r^n)\cdot \operatorname{Re}(\mu)]}, \tag{2.3}$$

where μ is any complex number with $\operatorname{Re}(\mu) \geqq 0$.

Corresponding results for $n = 1$ and $\mu = 0$ are due to Libera [3] and MacGregor [5].

PROOF. Let

$$f(z) = \frac{1-P(z)}{1+P(z)}. \tag{2.4}$$

Then $f(z)$ is regular in E and satisfies $|f(z)| < 1$ for z in E [7, p. 169]. Also $f(z)$ has a zero at $z = 0$ of order at least n and hence by Schwarz's lemma $|f(z)| \leqq |z|^n$. For such functions we have [1]

$$|f'(z)| \leqq \frac{nr^{n-1}}{(1-r^{2n})}(1-|f(z)|^2), \quad |z| = r < 1. \tag{2.5}$$

Substituting for $f(z)$ from (2.4) we obtain (2.2).

Further

$$\left|\frac{P'(z)}{P(z)+\mu}\right| \leqq \frac{|P'(z)|}{\operatorname{Re}(P(z))+\operatorname{Re}(\mu)} = \frac{|P'(z)|}{\operatorname{Re}(P(z))}\,\frac{1}{1+(\operatorname{Re}(\mu)/\operatorname{Re}(P(z)))},$$

which in view of (2.2) and the inequality [4]

$$\operatorname{Re}(P(z)) \leqq \frac{1+r^n}{1-r^n},$$

yields (2.3).

It is easy to show that equality holds in (2.2) and (2.3) for $\mu \geqq 0$ only for functions $P(z) = (1-\varepsilon z^n)/(1+\varepsilon z^n)$ where $|\varepsilon| = 1$ and for appropriate values of z.

LEMMA 2. *If $P(z)$ has power series expansion of the form* (2.1), *then for* $|z| = r < 1$ *and* $\mu \geqq 0$,

$$\left|\frac{1}{P(z)+\mu} - \frac{(\mu+1)-(\mu-1)r^{2n}}{(\mu+1)^2-(\mu-1)^2r^{2n}}\right| \leqq \frac{2r^n}{[(\mu+1)^2-(\mu-1)^2r^{2n}]}. \tag{2.6}$$

PROOF. Let $\psi(z) = 1/(P(z)+\mu)$. Substituting for $P(z)$ in terms of $\psi(z)$ in (2.4) and using the fact that $|f(z)| \leqq |z|^n$, we obtain for $|z| = r < 1$,

$$\left|\frac{\psi(z)-(1/(\mu+1))}{\psi(z)-(1/(\mu-1))}\right| \leqq \left|\frac{\mu-1}{\mu+1}\right| r^n.$$

This is equivalent to the inequality

$$\left|\psi(z) - \frac{(\mu+1)-(\mu-1)r^{2n}}{(\mu+1)^2-(\mu-1)^2r^{2n}}\right| \leqq \frac{2r^n}{[(\mu+1)^2-(\mu-1)^2r^{2n}]}.$$

It is easy to show that equality occurs in (2.6) only for functions $P(z) = (1-\varepsilon z^n)/(1+\varepsilon z^n)$ where $|\varepsilon| = 1$ and for appropriate values of z.

LEMMA 3. *If $P(z)$ has a power series expansion of the form*

$$P(z) = 1 + \sum_{m=1}^{\infty} c_{mk} z^{mk},$$

then for $|z| = r < 1$ *and* $\mu \geqq 0$,

$$\left|\frac{z^2P''(z)}{P(z)+\mu}\right| \leqq \frac{2kr^k}{(1-r^k)^2}\,\frac{(k-1)+(k+1)r^k}{(\mu+1)-(\mu-1)r^k}. \tag{2.7}$$

PROOF. Let $P(z) = g(z^k)$. Then $g(z)$ is regular in E and satisfies $g(0) = 1$,

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Re $g(z) > 0$ for z in E. Let ζ be a complex number such that $0 < |\zeta| < 1$. The function

$$G(z) = g\left(\frac{z+\zeta}{1+\bar{\zeta}z}\right) = g(\zeta)+(1-|\zeta|^2)g'(\zeta)z+\tfrac{1}{2}(1-|\zeta|^2)\{(1-|\zeta|^2)g''(\zeta)-2\bar{\zeta}g'(\zeta)\}z^2 + \cdots$$

is regular in E and satisfies Re $G(z) > 0$ for z in E. Therefore by the Carathéodory-Toeplitz theorem, we have

$$\left| g''(\zeta) - \frac{2\bar{\zeta}}{(1-|\zeta|^2)} g'(\zeta) \right| \leqq \frac{4|g(\zeta)|}{(1-|\zeta|^2)^2}.$$

This gives

$$\left| g''(z^k) - \frac{2\bar{z}^k}{(1-|z|^{2k})} g'(z^k) \right| \leqq \frac{4|g(z^k)|}{(1-|z|^{2k})^2},$$

for all z in E. Using the relation $P(z) = g(z^k)$, we obtain the inequality

$$\left| z^2P''(z) - \frac{((k-1)+(k+1)|z|^{2k})}{(1-|z|^{2k})} zP'(z) \right| \leqq \frac{4k^2|z|^{2k}}{(1-|z|^{2k})^2} |P(z)|.$$

Therefore

$$\left| \frac{z^2P''(z)}{P(z)+\mu} \right| \leqq \frac{2kr^k}{(1-r^k)[(\mu+1)-(\mu-1)r^k]}. \tag{2.8}$$

From lemma 1, we have for $|z| = r < 1$,

$$\left| \frac{zP'(z)}{P(z)+\mu} \right| \leqq \frac{2kr^k}{(1-r^k)[(\mu+1)-(\mu-1)r^k]}. \tag{2.9}$$

From lemma 2, we have for $|z| = r < 1$,

$$\left| \frac{P(z)}{P(z)+\mu} - \frac{(\mu+1)+(\mu-1)r^{2k}}{(\mu+1)^2-(\mu-1)^2r^{2k}} \right| \leqq \frac{2\mu r^k}{[(\mu+1)^2-(\mu-1)^2r^{2k}]}.$$

The above gives

$$\left| \frac{P(z)}{P(z)+\mu} \right| \leqq \frac{(1+r^k)}{[(\mu+1)-(\mu-1)r^k]}. \tag{2.10}$$

From (2.8), (2.9) and (2.10) we obtain (2.7).

It is easy to show that equality holds in (2.7) only for functions $P(z) = (1-\varepsilon z^k)/(1+\varepsilon z^k)$ where $|\varepsilon| = 1$ and for appropriate values of z.

THEOREM 1. *Suppose that* $f(z) = z+a_{k+1}z^{k+1}+a_{2k+1}z^{2k+1}+\cdots$ *is regular in E and satisfies* $|(f(z)|z)-\alpha| < \alpha\,(\alpha > \tfrac{1}{2})$ *for z in E. Then* $f(z)$ *maps* $|z| < r_\alpha$ *onto*

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a convex domain where r_α is the smallest positive root of the equation

$$\alpha^2-\alpha((2-\alpha)+(2\alpha-1)(2k+k^2))r^k+(1-\alpha)((1+\alpha)+(2\alpha-1)(2k-k^2))r^{2k} -(1-\alpha)^2r^{3k}=0.$$

This result is sharp in the sense that the number r_α cannot be replaced by any larger one.

PROOF. Let

$$\psi(z)=1-\frac{1}{\alpha}\frac{f(z)}{z}, \tag{2.11}$$

then $\psi(z)$ is regular in E, $|\psi(z)|<1$ for z in E and $\psi(0)=1-(1/\alpha)$. Let

$$F(z)=\frac{\psi(z)-\psi(0)}{1-\psi(0)\psi(z)}, \tag{2.12}$$

then $F(z)$ is regular in E, $|F(z)|<1$ for z in E and $F(0)=0$. Also $F(z)$ has a power series expansion of the form $F(z)=b_kz^k+b_{2k}z^{2k}+\cdots$. Such a function $F(z)$ can be represented as [7, p. 169]

$$F(z)=\frac{P(z)-1}{P(z)+1}. \tag{2.13}$$

Evidently $P(z)$ has a power series expansion of the form $P(z)=1+c_kz^k+c_{2k}z^{2k}+\cdots$. From (2.11), (2.12) and (2.13) we have

$$f(z)=\frac{2\alpha z}{1+(2\alpha-1)P(z)}. \tag{2.14}$$

The representation (2.14) yields the relation

$$1+\frac{zf''(z)}{f'(z)}=1-\frac{2zP'(z)}{P(z)+(1/(2\alpha-1))}-\frac{z^2P''(z)}{(1/(2\alpha-1))+P(z)-zP'(z)}. \tag{2.15}$$

From lemmas 1 and 3 we have for $|z|=r$, $0\leqq r<1$,

$$\left|\frac{zP'(z)}{P(z)+(1/(2\alpha-1))}\right|\leqq\frac{(2\alpha-1)kr^k}{\alpha-r^k-(\alpha-1)r^{2k}}=\frac{(2\alpha-1)kr^k}{(1-r^k)(\alpha-(1-\alpha)r^k)}, \tag{2.16}$$

and

$$\left|\frac{z^2P''(z)}{P(z)+(1/(2\alpha-1))}\right|\leqq\frac{(2\alpha-1)kr^k((k-1)+(k+1)r^k)}{(1-r^k)^2(\alpha-(1-\alpha)r^k)}. \tag{2.17}$$

Let

$$R=\left[\frac{(1+(2\alpha-1)k)-\{(2\alpha-1)((2\alpha-1)(1+k^2)+2k)\}^{\frac{1}{2}}}{2(1-\alpha)}\right]^{1/k}. \tag{2.18}$$

From (2.16) and (2.17) we have for $|z| = r, 0 \leqq r < R$,

$$(2.19) \quad \left| \frac{z^2P''(z)}{(1/(2\alpha-1))+P(z)-zP'(z)} \right| \leqq \frac{k(2\alpha-1)r^k((k+1)r^k+(k-1))}{(1-r^k)((1-\alpha)r^{2k}-(1+k(2\alpha-1))r^k+\alpha)}.$$

From (2.15), (2.16) and (2.19) we have for $|z| = r, 0 \leqq r < R$,

$$\begin{aligned} \operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) &\geqq 1-\frac{2(2\alpha-1)kr^k}{(1-r^k)(\alpha-(1-\alpha)r^k)} \\ &\quad -\frac{k(2\alpha-1)r^k((k+1)r^k+(k-1))}{(1-r^k)((1-\alpha)r^{2k}-(1+k(2\alpha-1))r^k+\alpha)} \\ &= \frac{p(r)}{(\alpha-(1-\alpha)r^k)((1-\alpha)r^{2k}-(1+k(2\alpha-1))r^k+\alpha)}, \end{aligned}$$

where

$$(2.20) \quad \begin{aligned} p(r) = \alpha^2-\alpha\{(2-\alpha)+(2\alpha-1)(2k+k^2)\}r^k \\ +(1-\alpha)\{(1+\alpha)+(2\alpha-1)(2k-k^2)\}r^{2k}-(1-\alpha)^2r^{3k}. \end{aligned}$$

The condition $\operatorname{Re}(1+(zf''(z)/f'(z))) > 0$ for $|z| < r$ is necessary and sufficient for $f(z)$ to map $|z| < r$ onto a convex domain. From the above estimate, we see that this condition is satisfied in $|z| < \min(r_\alpha, R)$ where r_α is the smallest positive root of the equation $p(r) = 0$. Writing $p(r)$ as

$$\begin{aligned} p(r) = (\alpha-(1-\alpha)r^k)((1-\alpha)r^{2k}-(1+k(2\alpha-1)r^k+\alpha) \\ +k(2\alpha-1)r^k((1-k)(1-\alpha)r^k-\alpha(1+k)) \end{aligned}$$

and using (2.18) it is easily verified that $r_\alpha < R$. Hence $f(z)$ is convex in $|z| < r_\alpha$.

The function

$$f_\alpha(z) = \frac{\alpha(z+z^{k+1})}{\alpha+(1-\alpha)z^k} = z+\left(2-\frac{1}{\alpha}\right)z^{k+1}+\cdots$$

satisfies the hypothesis of the above theorem but is not convex in $|z| < r$ with $r \geqq r_\alpha$.

Letting α tend to infinity and putting $k = 1$ in theorem 1, we get the result of Reade, Ogawa and Sakaguchi [8].

THEOREM 2. *Suppose that* $f(z) = z+a_nz^n+a_{n+1}z^{n+1}+\cdots$ *is regular in* E *and satisfies* $|(f(z)/z)-\alpha| < \alpha(\alpha > \frac{1}{2})$ *for* z *in* E. *Then* $f(z)$ *maps* $|z| < r_\alpha$ *onto a univalent and starlike domain where*

$$(2.21) \quad r_\alpha = \begin{cases} [\{2(1-\alpha)+(2\alpha-1)n \\ \quad -((2(1-\alpha)+(2\alpha-1)n)^2+4\alpha(\alpha-1))^{\frac{1}{2}}\}/2(1-\alpha)]^{1/(n-1)} & \text{if } \alpha \neq 1, \\ 2^{-1/(n-1)} \quad \text{if } \alpha = 1. \end{cases}$$

This result is sharp.

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PROOF. Proceeding as in theorem 1, we have

$$f(z) = \frac{2\alpha z}{1+(2\alpha-1)P(z)}, \tag{2.22}$$

where $P(z)$ has a power series expansion of the form $P(z) = 1+c_{n-1}z^{n-1}+\cdots$. The representation (2.21) yields

$$\frac{zf'(z)}{f(z)} = 1-\frac{(2\alpha-1)zP'(z)}{1+(2\alpha-1)P(z)}. \tag{2.23}$$

Taking real parts on both sides of (2.23) and using lemma 1 we have for $|z| = r$, $0 \leqq r < 1$,

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) \geqq \frac{\alpha-(2(1-\alpha)+(2\alpha-1)n)r^{n-1}+(1-\alpha)r^{2(n-1)}}{\alpha-r^{n-1}-(\alpha-1)r^{2(n-1)}}.$$

A necessary and sufficient condition for $f(z)$ to map $|z| < r$ onto a univalent and starlike domain is that $\operatorname{Re}(zf'(z)/f(z)) > 0$ for $|z| < r$. From the above estimate we see that this condition is satisfied for $|z| < r_\alpha$ where r_α is given by (2.21).

The function

$$f_\alpha(z) = \frac{\alpha(z+z^n)}{\alpha+(1-\alpha)z^{n-1}} = z+\left(2-\frac{1}{\alpha}\right)z^n+\cdots$$

satisfies the hypothesis of the above theorem but is not univalent in $|z| < r$ with $r \geqq r_\alpha$ for $f'_\alpha(z)$ vanishes at $z = r_\alpha \exp(i\pi/(n-1))$.

Let $F(z) = zf'(z)$. If $F(z)$ be starlike with respect to the origin in $|z| < r$ then $f(z)$ is convex in $|z| < r$ [7, p. 223]. Also if $f(z)$ satisfies $|f'(z)-\alpha| < \alpha$ for z in E, then $F(z)$ satisfies $|(F(z)/z)-\alpha| < \alpha$ for z in E. Therefore we arrive at

COROLLARY 2.1. *Suppose that $f(z) = z+a_nz^n+a_{n+1}z^{n+1}+\cdots$ is regular in E and satisfies $|f'(z)-\alpha| < \alpha(\alpha > \frac{1}{2})$ for z in E. Then $f(z)$ maps $|z| < r_\alpha$ onto a convex domain where r_α is given by* (2.21). *This result is sharp, the extremal function being*

$$f_\alpha(z) = \int_0^z \frac{\alpha(1+z^{n-1})}{\alpha+(1-\alpha)z^{n-1}}\,dz.$$

Letting α tend to infinity and putting $n = 2$ in theorem 2 and corollary 2.1, we get the results of MacGregor [4].

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THEOREM 3. *Suppose that $f(z)$ is regular in E and satisfies $f(0) = 0, f'(0) = 1$ and $|(f(z)/(z+az^2))-\alpha| < \alpha$ ($|a| \leqq |$, $\alpha > \frac{1}{2}$) for z in E. Then $f(z)$ maps $|z| < r_\alpha(|a|)$ onto a univalent and starlike domain where r_α ($|a|$) is the smallest positive root of*

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the equation

$$\alpha-2\alpha(1+|a|)r+((1-\alpha)+(2\alpha+1)|a|)r^2-2(1-\alpha)|a|r^3 = 0.$$

This result is sharp, the extremal function being

$$f(z) = \frac{1+ze^{i\gamma}}{1+\left(\frac{1}{\alpha}-1\right)ze^{i\gamma}}(z+az^2), \quad (\gamma = \arg a, |a| \leqq 1).$$

The number $r_\alpha(|a|)$ *decreases monotonically as* $|a|$ *increases from* 0 *to* 1.

The case $a = 0$ and $\alpha = 1$ is Theorem 1 of [6].

The proof of the above theorem is similar to that of theorem 2 and is therefore omitted.

THEOREM 4 (A) *Suppose that* $f(z)$ *is regular in* E *and satisfies* $f(0) = 0$, $f'(0) = 1$ *and* Re $(f(z)/(z+az^2)) > 0$ $(|a| \leqq 1)$ *for* z *in* E. *Then* $f(z)$ *maps* $|z| < r(|a|)$ *onto a convex domain where* $r(|a|)$ *is the smallest positive root of the equation*

$$1-(5+4|a|)r+(6|a|-3)r^2+(10|a|-1)r^3+4|a|r^4 = 0.$$

(B) *Suppose that* $f(z)$ *is regular in* E *and satisfies* $f(0) = 0$, $f'(0) = 1$ *and* $|(f(z)|(z+az^2))-1| < 1$ $(|a| \leqq 1)$ *for* z *in* E. *Then* $f(z)$ *maps* $|z| < r_1(|a|)$ *onto a convex domain where*

$$r_1(|a|) = \begin{cases} \frac{1}{4} & \text{if } a = 0, \\ [2(1+|a|)-\{4(1+|a|^2)-|a|\}^{\frac{1}{2}}]/(9|a|) & \text{if } a \neq 0. \end{cases}$$

The above estimates are sharp and decrease monotonically as $|a|$ *increases from* 0 *to* 1.

PROOF OF THEOREM 4(A). Let $g(z) = z+az^2$. It is easy to see that for $|z| = r$, $0 \leqq r < 1/(2|a|)$,

$$(3.1) \qquad \left|\frac{zg''(z)}{g(z)}\right| \leqq \frac{2|a|r}{1-|a|r}, \qquad \left|\frac{zg'(z)}{g(z)}\right| \geqq \frac{1-2|a|r}{1-|a|r}.$$

Let

$$(3.2) \qquad \frac{f(z)}{g(z)} = \frac{1}{P(z)}.$$

The representation (3.2) yields the relation

$$(3.3) \qquad 1+\frac{zf''(z)}{f'(z)} = 1-\frac{2zP'(z)}{P(z)}-\frac{\dfrac{z^2P''(z)}{P(z)}-\dfrac{z^2g''(z)}{g(z)}}{\dfrac{zg'(z)}{g(z)}-\dfrac{zP'(z)}{P(z)}}.$$

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Using (3.1) and lemma 1, we have for $|z| = r$, $0 \leqq r < 1/(2|a|)$,

$$(3.4)\quad \left| \frac{zg'(z)}{g(z)} - \frac{zP'(z)}{P(z)} \right| \geqq \frac{1-2|a|r}{1-|a|r} - \frac{2r}{1-r^2}$$
$$= \frac{1-2(1+|a|)r+(2|a|-1)r^2+2|a|r^3}{(1-r^2)(1-|a|r)}.$$

Let r_0 be the smallest positive root of the equation

$$\psi(r) \equiv 1-2(1+|a|)r+(2|a|-1)r^2+2|a|r^3 = 0.$$

It is easy to verify that $r_0 < 1/(2|a|)$. Therefore using lemmas 1 and 3 and the inequalities (3.1) and (3.4), we have for $|z| = r$, $0 \leqq r < r_0$,

$$\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) \geqq \frac{1-(5+4|a|)r+(6|a|-3)r^2+(10|a|-1)r^3+4|a|r^4}{(1+r)[1-2(1+|a|)r+(2|a|-1)r^2+2|a|r^3]}.$$

Let $r(|a|)$ be the smallest positive root of the equation

$$\chi(r) \equiv 1-(5+4|a|)r+(6|a|-3)r^2+(10|a|-1)r^3+4|a|r^4 = 0.$$

It is easy to verify that $0 < r(|a|) < \frac{1}{4}$. Also $\psi(r)$ is monotonically decreasing for $0 \leqq r \leqq \frac{1}{4}$ and $\psi(\frac{1}{4})$ is positive. Therefore $r_0 > \frac{1}{4}$. Thus we see that $\operatorname{Re}(1+(zf''(z)/f'(z))) > 0$ for $|z| = r < r(|a|)$, which implies that $f(z)$ maps $|z| < r(|a|)$ onto a convex domain.

The above estimate is sharp because the function

$$f(z) = \frac{1+ze^{i\gamma}}{1-ze^{i\gamma}}(z+az^2) \quad (\gamma = \arg a, |a| \leqq 1)$$

satisfies the hypothesis of the above theorem but is not convex in $|z| < r$ with $r \geqq r(|a|)$.

It is easy to verify that $r(|a|)$ decreases monotonically as $|a|$ increases from 0 to 1.

Theorem 4(B) can be proved in the same manner as theorem 4(A). The extremal function in this case is

$$f(z) = (1+e^{i\gamma}z)(z+az^2) \quad (\gamma = \arg a, |a| \leqq 1).$$

4

Typically-real functions The function $f(z) = z+a_2z^2+\cdots$, regular in E is called typically-real in E if it is real on the diameter $-1 < z < 1$ and if at other points of the circle E, $\operatorname{Im}(f(z))\cdot\operatorname{Im}(z) > 0$, see [11]. The radius of starlikeness of this class is $(\sqrt{2}-1)$, see [2]. We find below the radius of starlikeness of a subclass of the class of typically-real functions.

THEOREM 5. *Suppose that $f(z)$ is regular and real on the real axis in E and satisfies the conditions $f(0) = 0, f'(0) = 1, f''(0) = 0$ and*

$$\left|\frac{(1-z^2)}{z}f(z)-\alpha\right| < \alpha \quad (\alpha > \tfrac{1}{2}), \tag{4.1}$$

for all z in E. Then $f(z)$ is univalent and starlike in $|z| < r_\alpha$ where r_α is the smallest positive root of the equation

$$\alpha-(5\alpha-1)r^2-(5\alpha-4)r^4+(\alpha-1)r^6 = 0. \tag{4.2}$$

This estimate is sharp, the extremal function being

$$f_\alpha(z) = \frac{\alpha z(1+z^2)}{(1-z^2)(\alpha+(1-\alpha)z^2)} = z+\left(3-\frac{1}{\alpha}\right)z^3+\cdots.$$

REMARK. A necessary and sufficient condition that $f(z)$ be typically real in E is that Re $\{(1-z^1)/zf(z)\} > 0$ and $1-z^2/zf(z)$ is real on the real axis for z in E, [10]. Evidently the functions which satisfy the hypothesis of the above theorem satisfy this condition and therefore form a subclass of the class of typically real functions.

PROOF OF THEOREM 5. Proceeding as in theorem 1 we have

$$f(z) = \frac{2\alpha z}{(1-z^2)(1+(2\alpha-1)P(z))}, \tag{4.3}$$

where $P(z)$ has a power series expansion of the form $P(z) = 1+c_2z^2+\cdots$. The representation (4.3) yields the relation

$$\frac{zf'(z)}{f(z)} = \frac{1+z^2}{1-z^2} - \frac{(2\alpha-1)zP'(z)}{(1+(2\alpha-1)P(z))}. \tag{4.4}$$

From lemma 1, we have for $|z| = r < 1$,

$$\left|\frac{zP'(z)}{(1/(2\alpha-1))+P(z)}\right| \leqq \frac{2(2\alpha-1)r^2}{(1-r^2)(\alpha-(1-\alpha)r^2)}. \tag{4.5}$$

From (4.4) and (4.5) we have for $|z| = r < 1$,

$$\text{Re}\left(\frac{zf'(z)}{f(z)}\right) \geqq \frac{1-r^2}{1+r^2} - \frac{2(2\alpha-1)r^2}{(1-r^2)(\alpha-(1-\alpha)r^2)}$$
$$= \frac{\alpha-(5\alpha-1)r^2-(5\alpha-4)r^4+(\alpha-1)r^6}{(1-r^4)(\alpha-(1-\alpha)r^2)}.$$

Thus we see that Re $((zf'(z))/f(z)) > 0$ in $|z| < r_\alpha$ where r_α is the smallest positive root of (4.2).

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If $f(z)$ be typically-real, then $F(z) = zf'(z)$ is real on the real axis and convex in the direction of the imaginary axis [10]. Therefore we arrive at

COROLLARY 5.1. *Suppose that $f(z)$ is regular in E and satisfies the conditions* $t(0) = 0, f'(0) = 1, f''(0) = 0$ *and*

$$|(1-z^2)f'(z)-\alpha| < \alpha \quad (\alpha > \tfrac{1}{2}),$$

for $|z| < 1$. Then $f(z)$ is convex in $|z| < r_\alpha$ where r_α is the smallest positive root of fhe equation (5.2). *This result is sharp, the extremal function being*

$$f_\alpha(z) = \int_0^z \frac{\alpha(1+z^2)}{(1-z^2)(\alpha+(1-\alpha)z^2)}\, dz.$$

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