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WEIGHTED p-SIDON SETS

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Abstract

A weighted generalization of a p-Sidon set, called an (a, p)-Sidon set, is introduced and studied for infinite, non-abelian, connected, compact groups G. The entire dual object $\widehat{G}$ is shown never to be central $(p-1, p)$-Sidon for $1 \leq p < 2$, nor central $(1+\varepsilon, 2)$-Sidon for $\varepsilon > 0$. Local (p, p)-Sidon sets are shown to be identical to local Sidon sets. Examples are constructed of infinite non-Sidon sets which are central $(2p-1, p)$-Sidon, or $(p-1, p)$-Sidon, for $1 < p < 2$. Full m-fold FTR sets are proved not to be central $(a, 2m/(m+1))$-Sidon for any $a > 1$.

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0. Introduction

Suppose G is a compact group; we take its dual object $\widehat{G}$ to be a maximal set of pairwise inequivalent, continuous, irreducible, unitary representations of G. We define below a new class of lacunary subsets of $\widehat{G}$ which we call (a, p)-Sidon sets, where $a \in \mathbb{R}$ and $1 \leq p < \infty$. These sets arise by considering classical Sidonicity under a Fourier transform weighted by ath powers of representation degrees. The usual Fourier transform corresponds to the case $a = 1$ and has been extensively studied for both abelian and non-abelian G: $(1, 1)$-Sidon sets are usually called Sidon sets, and $(1, p)$-Sidon sets are known simply as p-Sidon sets. One motivation for studying such sets is the disparity in the classical setting between the abelian and non-abelian theory which is caused by the existence of irreducible representations of unbounded degree. For example, whereas every infinite set in the dual of an abelian group contains an infinite Sidon subset, the same is not true in the non-abelian case even for those groups which

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admit infinite Sidon sets. Moderating the effect of unbounded degrees by weighting should make these sets more plentiful.

The primary purpose of this paper is to initiate an investigation of those values of a and p for which non-trivial examples of (central) (a, p)-Sidon sets exist. We begin by listing the easy relationships between (a, p)-Sidon and (b, q)-Sidon sets, as well as those values of a and p for which either abundant or only trivial examples can be found. Generalizing [6], we show that if G is an infinite, non-abelian, connected, compact group then $\widehat{G}$ is never central $(p-1, p)$-Sidon for any $1 \leq p < 2$, nor central $(1+\varepsilon, 2)$-Sidon for any $\varepsilon > 0$, and these results are sharp. Such groups are seen to admit an infinite Sidon set if, and only if, they admit an infinite (p, p)-Sidon set: indeed, a set is local (p, p)-Sidon if, and only if, it is local Sidon.

Our main positive results are for infinite products of almost simple, simply-connected, compact Lie groups and are largely inspired by Blei's work (particularly [1, 2]) on p-Sidon sets for abelian groups. For each $1 \leq p < 2$ we construct examples of infinite sets which are central $(2p-1, p)$-Sidon but no better, and we produce examples of infinite, non-Sidon $(p-1, p)$-Sidon sets for each $p = 2n/(n+1)$, $n = 2, 3, \ldots$. These examples are particular subsets of n-fold products of FTR sets; FTR sets were introduced in [3] to characterize non-abelian Sidon sets. In contrast, we show that other subsets of an n-fold product of FTR sets can fail to be central $(a, 2n/(n+1))$-Sidon, for any $a > 1$. This requires the existence on $SU(\ell)$ of a central trigonometric polynomial, with coefficients ± 1 and sup-norm equal to the rank of the group, to be explicitly demonstrated.

Weighted lacunary sets have also been studied in [5, 7 and 17].

1. Preliminaries

NOTATION 1.1. For $E \subseteq \widehat{G}$ let $\mathscr{E}(E) = \prod_{\sigma \in E} M_{d(\sigma)}(\mathbb{C})$, and $\mathscr{E}^z(E) = \prod_{\sigma \in E} \mathbb{C}\, I_{d(\sigma)}$, where we write $d(\sigma)$ for the degree of $\sigma \in \widehat{G}$, and $M_d(\mathbb{C})$ denotes the vector space of $d \times d$ complex matrices. For $a \in \mathbb{R}$ and $A = (A_\sigma)_{\sigma \in E} \in \mathscr{E}(E)$, let

$$\|A\|_{a,p} = \begin{cases} \left(\sum_{\sigma \in E} d(\sigma)^a \operatorname{Tr}(|A_\sigma|^p) \right)^{1/p}, & 1 \leq p < \infty, \\ \sup_{\sigma \in E} d(\sigma)^{1-a} \|A_\sigma\|_\infty, & p = \infty, \end{cases}$$

where $\|X\|_\infty$ denotes the largest eigenvalue of $|X|$. (See [11, Appendix D] for properties of the von Neumann norms $\left(\operatorname{Tr}(|X|^p)\right)^{1/p}$, there denoted $\|X\|_{\phi_p}$.)

Write

$$\ell_{a,p}(E) = \{A \in \mathscr{E}(E) : \|A\|_{a,p} < \infty\}$$
$$\text{and } \ell^z_{a,p}(E) = \{A \in \mathscr{E}^z(E) : \|A\|_{a,p} < \infty\}.$$

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We denote by $\mathscr{T}_E(G)$ the space of trigonometric polynomials $f = \sum_{\sigma\in\widehat{G}} d(\sigma)\,\mathrm{Tr}\, A_\sigma\sigma$ on G having Fourier transform $\widehat{f}$ supported on E, where

$$\widehat{f}(\sigma) := \int_G f(x)\sigma(x^{-1})\,dx = A_\sigma.$$

We write $\mathscr{T}_E^z(G)$ for the corresponding space of central trigonometric polynomials.

DEFINITIONS 1.2. Let $a \in \mathbb{R}$ and $1 \le p < \infty$. We call $E \subseteq \widehat{G}$ an *(a, p)-Sidon set* if its (a, p)-Sidon constant

$$\kappa_{a,p}(E) := \sup\{\|\widehat{f}\|_{a,p} : f \in \mathscr{T}_E(G),\ \|f\|_\infty \le 1\}$$

is finite, and a *central (a, p)-Sidon set* if

$$\varkappa_{a,p}(E) := \sup\{\|\widehat{f}\|_{a,p} : f \in \mathscr{T}_E^z(G),\ \|f\|_\infty \le 1\}$$

is finite. If $\kappa_{a,p}^0(E) := \sup\left\{\kappa_{a,p}(\{\sigma\}) : \sigma \in E\right\}$ is finite we call E a *local (a, p)-Sidon set*, and if $\varkappa_{a,p}^0(E) := \sup\left\{\varkappa_{a,p}(\{\sigma\}) : \sigma \in E\right\}$ is finite we call E a *local central (a, p)-Sidon set*. Clearly if E consists of representations of bounded degree then the parameter a in these definitions becomes superfluous.

We suppress E in the case $E = \widehat{G}$, and we suppress a in the case $a = 1$. So for example we write ℓ_p for $\ell_{1,p}(\widehat{G})$ and κ_p for $\kappa_{1,p}$. A subscript E on a space of functions will denote the subspace of functions having Fourier transform supported on E, and a superscript z will always denote the subset of central elements.

Not surprisingly, there are many equivalent characterizations of (a, p)-Sidon sets; we list some below.

PROPOSITION 1.3. *Let G be a compact group and $E \subseteq \widehat{G}$. Let $a \in \mathbb{R}$, $1 \le p < \infty$ and $1/p + 1/p' = 1$. Write $x = p'(1 - a/p)$ for $p > 1$ and $x = a$ if $p = 1$. Then the following are equivalent:*

(i) *E is (central) (a, p)-Sidon;*
(ii) *there is $C \in \mathbb{R}$ such that $\|\widehat{f}\|_{a,p} \le C\|f\|_\infty$ for all $f \in L_E^{\infty(z)}(G)$;*
(iii) *whenever $\phi \in \ell_{x,p'}^{(z)}(E)$ there is a (central) measure μ on G (with $\mu \in L^1$ in the case $p > 1$) such that $\widehat{\mu}(\sigma) = \phi(\sigma)$ for all $\sigma \in E$ and $\|\mu\|_{M(G)} \le C\|\phi\|_{x,p'}$ where C is a constant depending only on a, p and E.*

PROOF. This is similar to those found in [11, 37.2] or [9, 1.2].

Pisier's work has made prominent the equivalence of Sidon and central Λ sets; (a, p)-Sidon sets also satisfy a Λ-like property:

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PROPOSITION 1.4. *Let G be a compact group, $a \in \mathbb{R}$ and $1 \le p < \infty$. Write $r = 2p/(3p-2)$ and $t = (3p-2a)/(3p-2)$. If $E \subseteq \widehat{G}$ is (a, p)-Sidon then there is an absolute constant C so that*

$$\|f\|_{2s} \le C\kappa_{a,p}(E)\sqrt{s}\,\|\widehat{f}\|_{t,r}$$

for all $1 < s < \infty$ and all $f \in \mathscr{T}_E(G)$.

PROOF. This is similar to the known proof for $a = 1$ [4, 5.1].

We now set down some basic estimates for Sidon constants. The following inequality is used frequently, so for the reader's convenience we give it here:

THEOREM 1.5 ([11, D.51]). *Let $A \in M_d(\mathbb{C})$ and $1 \le p < q < \infty$. Then*

$$\left(\operatorname{Tr}(|A|^q)\right)^{1/q} \le \left(\operatorname{Tr}(|A|^p)\right)^{1/p} \le d^{1/p-1/q}\left(\operatorname{Tr}(|A|^q)\right)^{1/q}.$$

PROPOSITION 1.6. *Let G be a compact group and $\sigma \in E \subseteq \widehat{G}$. Then*

(i) $\varkappa_{a,p}(\sigma) = d(\sigma)^{(a+1-2p)/p}$;

(ii) $\kappa_{a,p}(\sigma) \ge d(\sigma)^{(a-p)/p}$;

(iii) $\kappa_{a,p}(E) \le \begin{cases} \left(\sum_{\sigma\in E} d(\sigma)^{(2a+2-2p)/(2-p)}\right)^{(2-p)/2p}, & 1 \le p < 2; \\ \sup_{\sigma\in E} d(\sigma)^{(2a-p)/2p}, & 2 \le p < \infty. \end{cases}$

PROOF. For (i) consider $f = \operatorname{Tr}\sigma$, and for (ii) consider $f = \sigma_{11}$ (the trigonometric polynomial obtained by fixing a basis for the representation space of σ and thinking of σ as a matrix-valued function on G).

For part (iii), consider $f = \sum_{\sigma\in E} d(\sigma)\operatorname{Tr} A_\sigma\sigma \in \mathscr{T}_E(G)$. When $1 \le p < 2$ we have

$$\begin{aligned}
\|\widehat{f}\|_{a,p}^p &= \sum_{\sigma\in E} d(\sigma)^a \operatorname{Tr}|A_\sigma|^p \\
&\le \sum_{\sigma\in E} d(\sigma)^a d(\sigma)^{1-p/2}\left(\operatorname{Tr}|A_\sigma|^2\right)^{p/2} \qquad \text{(by 1.5)} \\
&= \sum_{\sigma\in E} d(\sigma)^{a+1-p}\left(d(\sigma)\operatorname{Tr}|A_\sigma|^2\right)^{p/2} \\
&\le \left(\sum_{\sigma\in E}\left(d(\sigma)^{a+1-p}\right)^{2/(2-p)}\right)^{(2-p)/2}\|\widehat{f}\|_2^p \quad \text{(by Hölder).}
\end{aligned}$$

Since $\|\widehat{f}\|_2 = \|f\|_2 \le \|f\|_\infty$, the desired estimate for $\kappa_{a,p}(E)$ obtains.

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When $2 \le p < \infty$ write $D = \sup_{\sigma \in E} d(\sigma)^{(2a-p)/2p}$. Then $d(\sigma)^{a/p} \le D \cdot d(\sigma)^{1/2}$ for each $\sigma \in E$, so we have

$$\begin{aligned} \|\widehat{f}\|_{a,p}^p &\le \sum_{\sigma \in E} \left(D \cdot d(\sigma)^{1/2} (\operatorname{Tr}|A_\sigma|^2)^{1/2}\right)^p \quad \text{(using 1.5)} \\ &\le D^p \|\widehat{f}\|_2^p \end{aligned}$$

and the stated estimate for $\kappa_{a,p}(E)$ follows.

Obviously it is easier to be an (a, p)-Sidon set as a decreases or as p increases. More generally,

PROPOSITION 1.7. *Let G be a compact group, $E \subseteq \widehat{G}$, and let $1 \le p \le q < \infty$. Then*

(i) *$\kappa_{b,q}(E) \le \kappa_{a,p}(E)$ whenever $b \le aq/p$; hence (local) (a, p)-Sidon implies (local) (b, q)-Sidon;*
(ii) *$\varkappa_{b,q}(E) \le \varkappa_{a,p}(E)$ whenever $(b+1)/q \le (a+1)/p$; hence central (a, p)-Sidon implies central (b, q)-Sidon;*
(iii) *$\varkappa^0_{b,q}(E) \le \varkappa^0_{a,p}(E)$ if, and only if, $(b+1)/q \le (a+1)/p$; hence local central (a, p)-Sidon implies local central (b, q)-Sidon.*

In the opposite direction we have

(iv) *$\kappa^0_{b,q}(E) \ge \kappa^0_{a,p}(E)$ whenever $(b+1)/q \ge (a+1)/p$; hence local (b, q)-Sidon implies local (a, p)-Sidon.*

PROOF. These follow easily using 1.5 and 1.6(i).

For certain values of (a, p) it is easy to produce abundant examples of (local) (central) (a, p)-Sidon sets, or else to show that there are only trivial ones.

COROLLARY 1.8. *Let G be a compact group, $1 \le p < \infty$, and $\varepsilon > 0$. Then*

(i) *$\widehat{G}$ is local central (a, p)-Sidon for $a \le 2p - 1$;*
(ii) *$\widehat{G}$ is local (a, p)-Sidon for $a \le p - 1$ and $p < 2$;*
(iii) *$\widehat{G}$ is (a, p)-Sidon for $a \le p/2$ and $p \ge 2$;*
(iv) *every (local) Sidon set is also a (local) (p, p)-Sidon set;*
(v) *every (local) central Sidon set is also a (local) central $(2p - 1, p)$-Sidon set;*
(vi) *every local central $(2p - 1 + \varepsilon, p)$-Sidon set and every local $(p + \varepsilon, p)$-Sidon set consists of representations of bounded degree;*
(vii) *every infinite subset of $\widehat{G}$ contains an infinite (a, p)-Sidon subset for $a < p-1$ and $p < 2$;*
(viii) *if G is a Lie group and $1 \le p < 2$, every local $(p - 1 + \varepsilon, p)$-Sidon set consists of representations of bounded degree.*

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PROOF. These follow mainly from 1.6 and 1.7. For (vii) use [13, 2.5] if the degrees are bounded (in which case we obtain an infinite Sidon set), and otherwise choose a subset with degrees growing sufficiently fast to make the sum in 1.6(iii) converge. For (viii) imitate [8], where it is shown that sets of representations of unbounded degree cannot be local p-Sidon for any $1 \leq p < 2$.

The following fundamental facts often allow results for Lie groups to be carried over to connected compact groups.

STRUCTURE THEOREM 1.9 ([15, 6.5.6]). *Let G be a connected compact group and write T for the connected component of the identity of the centre of G. Then there exist a group $\mathscr{G} = \prod_{\iota \in I} G_\iota$, where $(G_\iota)_{\iota \in I}$ is a family of almost simple, simply-connected, compact Lie groups, and a continuous epimorphism $\pi : T \times \mathscr{G} \to G$ whose kernel is a totally disconnected closed subgroup of the centre of $T \times \mathscr{G}$ with the property that $(\ker \pi) \cap T$ contains only the identity. Consequently there is also a continuous epimorphism $\phi : G \to \prod_{\iota \in I} G_\iota / Z(G_\iota)$.*

PROPOSITION 1.10 (see [3, 2.2]). *Let $\phi : G \to H$ be a continuous epimorphism of compact groups. Let $E \subseteq \widehat{H}$ and put $E \circ \phi = \{\sigma \circ \phi : \sigma \in E\} \subseteq \widehat{G}$.*
Then $\kappa_{a,p}(E \circ \phi) = \kappa_{a,p}(E)$ (and similarly for $\kappa^0_{a,p}$, $\varkappa_{a,p}$, $\varkappa^0_{a,p}$).

We now show that $\widehat{G}$ being $(1, 2)$-Sidon cannot be improved upon for infinite, connected, compact groups.

PROPOSITION 1.11. *Let G be an infinite, connected, compact group. Then $\widehat{G}$ is not central $(p-1, p)$-Sidon for any $1 \leq p < 2$. If $\sup\{d(\sigma) : \sigma \in \widehat{G}\} = \infty$ then $\widehat{G}$ is not central $(a, 2)$-Sidon for any $a > 1$.*

REMARK. Previously it was known that $\widehat{G}$ could not be central p-Sidon for any $p < 2$ [6, Corollary 7].

We require a lemma, which, as noted by Dooley, can be proved by arguments similar to [11, 36.15], using [6, Theorem 1].

LEMMA 1.12. *If G is a connected compact Lie group, then for $1 \leq p \leq 2$ the set $M(C^z(G), \ell^z_p)$ of multipliers is isomorphic to $\ell^z_{2p/(2-p)}$.*

PROOF OF PROPOSITION 1.11. If G is abelian the result follows from [9, Theorem 3.1]. So we assume G is non-abelian and suppose firstly that G is a connected compact Lie group. If h and f are central trigonometric polynomials then also $h * f \in \mathscr{T}^z(G)$, whence

$$\|h\|_1 \, \|f\|_\infty \geq \|h * f\|_\infty \geq \frac{1}{\varkappa_{a,p}(\widehat{G})} \|\widehat{h * f}\|_{a,p}.$$

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Define $H \in \mathscr{T}^z(G)$ by $\widehat{H}(\sigma) = d(\sigma)^{(a-1)/p}\,\widehat{h}(\sigma)$. Then

$$\|h\|_1 \|f\|_\infty \geq \frac{1}{\varkappa_{a,p}(\widehat{G})} \|\widehat{H * f}\|_p,$$

so that $H \in M(C^z(G), \ell_p^z)$, with operator norm at most $\varkappa_{a,p}(\widehat{G})\|h\|_1$. Combining this with 1.12 we can assert the existence of a constant K, depending on G and p but independent of h, so that

$$\|h\|_1 \geq \frac{K}{\varkappa_{a,p}(\widehat{G})} \|\widehat{H}\|_{2p/(2-p)}.$$

By choosing h to be an appropriate bounded approximate identity we can derive the conclusion that

$$1 \geq \begin{cases} \dfrac{K}{2\varkappa_{a,p}(\widehat{G})} \left(\displaystyle\sum_{\sigma \in \widehat{G}} d(\sigma)^{(2+2a-2p)/(2-p)} \right)^{(2-p)/2p}, & 1 \leq p < 2, \\ \dfrac{K}{2\varkappa_{a,2}(\widehat{G})} \displaystyle\sup_{\sigma \in \widehat{G}} d(\sigma)^{(a-1)/2}, & p = 2, \end{cases}$$

which if $\widehat{G}$ is central (a, p)-Sidon is clearly impossible unless $a < p - 1$ in the first case or $a \leq 1$ in the second.

In the case G is a non-abelian, connected, compact, non-Lie group, it follows from the structure theorem 1.9 that there is an epimorphism $\phi\colon G \to G'$ where G' is a connected compact Lie group. It then follows from 1.10 that central (a, p)-Sidonicity of $\widehat{G}$ implies the same for $\widehat{G'}$, and the result is established.

COROLLARY 1.13. *Suppose G is a connected compact Lie group, and let $1 \leq p < 2$. Then $\widehat{G}$ is (a, p)-Sidon if, and only if, $\widehat{G}$ is central (a, p)-Sidon.*

PROOF. The calculation in the proof of 1.11 above, together with 1.6(iii), shows that

$$\varkappa_{a,p}(\widehat{G}) \geq \frac{K}{2} \kappa_{a,p}(\widehat{G}) \geq \frac{K}{2} \varkappa_{a,p}(\widehat{G}).$$

EXAMPLE 1.14. It follows from the Weyl dimension formula [12, 24.3] and formulæ for the positive roots of $SU(n+1)$ [18, p. 26] that

$$D = \sum_{\sigma \in SU(n+1)^\frown} d(\sigma)^{(2a+2-2p)/(2-p)} = K(n, a, p) \sum_{m_1, \dots, m_n \geq 1} \prod_{1 \leq i \leq j \leq n} \left(\sum_{\ell=i}^{j} m_\ell \right)^{(2a+2-2p)/(2-p)}.$$

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Write $b = (2a+2-2p)/(2-p)$ and assume $b < 0$. Then since $\sum_{\ell=i}^{j} m_\ell \geq \prod_{\ell=i}^{j} m_\ell^{1/(j+1-i)}$ we obtain

$$D \leq K \prod_{k=1}^{n} \sum_{m_k \geq 1} m_k^{bS_k}, \quad \text{where} \quad S_k = \sum_{s=1}^{\min(k,n+1-k)} \sum_{t=s}^{n+1-s} \frac{1}{t};$$

D is thus certainly finite whenever

$$\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right) \frac{2a+2-2p}{2-p} < -1.$$

Now if $1 \leq p < 2$ and $\varepsilon > 0$ it follows from 1.6(iii) that $\widehat{SU(n+1)}$ is $(p-1-\varepsilon, p)$-Sidon whenever $1 + 1/2 + \cdots + 1/n > (2-p)/2\varepsilon$; thus the first statement of 1.11 is sharp.

2. Products of central Sidon sets

For abelian groups, examples of p-Sidon, non-Sidon sets were constructed by taking m-fold products of dissociate sets, with $p \geq 2m/(m+1)$ (see [9, 14, 1]). As dissociate sets are Sidon it is natural to conjecture that if $E_j \subseteq \widehat{G}_j$ are (central) Sidon, then $E_1 \times \cdots \times E_m$ should be a (central) p-Sidon subset of $\widehat{(G_1 \times \cdots \times G_m)}$ provided $p \geq 2m/(m+1)$. In the case where the E_j are central Sidon, we show that such products are central $(2p-1, p)$-Sidon for $p \geq 2m/(m+1)$ and for no smaller p (so in particular these sets are not central Sidon if $m > 1$), while in the case where the E_j are Sidon, we show in Section 3 below that $E_1 \times \cdots \times E_m$ is $(p-1, p)$-Sidon for $p \geq 2m/(m+1)$.

We require ideas of Blei [1] regarding 'fractional cartesian products'.

NOTATION. Let $J \geq K$ be positive integers and let $N = \binom{J}{K}$. Let $G = \prod_{i=1}^{N} G_i$ be a product of compact groups, and suppose $E_i \subseteq \widehat{G}_i$ is infinite for each $1 \leq i \leq N$. Let $\{P_1, \ldots, P_N\}$ be the collection of natural projections from $\mathbb{N}^J$ to $\mathbb{N}^K$ (see [1, p. 80]), endow each E_i with a K-fold enumeration $E_i = \{\gamma_j^{(i)}\}_{j \in \mathbb{N}^K}$, and put

$$E^{J,K} = \left\{\gamma_{P_1(j)}^{(1)} \times \cdots \times \gamma_{P_N(j)}^{(N)} : j \in \mathbb{N}^J\right\} \subseteq \widehat{G}.$$

For example, when $K = 1$ we have $N = J$ and $E^{J,1}$ is simply $E_1 \times \cdots \times E_N$.

THEOREM 2.1. *If $E_i \subseteq \widehat{G}_i$ is central Sidon for each $1 \leq i \leq N$ then $E^{J,K}$ is central $(2p-1, p)$-Sidon if, and only if, $p \geq 2J/(J+K)$.*

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PROOF. We first prove sufficiency.

Let $I = \{\pi_i : i \in \mathbb{N}^K\}$ be the set of projections on $\mathbb{T}^{\mathbb{N}}$, endowed with a K-fold enumeration, and consider the set $I^{J,K} \subseteq \widehat{\left(\mathbb{T}^{\mathbb{N}} \times \cdots \times \mathbb{T}^{\mathbb{N}}\right)}$. Let

$$f(x_1, \ldots, x_N) = \sum_{j \in \mathbb{N}^J} a_j \operatorname{Tr}\left(\gamma^{(1)}_{P_1(j)}(x_1) \otimes \cdots \otimes \gamma^{(N)}_{P_N(j)}(x_N)\right) \in \mathscr{T}^z_{E^{J,K}}(G),$$

and denote by $d_{j(k)}$ the degree of $\gamma^{(k)}_{P_k(j)}$. It is useful to introduce the $I^{J,K}$-polynomial

$$g(\omega_1, \ldots, \omega_N) = \sum_{j \in \mathbb{N}^J} d_{j(1)} \cdots d_{j(N)}\, a_j\, \pi_{P_1(j)}(\omega_1) \cdots \pi_{P_N(j)}(\omega_N)$$

where $\omega = (\omega_1, \ldots, \omega_N) \in \mathbb{T}^{\mathbb{N}} \times \cdots \times \mathbb{T}^{\mathbb{N}}$. Choose ω with $|g(\omega)| \geq \frac{1}{2}\|g\|_\infty$.

Since the sets E_i are central Sidon sets there are central measures μ_i on G_i satisfying $\operatorname{Tr}\widehat{\mu_i}\left(\gamma^{(i)}_{P_i(j)}\right) = \pi_{P_i(j)}(\omega_i)\, d_{j(i)}$ with $\|\mu_i\|_{\mathrm{M}(G)} \leq \varkappa(E_i)$. If $\mu = \mu_1 \times \cdots \times \mu_N$ is the product measure on G, an application of Fubini's Theorem shows that

$$\operatorname{Tr}\widehat{\mu}\left(\gamma^{(1)}_{P_1(j)} \times \cdots \times \gamma^{(N)}_{P_N(j)}\right) = \prod_{i=1}^{N} \operatorname{Tr}\widehat{\mu_i}\left(\gamma^{(i)}_{P_i(j)}\right)$$

and $\quad \|\mu\|_{\mathrm{M}(G)} \leq \varkappa(E_1) \times \cdots \times \varkappa(E_N) \leq \varkappa \quad$ (say).

Thus $$\|f\|_\infty \geq \frac{1}{\varkappa}\left|\sum_{j \in \mathbb{N}^J} d_{j(1)} \cdots d_{j(N)}\, a_j\, \pi_{P_1(j)}(\omega_1) \cdots \pi_{P_N(j)}(\omega_N)\right| \geq \frac{1}{2\varkappa}\|g\|_\infty.$$

As $I^{J,K}$ is known to be a p-Sidon set if, and only if, $p \geq 2J/(J+K)$ [1, Theorem 1.6] the result easily follows.

Suppose now that $p < 2J/(J+K)$. Since $I^{J,K}$ is not p-Sidon, given any positive number M there is an $I^{J,K}$ polynomial g with $\|g\|_\infty = 1$ but $\|\widehat{g}\|_p > M$. Choose a finite subset $F \subseteq I$ with $\operatorname{supp}\widehat{g} \subseteq F^N$.

Let D denote the closed unit ball in $\mathbb{C}$, and let $b_j = \widehat{g}(\pi_{P_1(j)} \times \cdots \times \pi_{P_N(j)})$. For any $j \in \mathbb{N}^J$ with $b_j \neq 0$ define $\pi_{P_k(j)}$ on $D^{|F|}$ in the obvious way: $\pi_{P_k(j)}(z) = z_{P_k(j)}$. (Here F has the inherited K-fold enumeration.) Define h: $D^{|F|} \times \cdots \times D^{|F|} \to \mathbb{C}$ by

$$h(z_1, \ldots, z_N) = \sum_{j \in \mathbb{N}^J} b_j\, \pi_{P_1(j)}(z_1) \cdots \pi_{P_N(j)}(z_N).$$

Since h is a polynomial in several complex variables, the maximum principle implies that it attains its maximum at a point all of whose coordinates are of modulus one. Thus $\|h\|_\infty = \|g\|_\infty = 1$.

Finally, define a central $E^{J,K}$-polynomial f by

$$f(x_1, \ldots, x_N) = \sum_{j \in \mathbb{N}^J} \left(d_{j(1)} \cdots d_{j(N)}\right)^{-1} b_j \operatorname{Tr}\left(\gamma^{(1)}_{P_1(j)}(x_1) \otimes \cdots \otimes \gamma^{(N)}_{P_N(j)}(x_N)\right).$$

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Notice that $f(x_1, \dots, x_N) = h(z_1, \dots, z_N)$ where the $P_k(j)$ component of z_k is $d_{j(k)}^{-1} \operatorname{Tr} \gamma_{P_k(j)}^{(k)}(x_k)$, so that $\|f\|_\infty \le 1$. But

$$\|\widehat{f}\|_{2p-1,p} = \Big(\sum_{j\in\mathbb{N}^J} |b_j|^p\Big)^{1/p} > M,$$

and since M was arbitrary, $E^{J,K}$ cannot be central $(2p-1, p)$-Sidon.

REMARKS. This theorem is sharp in the sense that for $a > 2p-1$ the only central (a, p)-Sidon sets consist entirely of representations of bounded degree 1.8(vi). We do not know if, for all such sets E_i, $E^{J,K}$ is central (b, p)-Sidon for some $p < 2J/(J+K)$ and $b < 2p-1$. It does follow easily that (using the notation of the proof)

$$|a_j| \le \frac{2\varkappa\|f\|_\infty}{d_{j(1)} \cdots d_{j(N)}};$$

thus $E^{J,K}$ can be seen to be central $(b, 1)$-Sidon for certain $b < 1$, provided the degrees of the representations in $E^{J,K}$ tend sufficiently fast to infinity.

Similar arguments to those in the theorem can be used to prove that if $E_i \subseteq \widehat{G}_i$ are central $(a, 1)$-Sidon, then $E^{J,K}$ is central $((1+a)p-1, p)$-Sidon for $p \ge 2J/(J+K)$. Dooley [7] has shown that the dual of every semisimple, connected, compact Lie group contains an infinite set which is central p-Sidon for all $p > 1$. (In contrast, such groups admit no infinite central Sidon sets [16].) By making the obvious modifications to Dooley's proof, the set he constructs can also be shown to be central (a, p)-Sidon for all $a < 2p-1$ and $p \ge 1$.

DEFINITION 2.2 ([3, 5.1], [19, 3.1]). Let G_ι be an almost simple, simply-connected, compact Lie group of rank ℓ_ι with fundamental weights $\lambda_1, \dots, \lambda_{\ell_\iota}$; we identify a representation of G_ι with its highest weight. We define

$$\mathrm{FTR}(G_\iota) = \begin{cases} \{\lambda_1, \lambda_{\ell_\iota}\}, & G_\iota \text{ of type } A_{\ell_\iota}; \\ \{\lambda_1\}, & G_\iota \text{ of type } B_{\ell_\iota} \text{ or } C_{\ell_\iota}\ (\ell_\iota \ge 3),\ \text{or } D_{\ell_\iota}\ (\ell_\iota \ge 5); \\ \{\lambda_1, \lambda_3, \lambda_4\}, & G_\iota \text{ of type } D_4; \\ \emptyset, & \text{otherwise.} \end{cases}$$

Let G be a compact connected group, with covering epimorphism $\pi: T \times \mathscr{G} \to G$ as in 1.9. Write $\pi_\iota: \mathscr{G} \to G_\iota$ for the canonical projection. We define the *Figà-Talamanca–Rider set of G* by

$$\mathrm{FTR}(\mathscr{G}) = \bigcup\nolimits_{\iota\in I} \mathrm{FTR}(G_\iota) \circ \pi_\iota;$$
$$\mathrm{FTR}(T \times \mathscr{G}) = 1 \times \mathrm{FTR}(\mathscr{G});$$
$$\mathrm{FTR}(G) \circ \pi = (\widehat{G} \circ \pi) \cap \mathrm{FTR}(T \times \mathscr{G}).$$

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Note that $\kappa\big(\text{FTR}(G)\big) \le \kappa\big(\text{FTR}(\mathscr{G})\big) \le 32$ ([19, 3.4]); that is, FTR sets are Sidon.

COROLLARY 2.3. *Let $G = \prod_{i=1}^{N} G_i$, where each G_i is a product of infinitely many almost simple, simply-connected, compact Lie groups. Let $E_i = \text{FTR}(G_i)$, $1 \le i \le N$. Then $E_1 \times \cdots \times E_N$ is central $(2p-1, p)$-Sidon if, and only if, $p \ge 2N/(N+1)$.*

3. Products of Sidon sets

DEFINITION 3.1. Let G be a compact group and $2 < p < \infty$. A subset E of $\widehat{G}$ is called a $\Lambda(p)$ *set* if there is a constant η_p such that $\|f\|_p \le \eta_p \|f\|_2$ whenever $f \in \mathscr{T}_E(G)$. Local and central $\Lambda(p)$ sets are defined in the obvious way.

THEOREM 3.2. *Let $\mathscr{G} = \prod_{\iota \in I} G_\iota$ where $(G_\iota)_{\iota \in I}$ is a family of almost simple, simply-connected, compact Lie groups, and let $m \in \mathbb{Z}^+$. Let $E \subseteq \prod_{\iota \in I} \text{FTR}(G_\iota)$ consist of those representations having at most m non-trivial components. Then E is $\Lambda(2s)$ for all $s \in \mathbb{Z}^+$.*

REMARK. A similar result, for G_ι of type A_{ℓ_ι} only, was given in [10, 4.8]; it is required in the proof below.

PROOF. In view of [10, 4.4] and [16, Theorem 5] we need only prove that E is local $\Lambda(2s)$. Further, without loss of generality we may suppose $\text{FTR}(G_\iota) \ne \emptyset$ for each $\iota \in I$. Write d_ι for the common degree of all elements of $\text{FTR}(G_\iota)$, and put $\mathscr{U} = \prod_{\iota \in I} SU(d_\iota)$.

Let $\sigma \in E$, and let $\{\iota \in I : \sigma\big|_{G_\iota} \ne 1\} = \{1, \dots, m\}$. Consider $f(x) = d\ \text{Tr}\big(A\,\sigma_1(x_1) \otimes \cdots \otimes \sigma_m(x_m)\big) \in \mathscr{T}_{\{\sigma\}}(\mathscr{G})$, where we write $d = d_1 \times \cdots \times d_m$. For each $\omega = (\omega_\iota)_{\iota \in I} \in \mathscr{U}$ define

$$F(\omega, x) = d\ \text{Tr}\big(A\,\omega_1 \otimes \cdots \otimes \omega_m\, \sigma_1(x) \otimes \cdots \otimes \sigma_m(x)\big).$$

Since $\{1\} \cup \text{FTR}(G_j)$ is a Sidon set with Sidon constant at most $\kappa = \kappa\big(\{1\} \cup \text{FTR}(\mathscr{G})\big) < \infty$, for each $\omega \in \mathscr{U}$ there are measures $\mu_j^{(\omega)}$ on G_j with $\|\mu_j^{(\omega)}\| \le \kappa$, $\widehat{\mu_j^{(\omega)}}(1) = 1$ and $\widehat{\mu_j^{(\omega)}}(\sigma_j) = \omega_j^*$ for $j = 1, \dots, m$.

Consider the measure μ_ω on $\mathscr{G}$ defined by

$$\int_{\mathscr{G}} g(x)\,d\mu_\omega \equiv \int_{G_m} \cdots \int_{G_1} g\Big|_{G_1 \times \cdots \times G_m} (x_1, \dots, x_m)\, d\mu_1^{(\omega)}(x_1) \cdots d\mu_m^{(\omega)}(x_m).$$

Clearly $\|\mu_\omega\| \le \kappa^m$ and as μ_ω is a product measure it is easy to prove that

$$\widehat{\mu_\omega}(\sigma_1 \times \cdots \times \sigma_m) = \omega_1^* \otimes \cdots \otimes \omega_m^*.$$

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Let $f_\omega(x) = F(\omega, x)$. Comparing Fourier transforms we see that $f_\omega * \mu_\omega = f$. Thus

$$\|f\|_{2s}^{2s} \leq \|f_\omega\|_{2s}^{2s}\ \|\mu_\omega\|^{2s} \leq \|f_\omega\|_{2s}^{2s}\ \kappa^{2sm}$$

for all $\omega \in \mathscr{U}$. Hence

$$\|f\|_{2s}^{2s} = \int_{\mathscr{U}} \|f\|_{2s}^{2s}\, d\omega \leq \kappa^{2sm} \int_{\mathscr{G}} \left(\int_{\mathscr{U}} |F(\omega, x)|^{2s}\, d\omega \right) dx.$$

If $g_x(\omega) \equiv F(\omega, x)$ and π_ι denotes the self-representation of $SU(d_\iota)$, then we can write

$$g_x(\omega) = d\ \operatorname{Tr}\big(\sigma(x)\, A\, (\pi_1 \times \cdots \times \pi_m)(\omega)\big).$$

Since ([10, 4.8]) $\{\pi_{\iota_1} \times \cdots \times \pi_{\iota_m} : \{\iota_1, \ldots, \iota_m\} \subseteq I\} \subseteq \widehat{\mathscr{U}}$ is a $\Lambda(2s)$ set with constant $\eta = \eta(m)$, we have

$$\|g_x\|_{2s}^{2s} \leq \eta^{2s} \|g_x\|_2^{2s} = \eta^{2s} \big(d\ \operatorname{Tr}|A|^2\big)^s,$$

using the fact that $\sigma(x) = \sigma_1(x_1) \otimes \cdots \otimes \sigma_m(x_m)$ is a unitary matrix. We conclude that

$$\|f\|_{2s} \leq \left(\kappa^{2sm} \int_{\mathscr{G}} \eta^{2s}\, \big(d\ \operatorname{Tr}|A|^2\big)^s\, dx\right)^{1/2s} = \kappa^m \eta \|f\|_2,$$

which proves the theorem.

THEOREM 3.3. *Let* $\mathscr{G} = \prod_{i=1}^n \mathscr{G}_i$, *where each* $\mathscr{G}_i$ *is a product of almost simple, simply-connected, compact Lie groups, and let* $E_i = \mathrm{FTR}(\mathscr{G}_i)$. *Then* $E = E_1 \times \cdots \times E_n$ *is* $(p-1, p)$*-Sidon for* $p = 2n/(n+1)$. *In fact, there is* $0 < K < \infty$ *such that*

$$(*_n) \qquad \|f\|_\infty \geq K \Big(\sum_{\sigma \in E} d(\sigma)^{p-1} \sum_{1 \leq i,j \leq d(\sigma)} |(A_\sigma)_{ij}|^p \Big)^{1/p}$$

for any $f = \sum_{\sigma \in E} d(\sigma) \operatorname{Tr} A_\sigma \sigma \in \mathscr{T}_E(\mathscr{G})$.

REMARKS. These sets are known to not be Sidon if at least two of the $\mathrm{FTR}(\mathscr{G}_i)$ are infinite [3, 5.4.2].

The proof of the theorem follows a method similar to that used in [2, Chapter 1] for the abelian case.

That E is $(p-1, p)$-Sidon follows directly from $(*_n)$ together with

LEMMA 3.4. *Let* $A = (A_{ij})$ *be a* $d \times d$ *matrix. Then for any* $1 \leq p \leq 2$

$$\operatorname{Tr}|A|^p \leq 2^{2-p} \sum_{i,j=1}^d |A_{ij}|^p.$$

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PROOF. Obviously this is true for $p = 2$. The case $p = 1$ is straightforward if A is assumed to be normal, and extends easily to arbitrary A using the triangle inequality and noting that $A \pm A^*$ are normal. By the Riesz–Thorin interpolation theorem and the interpolation theorem for the Schatten p-classes [21, p. 31] the inequality holds for $1 \leq p \leq 2$.

PROOF OF THEOREM. We prove $(*_n)$ by induction on n: consider $n = 1 = p$. We know

$$\|f\|_\infty \geq \frac{1}{\kappa(E)} \sum_{\sigma \in E} d(\sigma) \operatorname{Tr} |A_\sigma|;$$

an easy calculation shows that for any $d \times d$ matrix A

$$\operatorname{Tr} |A| \geq \frac{1}{d} \sum_{i,j=1}^{d} |A_{ij}|.$$

It follows that

$$\|f\|_\infty \geq \frac{1}{\kappa(E)} \sum_{\sigma \in E} \sum_{1 \leq i,j \leq d(\sigma)} \left|(A_\sigma)_{ij}\right|,$$

which is $(*_1)$.

Assume the result for $(*_{n-1})$. We first obtain two preliminary lower bounds for $\|f\|_\infty$.

Write

$$f(x_1, \dots, x_n) = \sum_{\sigma = \sigma_1 \times \cdots \times \sigma_n \in E} d(\sigma) \operatorname{Tr} \big(A_\sigma\, \sigma_1(x_1) \otimes \cdots \otimes \sigma_n(x_n)\big),$$

and label the entries of A_σ as $(A_\sigma)^{i_1 j_1}_{\vdots\, i_n j_n}$ in accordance with the tensor product multiplication. We write $d_k = d(\sigma_k)$, and always assume that $1 \leq i_k, j_k \leq d_k$, $1 \leq k \leq n$. In what follows K will denote a constant which may vary from line to line.

LEMMA 3.5. *We have*

$$\|f\|_\infty \geq K \left[\sum_{\substack{\sigma_k \in E_k;\ i_k, j_k \\ 1 \leq k \leq n-1}} (d_1 \cdots d_{n-1})^{(n-2)/n} \Big(\sum_{\substack{\sigma_n \in E_n; \\ i_n, j_n}} d_n \big|(A_\sigma)^{i_1 j_1}_{\vdots\, i_n j_n}\big|^2 \Big)^{(n-1)/n} \right]^{n/(2(n-1))}.$$

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LEMMA 3.6. *We have*

$$\|f\|_\infty \geq K \sum_{\substack{\sigma_n \in E_n; \\ i_n, j_n}} \Bigg(\sum_{\substack{\sigma_k \in E_k;\ i_k, j_k \\ 1 \leq k \leq n-1}} d_1 \cdots d_{n-1} \Big| (A_\sigma)^{i_1 j_1}_{\ddots\, i_n j_n} \Big|^2 \Bigg)^{1/2}.$$

Once the two lemmas are proved, the theorem is obtained as follows:

$$\sum_{\substack{\sigma_k \in E_k;\ i_k, j_k \\ 1 \leq k \leq n}} (d_1 \cdots d_n)^{(n-1)/(n+1)} \Big| (A_\sigma)^{i_1 j_1}_{\ddots\, i_n j_n} \Big|^{2n/(n+1)}$$

$$= \sum_{\substack{\sigma_k \in E_k;\ i_k, j_k \\ 1 \leq k \leq n-1}} (d_1 \cdots d_{n-1})^{(n-1)/(n+1)} \Bigg[\sum_{\substack{\sigma_n \in E_n; \\ i_n, j_n}} d_n^{(n-1)/(n+1)} \Big| (A_\sigma)^{i_1 j_1}_{\ddots\, i_n j_n} \Big|^{2(n-1)/(n+1)} \Big| (A_\sigma)^{i_1 j_1}_{\ddots\, i_n j_n} \Big|^{2/(n+1)} \Bigg]$$

$$\leq \sum_{\substack{\sigma_k \in E_k;\ i_k, j_k \\ 1 \leq k \leq n-1}} (d_1 \cdots d_{n-1})^{(n-1)/(n+1)} \Bigg[\sum_{\substack{\sigma_n \in E_n; \\ i_n, j_n}} d_n \Big| (A_\sigma)^{i_1 j_1}_{\ddots\, i_n j_n} \Big|^2 \Bigg]^{(n-1)/(n+1)} \Bigg[\sum_{\substack{\sigma_n \in E_n; \\ i_n, j_n}} \Big| (A_\sigma)^{i_1 j_1}_{\ddots\, i_n j_n} \Big| \Bigg]^{2/(n+1)}$$

(by Hölder with $(n+1)/(n-1)$ and $(n+1)/2$)

$$= \sum_{\substack{\sigma_k \in E_k;\ i_k, j_k \\ 1 \leq k \leq n-1}} \Bigg((d_1 \cdots d_{n-1})^{(n-2)/(n+1)} \Bigg[\sum_{\substack{\sigma_n \in E_n; \\ i_n, j_n}} d_n \Big| (A_\sigma)^{i_1 j_1}_{\ddots\, i_n j_n} \Big|^2 \Bigg]^{(n-1)/(n+1)} \Bigg) \cdot \Bigg(\sum_{\substack{\sigma_n \in E_n; \\ i_n, j_n}} (d_1 \cdots d_{n-1})^{1/2} \Big| (A_\sigma)^{i_1 j_1}_{\ddots\, i_n j_n} \Big| \Bigg)^{2/(n+1)}$$

$$\leq \Bigg(\sum_{\substack{\sigma_k \in E_k;\ i_k, j_k \\ 1 \leq k \leq n-1}} (d_1 \cdots d_{n-1})^{(n-2)/n} \Bigg[\sum_{\substack{\sigma_n \in E_n; \\ i_n, j_n}} d_n \Big| (A_\sigma)^{i_1 j_1}_{\ddots\, i_n j_n} \Big|^2 \Bigg]^{(n-1)/n} \Bigg)^{n/(n+1)} \cdot \Bigg(\sum_{\substack{\sigma_k \in E_k;\ i_k, j_k \\ 1 \leq k \leq n-1}} \Bigg[\sum_{\substack{\sigma_n \in E_n; \\ i_n, j_n}} (d_1 \cdots d_{n-1})^{1/2} \Big| (A_\sigma)^{i_1 j_1}_{\ddots\, i_n j_n} \Big| \Bigg]^2 \Bigg)^{1/(n+1)}$$

(by Hölder with $(n+1)/n$ and $n+1$)

$$\leq K \|f\|_\infty^{2(n-1)/(n+1)} \Bigg(\sum_{\substack{\sigma_n \in E_n; \\ i_n, j_n}} \Bigg[\sum_{\substack{\sigma_k \in E_k;\ i_k, j_k \\ 1 \leq k \leq n-1}} d_1 \cdots d_{n-1} \Big| (A_\sigma)^{i_1 j_1}_{\ddots\, i_n j_n} \Big|^2 \Bigg]^{1/2} \Bigg)^{2/(n+1)}$$

(by 3.5 and Minkowski's inequality). Now apply 3.6 and note that when $p = 2n/(n+1)$ we have $p - 1 = (n-1)/(n+1)$ so this proves $(*_n)$.

It remains to prove the two lemmas.

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PROOF OF LEMMA 3.5. Expanding out the matrix multiplication, we find that

$$f(x_1,\dots,x_n)=\sum_{\substack{\sigma_k\in E_k;\ i_k,j_k\\ 1\le k\le n-1}} d_1\cdots d_{n-1}\Big(\sum_{\substack{\sigma_n\in E_n;\\ i_n,j_n}} d_n(A_\sigma)^{i_1j_1}_{\ddots\, i_nj_n}\big(\sigma_n(x_n)\big)_{i_nj_n}\Big)\cdot \big(\sigma_1(x_1)\big)_{i_1j_1}\cdots\big(\sigma_{n-1}(x_{n-1})\big)_{i_{n-1}j_{n-1}}.$$

If we denote by $B=B(\sigma_1\dots,\sigma_{n-1};x_n)$ the matrix whose $i_1j_1,\dots,i_{n-1}j_{n-1}$ entry is

$$\sum_{\substack{\sigma_n\in E_n;\\ i_n,j_n}} d_n(A_\sigma)^{i_1j_1}_{\ddots\, i_nj_n}\big(\sigma_n(x_n)\big)_{i_nj_n}$$

then we can write

$$f(x_1,\dots,x_n)=\sum_{\substack{\sigma_k\in E_k\\ 1\le k\le n-1}} d_1\cdots d_{n-1}\operatorname{Tr}\big(B\sigma_1(x_1)\otimes\cdots\otimes\sigma_{n-1}(x_{n-1})\big).$$

Applying the induction hypothesis $(*_{n-1})$ we see that

$$\begin{aligned}\|f\|_\infty^{2(n-1)/n}&=\sup_{x_n}\Big(\sup_{x_1,\dots,x_{n-1}}|f(x_1,\dots,x_n)|\Big)^{2(n-1)/n}\\ &\ge K\sup_{x_n}\sum_{\substack{\sigma_k\in E_k\\ 1\le k\le n-1}}(d_1\cdots d_{n-1})^{(n-2)/n}\sum_{\substack{i_k,j_k\\ 1\le k\le n-1}}\Big|B^{i_1j_1}_{\ddots\, i_{n-1}j_{n-1}}\Big|^{2(n-1)/n}\\ &=K\sup_{x_n}\sum_{\substack{\sigma_k\in E_k\\ 1\le k\le n-1}}(d_1\cdots d_{n-1})^{(n-2)/n}\sum_{\substack{i_k,j_k\\ 1\le k\le n-1}}\Big|\sum_{\substack{\sigma_n\in E_n;\\ i_n,j_n}} d_n(A_\sigma)^{i_1j_1}_{\ddots\, i_nj_n}\big(\sigma_n(x_n)\big)_{i_nj_n}\Big|^{2(n-1)/n}.\end{aligned}$$

Now writing $C_{\sigma_n}=C_{\sigma_n}(\sigma_1,\dots,\sigma_{n-1};i_1j_1,\dots,i_{n-1}j_{n-1})$ for the matrix whose i_nj_n entry is $(A_\sigma)^{i_1j_1}_{\ddots\, i_nj_n}$ the innermost sum becomes $\sum_{\sigma_n\in E_n}d_n\operatorname{Tr}\big(C_{\sigma_n}\sigma_n(x_n)\big)$. Because $\mathrm{FTR}(\mathscr{G}_n)$ is a $\Lambda(2s)$ set for all $s\in\mathbb{Z}^+$ it follows that

$$\begin{aligned}\|f\|_\infty^{2(n-1)/n}&\ge K\sum_{\substack{\sigma_k\in E_k;\ i_k,j_k\\ 1\le k\le n-1}}(d_1\cdots d_{n-1})^{(n-2)/n}\int_{\mathscr{G}_n}\Big|\sum_{\sigma_n\in E_n}d_n\operatorname{Tr}C_{\sigma_n}\sigma_n(x_n)\Big|^{2(n-1)/n}dx_n\\ &\ge K\sum_{\substack{\sigma_k\in E_k;\ i_k,j_k\\ 1\le k\le n-1}}(d_1\cdots d_{n-1})^{(n-2)/n}\Big(\sum_{\sigma_n\in E_n}d_n\operatorname{Tr}|C_{\sigma_n}|^2\Big)^{(n-1)/n}.\end{aligned}$$

Since $\operatorname{Tr}|C_{\sigma_n}|^2=\sum_{i_nj_n}|(C_{\sigma_n})_{i_nj_n}|^2$, this completes the proof of 3.5.

PROOF OF LEMMA 3.6. Let $B_{\sigma_n}=B_{\sigma_n}(x_1,\dots,x_{n-1})$ denote the matrix whose i_nj_n entry is

$$\sum_{\substack{\sigma_k\in E_k;\ i_k,j_k\\ 1\le k\le n-1}} d_1\cdots d_{n-1}(A_\sigma)^{i_1j_1}_{\ddots\, i_nj_n}\big(\sigma_1(x_1)\big)_{i_1j_1}\cdots\big(\sigma_{n-1}(x_{n-1})\big)_{i_{n-1}j_{n-1}};$$

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then $f(x_1, \dots, x_n) = \sum_{\sigma_n \in E_n} d_n \operatorname{Tr}\left(B_{\sigma_n}\sigma_n(x_n)\right)$. As E_n is Sidon, we obtain

$$\begin{aligned}\|f\|_\infty &\geq K \sup_{x_1,\dots,x_{n-1}} \sum_{\sigma_n \in E_n} d_n \operatorname{Tr}|B_{\sigma_n}| \\ &\geq K \sup_{x_1,\dots,x_{n-1}} \sum_{\substack{\sigma_n \in E_n;\\ i_n, j_n}} \left|(B_{\sigma_n})_{i_n j_n}\right|.\end{aligned}$$

Since 3.2 $E_1 \times \cdots \times E_{n-1}$ is $\Lambda(2s)$, it follows that

$$\begin{aligned}\|f\|_\infty \geq K \sum_{\substack{\sigma_n \in E_n;\\ i_n, j_n}} \int_{\mathscr{G}_1} \cdots \int_{\mathscr{G}_{n-1}} \Bigg| \sum_{\substack{\sigma_k \in E_k;\ i_k, j_k\\ 1 \leq k \leq n-1}} d_1 \cdots d_{n-1} (A_\sigma)^{i_1 j_1}_{\ddots\; i_n j_n} \cdot \\ \left(\sigma_1(x_1)\right)_{i_1 j_1} \cdots \left(\sigma_{n-1}(x_{n-1})\right)_{i_{n-1} j_{n-1}} \Bigg| \, dx_1 \dots dx_{n-1} \\ \geq K \sum_{\substack{\sigma_n \in E_n;\\ i_n, j_n}} \Bigg(\sum_{\substack{\sigma_k \in E_k;\ i_k, j_k\\ 1 \leq k \leq n-1}} d_1 \cdots d_{n-1} \left|(A_\sigma)^{i_1 j_1}_{\ddots\; i_n j_n}\right|^2 \Bigg)^{1/2}\end{aligned}$$

which completes the proof.

4. m-fold FTR sets need not be central (a, p)-Sidon

In [10, §4] it was shown that m-fold FTR sets are central $\Lambda(p)$ for all $p > 2$, and partial m-fold FTR sets are $\Lambda(p)$ for all $p > 2$. We show now that, unlike the examples of Section 2, these sets need not be central $(2p-1, p)$-Sidon if $m > 1$, nor central $(a, 2m/(m+1))$-Sidon for any $a > 1$. In contrast to the previous section, where products of representations across different factors of a product group were considered, here we consider products within the separate factors.

Throughout this section, let G_ℓ denote $SU(\ell+1)$ and $\lambda_1, \dots, \lambda_\ell$ the fundamental weights. We write $\lambda_0 = \lambda_{\ell+1} = 0$ and identify representations of G_ℓ with their highest weights. For $m \leq \ell$ we write

$$E_\ell^m = \{(m-k)\lambda_1 + \lambda_k : 1 \leq k \leq m\}.$$

Observe that $k\lambda_1$ and λ_k both occur in λ_1^k for $1 \leq k \leq m$: hence $(m-k)\lambda_1 + \lambda_k$ occurs in $\lambda_1^{m-k} \otimes \lambda_1^k = \lambda_1^m$ for $1 \leq k \leq m$ and it follows that

$$E_\ell^m \subseteq \{\lambda \in \widehat{G}_\ell : \lambda \leq \lambda_1^m\}.$$

Thus E_ℓ^m is a subset of the m-fold FTR set of G_ℓ.

PROPOSITION 4.1. *Let $m \in \mathbb{Z}^+$. Then for any $\ell \geq m$ there is a central trigonometric polynomial f on G_ℓ, with m non-zero terms, coefficients ± 1, and $\|f\|_\infty = \ell + 1$.*

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PROOF. Consider

$$f := \sum_{k=1}^{m} (-1)^{k-1} \operatorname{Tr}\big((m-k)\lambda_1 + \lambda_k\big) \in \mathscr{T}^{z}_{E_\ell}(G_\ell);$$

we must show $\|f\|_\infty = \ell + 1$. Now for $k < m$ we have [20, §79 Example 3]

$$(m-k)\lambda_1 \otimes \lambda_k = \big((m-k)\lambda_1 + \lambda_k\big) \oplus \big((m-k-1)\lambda_1 + \lambda_{k+1}\big),$$

so that

$$\operatorname{Tr}\big((m-k)\lambda_1\big)\operatorname{Tr}(\lambda_k) = \operatorname{Tr}\big((m-k)\lambda_1 + \lambda_k\big) + \operatorname{Tr}\big((m-k-1)\lambda_1 + \lambda_{k+1}\big),$$

or

$$\operatorname{Tr}\big((m-k)\lambda_1 + \lambda_k\big) = \operatorname{Tr}\big((m-k)\lambda_1\big)\operatorname{Tr}(\lambda_k) - \operatorname{Tr}\big((m-k-1)\lambda_1 + \lambda_{k+1}\big).$$

Applying this recursively we obtain

$$\operatorname{Tr}\big((m-k)\lambda_1 + \lambda_k\big) = \sum_{j=0}^{m-k} (-1)^j \operatorname{Tr}\big((m-k-j)\lambda_1\big)\operatorname{Tr}(\lambda_{k+j})$$

whence

$$f = \sum_{k=1}^{m} (-1)^{k-1} k \operatorname{Tr}\big((m-k)\lambda_1\big)\operatorname{Tr}(\lambda_k). \tag{1}$$

Let $\boldsymbol{x} = \operatorname{diag}(x_1, \dots, x_{\ell+1}) \in G_\ell$, and note that the diagonal elements of G_ℓ form a maximal torus. Then [20, §79 Example 5, §75]

$$\prod_{j=1}^{\ell+1}(1 + x_j t) = \sum_{j=0}^{\ell+1} \operatorname{Tr}(\lambda_j)(\boldsymbol{x})\, t^j, \tag{2}$$

$$\prod_{j=1}^{\ell+1}(1 - x_j t)^{-1} = \sum_{j\ge 0} \operatorname{Tr}(j\lambda_1)(\boldsymbol{x})\, t^j. \tag{3}$$

Differentiating (2) with respect to t gives

$$\Big(\prod_{j=1}^{\ell+1}(1 + x_j t)\Big) \sum_{j=1}^{\ell+1} \frac{x_j}{1 + x_j t} = \sum_{j=1}^{\ell+1} j \operatorname{Tr}(\lambda_j)(\boldsymbol{x})\, t^{j-1}; \tag{4}$$

using (3) and evaluating (4) at $-t$ we obtain

$$\sum_{j=1}^{\ell+1} \frac{x_j}{1 - x_j t} = \sum_{j=1}^{\ell+1} (-1)^{j-1} j \operatorname{Tr}(\lambda_j)(\boldsymbol{x})\, t^{j-1} \sum_{j\ge 0} \operatorname{Tr}(j\lambda_1)(\boldsymbol{x})\, t^j.$$

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Equating coefficients of t^{m-1} and using (1) we find

$$\sum_{j=1}^{\ell+1} x_j^m = \sum_{k=1}^{m} (-1)^{k-1} k \operatorname{Tr}\big((m-k)\lambda_1\big)(\boldsymbol{x}) \operatorname{Tr}(\lambda_k)(\boldsymbol{x}) = f(\boldsymbol{x}). \tag{5}$$

Since f is central and all elements of G_ℓ are conjugate to an element of the torus, f assumes its maximum on the torus and we have $\|f\|_\infty = \ell + 1$.

REMARK. We thank D. Ž. Djoković for showing us how to prove equation (5) above.

We require next an estimate for the degrees of representations belonging to the m-fold FTR set of G_ℓ.

PROPOSITION 4.2. *Let* $\lambda = \sum_{j=1}^{\ell} m_j \lambda_j \in \widehat{G}_\ell$. *Write*

$$m = \sum_{j=1}^{[\ell/2]} j m_j + \sum_{[\ell/2]+1}^{\ell} (\ell + 1 - j) m_j,$$

and suppose that $m \leq \ell/2$. *Then* $d(\lambda) \geq \binom{\ell+1}{m}$.

PROOF. The positive roots are given by $\beta_{ki} = \sum_{j=k}^{i} \alpha_j$, $1 \leq k \leq i \leq \ell$, [18, p. 26] so by the Weyl dimension formula [12, 24.3] we have $d(\lambda) = \prod_{k=1}^{\ell} \rho_k$, where we write $\rho_k = \prod_{i=k}^{\ell} \langle \lambda + \delta, \beta_{ki} \rangle / \langle \delta, \beta_{ki} \rangle$. Write $\mu = \sum_{j=1}^{[\ell/2]} m_j \lambda_j$ and let $\nu = \lambda - \mu$. Let k_0 denote the largest index $j \leq \ell/2$ for which $m_j \neq 0$, and k_1 denote the smallest index $j > \ell/2$ for which $m_j \neq 0$. Thus $\mu = \sum_{j=1}^{k_0} m_j \lambda_j$ and $\nu = \sum_{j=k_1}^{\ell} m_j \lambda_j$.

Suppose first that $\nu = 0$, so that $m = \sum_{j=1}^{k_0} j m_j$. Now $d(\mu) = \prod_{k=1}^{k_0} \rho_k$, and

$$\begin{aligned} \rho_k = {} & \frac{1 + m_k}{1} \times \frac{2 + m_k + m_{k+1}}{2} \times \frac{3 + m_k + m_{k+1} + m_{k+2}}{3} \times \cdots \\ & \times \frac{k_0 + 1 - k + \sum\limits_{j=k}^{k_0} m_j}{k_0 + 1 - k} \times \frac{k_0 + 2 - k + \sum\limits_{j=k}^{k_0} m_j}{k_0 + 2 - k} \times \cdots \times \frac{\ell + 1 - k + \sum\limits_{j=k}^{k_0} m_j}{\ell + 1 - k}. \end{aligned}$$

Since $k_0 + 1 - k + \sum_{j=k}^{k_0} m_j \leq 1 - k + 2\sum_{j=k}^{k_0} j m_j \leq \ell + 1 - k$, there is cancellation in the above expression for ρ_k, and the numerator reduces to

$$(\ell + 2 - k) \times \cdots \times \left(\ell + 1 - k + \sum\nolimits_{j=k}^{k_0} m_j\right).$$

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There are a corresponding $\sum_{j=k}^{k_0} m_j$ factors remaining in the denominator, the largest of which could be $k_0 - k + \sum_{j=k}^{k_0} m_j$, which is certainly at most

$$\sum_{j=k}^{k_0} (j+1-k)m_j \leq m - \sum_{i=1}^{k-1} \sum_{j=i}^{k_0} m_j$$

and so the denominator of ρ_k is at most

$$\left(m - \sum_{i=1}^{k-1} \sum_{j=i}^{k_0} m_j\right) \times \cdots \times \left(m + 1 - \sum_{i=1}^{k} \sum_{j=i}^{k_0} m_j\right).$$

It follows that the denominator of $\prod_{k=1}^{k_0} \rho_k$ is at most $m!$. Since the smallest factor remaining in the numerator of ρ_k is $\ell + 2 - k$, and the number of terms in the numerator of $\prod_{k=1}^{k_0} \rho_k$ is $\sum_{k=1}^{k_0} \sum_{j=k}^{k_0} m_j = m$, the numerator of $\prod_{k=1}^{k_0} \rho_k$ is at least $(\ell+1)!/(\ell+1-m)!$, and so $d(\mu) \geq \binom{\ell+1}{m}$.

If $\mu = 0$, then since $d(\nu) = d(\overline{\nu})$, we obtain $d(\nu) \geq \binom{\ell+1}{m}$ by the same argument.

Suppose finally that neither μ nor ν is zero, and write μ' for μ considered as a representation in $\widehat{G}_{k_1-1}$, and ν' for ν considered as a representation in $\widehat{G}_{\ell-k_0}$. Let $m' = \sum_{j=1}^{k_0} j m_j$ and $m'' = m - m'$. Then $m'' = \sum_{j=k_1}^{\ell} (\ell + 1 - j)m_j \geq \ell + 1 - k_1$ so we have

$$m' \leq \ell/2 - m'' \leq \frac{k_1 - 1}{2} + \frac{k_1 - 1 - \ell}{2} \leq \frac{k_1 - 1}{2}$$

and the first part of the proof yields $d(\mu') \geq \binom{k_1}{m'}$. Similarly, since $m' \geq k_0$ we obtain $m'' \leq (\ell - k_0)/2$ and hence $d(\nu') \geq \binom{\ell-k_0+1}{m''}$.

Now observe that for $k_0 < k \leq i < k_1$ we have $\langle \lambda, \beta_{ki} \rangle = 0$, so that

$$\begin{aligned} d(\lambda) &= \prod_{1 \leq k \leq i < k_1} \frac{\langle \lambda + \delta, \beta_{ki} \rangle}{\langle \delta, \beta_{ki} \rangle} \times \prod_{k_0 < k \leq i \leq \ell} \frac{\langle \lambda + \delta, \beta_{ki} \rangle}{\langle \delta, \beta_{ki} \rangle} \times \prod_{1 \leq k \leq k_0} \prod_{k_1 \leq i \leq \ell} \frac{\langle \lambda + \delta, \beta_{ki} \rangle}{\langle \delta, \beta_{ki} \rangle} \\ &\geq d(\mu') \times d(\nu') \times 1 \\ &\geq \binom{k_1}{m'} \binom{\ell - k_0 + 1}{m''}; \end{aligned}$$

to show this exceeds $\binom{\ell+1}{m'+m''}$ it suffices to prove

$$\frac{k_1 \times (k_1 - 1) \times \cdots \times (k_1 + 1 - m')}{1 \times 2 \times \cdots \times m'} \geq \frac{(\ell+1) \times \ell \times (\ell - 1) \times \cdots \times (\ell + 2 - m')}{(m''+1) \times (m''+2) \times \cdots \times (m'' + m')}:$$

we show

$$\frac{k_1 + 1 - m' + j}{1 + j} \geq \frac{\ell + 2 - m' + j}{m'' + 1 + j}$$

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for $j \geq 0$. Since $m_{k_1} \neq 0$ and we have $m'' \geq \ell+1-k_1$, it follows that $k_1+2+m''-m' \geq \ell+3-m'$; also $k_1+1-m' \geq 1$ since $m' \leq (k_1-1)/2$, giving $m''(k_1+1-m') \geq \ell+1-k_1$.

Thus $(m''+1)(k_1+1-m') \geq \ell+2-m'$ and, for $j \geq 0$,

$$(m''+1)(k_1+1-m')+j(k_1+2+m''-m')+j^2 \geq \ell+2-m'+j(\ell+3-m')+j^2,$$

that is,

$$(m''+1+j)(k_1+1-m'+j) \geq (\ell+2-m'+j)(1+j),$$

as required.

REMARK. The bound is sharp: consider $d(\lambda_j)$.

COROLLARY 4.3. *Fix $m \in \mathbb{Z}^+$. Let $\mathcal{G} = \prod_{\ell \geq 1} G_\ell$, and let $E = \bigcup_{\ell \geq 2m} E_\ell^m \circ \pi_\ell \subseteq \widehat{\mathcal{G}}$, where $\pi_\ell : \mathcal{G} \to G_\ell$ denotes the canonical projection.*

Then $\varkappa_{a,p}(E) < \infty$ only if $p \geq (a+1)m/(m+1)$.

PROOF. Let $\ell \geq 2m$ and let f be the polynomial considered in 4.1. Then 4.2 gives

$$\|\widehat{f}\|_{a,p}^p = \sum_{k=1}^m d\big((m-k)\lambda_1+\lambda_k\big)^{a+1-p} \geq m\binom{\ell+1}{m}^{a+1-p},$$

whence there is a constant C depending only on m and p so that

$$\varkappa(E_\ell^m) \geq \|\widehat{f}\|_{a,p} \big/ \|f\|_\infty \geq C\,\ell^{((a+1)m-(m+1)p)/p}.$$

Since ℓ is unbounded, the result follows.

REMARK. The set E is a subset of a partial m-fold FTR set of $\mathcal{G}$ with the property that each representation in E is supported by a single factor of the product group.

5. Characterization of (p, p)-Sidon sets

Somewhat surprisingly, for connected compact groups the concept of (p, p)-Sidonicity essentially coincides with the classical concept of Sidonicity. This comes about because local (p, p)-Sidon sets on product groups have the same kind of structure as exhibited for local Sidon sets in [3, 5.5].

PROPOSITION 5.1. *Let $\mathcal{G} = \prod_{\iota \in I} G_\iota$ be a product of compact groups G_ι.*

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(i) *Let* $E_\iota \subseteq \widehat{G}_\iota$ *and write* $\pi_\iota : \mathscr{G} \to G_\iota$ *for the canonical projection. Then* $E = \bigcup_{\iota \in I} E_\iota \circ \pi_\iota$ *is* (a, p)*-Sidon if, and only if,* $\sup_{\iota \in I} \kappa_{a,p}(E_\iota) < \infty$. *(Indeed,* $\kappa_{a,p}(E)/4 \leq \sup_{\iota \in I} \kappa_{a,p}(E_\iota) \leq \kappa_{a,p}(E)$*).*

(ii) *Let* $\sigma = (\sigma_\iota)_{\iota \in I} \in \widehat{\mathscr{G}}$. *Then* $\kappa_{a,p}(\sigma) \geq \prod_{\iota \in I} \kappa_{a,p}(\sigma_\iota)$.

PROOF. For (i), mimic the proof of [3, 5.2]; (ii) is [3, 5.4.1].

Adapting [19, 7.5], and noting in particular that for B as in [19, 7.5], $|B|$ has exactly one non-zero entry, so that $\left(\operatorname{Tr}|B|^p\right)^{1/p} = \operatorname{Tr}|B|$, we obtain

PROPOSITION 5.2. *Let* G *and* H *be compact groups,* $\sigma \in E \subseteq \widehat{G}$ *and* $\tau \in E' \subseteq \widehat{H}$. *Then*

$$\kappa_{a,p}(\sigma \times \tau) \geq \min\left(d(\sigma), d(\tau)\right)^{1/2} \left(d(\sigma)d(\tau)\right)^{(a-p)/p}.$$

Using 1.4 and following the argument of [3, 4.2] yields

PROPOSITION 5.3. *Let* G *be an almost simple, connected, compact Lie group. Let* $\sigma \in \widehat{G}$ *and write* $M(\sigma)$ *for the number of positive roots having non-zero inner product with the highest weight of* σ. *Then there is an absolute constant* $C > 0$ *such that for any* $0 < \varepsilon < 1/2$ *we have*

$$\kappa_{a,p}(\sigma) \geq \frac{\varepsilon}{C} \frac{d(\sigma)^{(2a-p)/2p - \varepsilon}}{M(\sigma)^{\varepsilon + 1/2}}.$$

PROPOSITION 5.4. *Let* $\mathscr{G}$ *be a product of almost simple, simply-connected, compact Lie groups. Then* $E \subseteq \widehat{\mathscr{G}}$ *is local* (p, p)*-Sidon if, and only if,* E *is local Sidon.*

PROOF. By 1.7(i) we have $\kappa^0_{p,p}(E) \leq \kappa^0(E)$, so suppose E is local (p, p)-Sidon. Observe that the estimate for $\kappa_{p,p}(\sigma)$ afforded by 5.3 is independent of p, and essentially the same as [3, 4.2]. Consequently the conclusions of [3, Proposition 4.3, 4.4] hold for $\kappa_{p,p}$. The estimates for $\kappa_{p,p}$ provided by 1.6(ii), 5.1(ii) and 5.2 are identical to those for κ used in [3, 5.5], and so the same argument shows that if $\mathscr{G} = \prod_{\iota \in I} G_\iota$ then there is a partition $I = I_1 \cup I_2 \cup I_3$ of I and a subset $E_2 \subseteq \widehat{\mathscr{G}_2}$ (where $\mathscr{G}_j = \prod_{\iota \in I_j} G_\iota$, $j = 1, 2, 3$) of representations of bounded degree such that $E \subseteq \{1\} \times E_2 \times \left(\{1\} \cup \mathrm{FTR}(\mathscr{G}_3)\right)$. Hence [3, 5.5] E is local Sidon.

COROLLARY 5.5. *Let* G *be a connected compact group. Then* $E \subseteq \widehat{G}$ *is local* (p, p)*-Sidon if, and only if,* E *is local Sidon.*

PROOF. In view of 1.10, we may suppose that G is its own covering group: $G = T \times \mathscr{G}$ where T is compact, connected and abelian, and $\mathscr{G}$ is as in 5.4. Moreover, in view of 1.7(i) we need only consider the case of $E \subseteq \widehat{G}$ local (p, p)-Sidon.

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Write $E = \bigcup_{\chi \in \widehat{T}} \chi \times E_\chi$, and observe that $\kappa_{a,q}(\chi \times \sigma) = \kappa_{a,q}(\sigma)$, so that $\kappa_{a,q}^{\,0}(E) = \kappa_{a,q}^{\,0}(E')$ where $E' = \bigcup_{\chi \in \widehat{T}} E_\chi \subseteq \widehat{\mathscr{G}}$. Thus E' is local (p, p)-Sidon, hence local Sidon by 5.4, and it follows that E is also local Sidon.

Combining this with [3, 6.2] yields our final result:

THEOREM 5.6. *Let G be a connected, compact group. Then the following are equivalent:*

(i) *G admits an infinite Sidon set;*
(ii) *G admits an infinite local Sidon set;*
(iii) *G admits an infinite (p, p)-Sidon set;*
(iv) *G admits an infinite local (p, p)-Sidon set.*

Structural criteria for a compact group G to admit infinite Sidon sets are described in [3, §6] and [19, §5].

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