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Lie Elements and Knuth Relations

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Abstract. A coplactic class in the symmetric group $\mathcal{S}_n$ consists of all permutations in $\mathcal{S}_n$ with a given Schensted Q-symbol, and may be described in terms of local relations introduced by Knuth. Any Lie element in the group algebra of $\mathcal{S}_n$ which is constant on coplactic classes is already constant on descent classes. As a consequence, the intersection of the Lie convolution algebra introduced by Patras and Reutenauer and the coplactic algebra introduced by Poirier and Reutenauer is the direct sum of all Solomon descent algebras.

1 Introduction

In 1995, Malvenuto and Reutenauer introduced the structure of a graded Hopf algebra on the direct sum

$$\mathcal{P} = \bigoplus_{n\geq 0} \mathbb{Z}\mathcal{S}_n$$

of all symmetric group algebras $\mathbb{Z}\mathcal{S}_n$ over the ring $\mathbb{Z}$ of integers ([MR95]). Apart from this *convolution algebra of permutations* $\mathcal{P}$ itself ([AS, DHT]) several subalgebras of $\mathcal{P}$ turned out to be of particular algebraic and combinatorial interest and have been studied intensively; for instance, the Rahmenalgebra ([Jöl99]), the Hopf algebra of the planar binary trees ([LR98, Cha00]), the Lie convolution algebra $\mathcal{L}$ ([PR01]), the coplactic algebra $\mathcal{Q}$ ([PR95][1], [BS]), and the direct sum $\mathcal{D}$ of the Solomon descent algebras ([Sol76, GR89, Reu93, MR95, GKL+95, BL96, JR01]).

Here, the *relation* between the algebras $\mathcal{L}$ and $\mathcal{Q}$ shall be investigated. The latter is defined combinatorially as the linear span of the sums of permutations with given Schensted Q-symbol ([Sch61]), or, equivalently, of the sums of equivalence classes arising from the coplactic relations in $\mathcal{S}_n$, $n \geq 0$, introduced by Knuth ([Knu70]). The Lie convolution algebra $\mathcal{L}$ is generated (as an algebra) by all Lie elements in $\mathcal{P}$. Both $\mathcal{L}$ and $\mathcal{Q}$ contain $\mathcal{D}$. Combinatorial descriptions of the algebras $\mathcal{D}$ and $\mathcal{Q}$, and the set of Lie elements in $\mathcal{P}$, follow in Section 2. The main goal of this paper is to show the following.

Theorem 1 $\mathcal{L} \cap \mathcal{Q} = \mathcal{D}$.

This result (once more) points out the exceptional role played by the descent algebras. The proof is given in Section 2, and is essentially based on the fact that any

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[1]The algebra $(\mathbb{Z}C, *, \delta)$ introduced in [PR95] is the dual algebra of the algebra $\mathcal{Q}$ considered here (see [PR95, Théorème 3.4]).

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Lie element in $\mathcal{P}$ which is constant on coplactic classes is already contained in $\mathcal{D}$ (see Section 3), which is combinatorially interesting for its own sake.

One might be tempted to conjecture that a lack of co-commutativity of $\mathcal{Q}$ is the deeper reason for Theorem 1, since $\mathcal{L}$ is—at least in comparison to $\mathcal{D}$—a "large" co-commutative subalgebra of $\mathcal{P}$; but this is false. The domain of co-commutativity of $\mathcal{Q}$ *strictly* contains $\mathcal{D}$. Some comments concerning this can be found at the end of Section 3.

2 Descent, Coplactic, and Lie Relations

In this section, combinatorial descriptions of the algebras $\mathcal{D}$ and $\mathcal{Q}$, and of the Lie elements in $\mathcal{P}$, are recalled briefly, and a proof of Theorem 1 is given.

Let $\mathbb{N}$ (respectively, $\mathbb{N}_0$) be the set of positive (respectively, nonnegative) integers and set

$$[n] := \{i \in \mathbb{N} \mid i \leq n\}$$

for all integers n. For any $\pi \in \mathcal{S}_n$, $\mathrm{Des}(\pi) := \{i \in [n-1] \mid \pi(i) > \pi(i+1)\}$ is the *descent set* of π. The Solomon descent algebra $\mathcal{D}_n$ is the linear span of the sums $\sum_{\substack{\pi \in \mathcal{S}_n \\ \mathrm{Des}(\pi)=D}} \pi$, where $D \subseteq [n-1]$. According to Malvenuto and Reutenauer, $\mathcal{D} = \bigoplus_{n \geq 0} \mathcal{D}_n$ is a Hopf subalgebra of $\mathcal{P}$ ([MR95]); and as such, $\mathcal{D}$ is isomorphic to the algebra of noncommutative symmetric functions ([GKL$^+$95]). We mention that $\mathcal{D}_n$ is also a subalgebra of the group algebra $\mathbb{Z}\mathcal{S}_n$, according to a remarkable result of Solomon ([Sol76]), although this is not of relevance here.

Let $\mathbb{N}^*$ be a free monoid over the alphabet $\mathbb{N}$ and denote by $\varnothing$ the empty word in $\mathbb{N}^*$. The mapping $\pi \mapsto \pi(1) \cdots \pi(n)$ extends to a linear embedding of $\mathbb{Z}\mathcal{S}_n$ into the semi-group algebra $\mathbb{Z}\mathbb{N}^*$. As is convenient for our purposes, elements of $\mathbb{Z}\mathcal{S}_n$ will be identified with the corresponding elements of $\mathbb{Z}\mathbb{N}^*$. Furthermore, products $\sigma\pi$ of permutations $\sigma, \pi \in \mathcal{S}_n$ are to be read from right to left: first π, then σ.

The following combinatorial characterization of $\mathcal{D}_n$ was given in [BL93, 4.2].

Proposition 2.1 (Descent Relations) *Let $\varphi = \sum_{\pi \in \mathcal{S}_n} k_\pi \pi \in \mathbb{Z}\mathcal{S}_n$, then $\varphi \in \mathcal{D}_n$ if and only if*

$$k_{uaw(a+1)v} = k_{u(a+1)wav}$$

for all $a \in [n-1]$, $u, v, w \in \mathbb{N}^*$ such that $\pi = uaw(a+1)v \in \mathcal{S}_n$ and $w \neq \varnothing$.

Let $Q(\pi)$ denote the Schensted Q-symbol of π, for all $\pi \in \mathcal{S}_n$ ([Sch61]), then the set of all $\sigma \in \mathcal{S}_n$ such that $Q(\pi) = Q(\sigma)$ is a *coplactic class* in $\mathcal{S}_n$[2]. The coplactic algebra $\mathcal{Q}$ is the linear span of all sums of coplactic classes in $\mathcal{P}$:

$$\mathcal{Q} = \Big\langle \Big\{ \sum_{Q(\sigma)=Q(\pi)} \sigma \;\Big|\; \pi \in \mathcal{S}_n, n \in \mathbb{N}_0 \Big\} \Big\rangle_{\mathbb{Z}}.$$

[2]According to Schützenberger ([Sch63]), $P(\pi) = Q(\pi^{-1})$ is the Schensted P-symbol of π; and the equivalence arising from equality of P-symbols leads to the *plactic monoid* ([LS81]). This is the reason why the word coplactic is used here.

Accordingly, each element $\varphi \in \mathcal{Q}$ is called *coplactic*. According to Poirier and Reutenauer, $\mathcal{Q}$ is a Hopf subalgebra of $\mathcal{P}$ ([PR95]). The following characterization of $\mathcal{Q}_n := \mathcal{Q} \cap \mathbb{Z}\mathcal{S}_n$ is due to Knuth ([Knu70]).

Proposition 2.2 (Coplactic Relations) *Let $\varphi = \sum_{\pi\in\mathcal{S}_n} k_\pi \pi \in \mathbb{Z}\mathcal{S}_n$. Then $\varphi \in \mathcal{Q}_n$ if and only if*

$$k_{uaw(a+1)v} = k_{u(a+1)wav}$$

for all $a \in [n-1]$, $u, v, w \in \mathbb{N}^*$ such that $\pi = uaw(a+1)v \in \mathcal{S}_n$ and w contains the letter $a-1$ or the letter $a+2$.

Combining Propositions 2.1 and 2.2 implies, in particular, $\mathcal{D} \subseteq \mathcal{Q}$.

Let

$$\omega_n = \sum_\nu (-1)^{\nu^{-1}(1)-1} \nu \in \mathbb{Z}\mathcal{S}_n,$$

where the sum is taken over all *valley permutations* $\nu \in \mathcal{S}_n$, which are defined by the property $\nu(1) > \cdots > \nu(k-1) > \nu(k) < \nu(k+1) < \cdots < \nu(n)$, where $k := \nu^{-1}(1)$. For instance, $\omega_3 = 123-213-312+321$. The element ω_n projects $\mathbb{Z}\mathcal{S}_n$ onto the multilinear part of the free Lie algebra, by right multiplication ([Dyn47, Spe48, Wev49], see [BL93]). Accordingly,

$$\mathrm{Lie}_n := \mathbb{Z}\mathcal{S}_n\omega_n$$

is the set of *Lie elements* in $\mathbb{Z}\mathcal{S}_n$ for all $n \in \mathbb{N}_0$. Each $\varphi \in \mathrm{Lie} := \bigoplus_{n\geq 0} \mathrm{Lie}_n$ is a primitive element of the Hopf algebra $\mathcal{P}$ ([PR01]). The Lie convolution algebra $\mathcal{L}$ is the (co-commutative) Hopf subalgebra of $\mathcal{P}$ generated by Lie; there is also the relation $\mathcal{D} \subseteq \mathcal{L}$ ([PR01]).

In view of a Proof of Theorem 1, consider the corresponding algebras $\mathcal{D}_\mathbb{Q}$, $\mathcal{L}_\mathbb{Q}$, $\mathcal{Q}_\mathbb{Q}$, and $\mathcal{P}_\mathbb{Q}$ over the field $\mathbb{Q}$ of rational numbers, then $\mathcal{D}_\mathbb{Q}$ is contained in $\mathcal{L}_\mathbb{Q} \cap \mathcal{Q}_\mathbb{Q}$; the latter is a co-commutative Hopf subalgebra of $\mathcal{P}_\mathbb{Q}$, hence generated by its primitive elements, according to Milnor and Moore ([MM65]). But each primitive element in $\mathcal{L}_\mathbb{Q} \cap \mathcal{Q}_\mathbb{Q}$ is, in particular, a primitive element in $\mathcal{L}_\mathbb{Q}$ and therefore contained in Lie. In Section 3, it will be shown that any coplactic Lie element $\varphi \in \mathrm{Lie} \cap \mathcal{Q}$ is contained in $\mathcal{D}$ (Theorem 2). This implies $\mathcal{L}_\mathbb{Q} \cap \mathcal{Q}_\mathbb{Q} \subseteq \mathcal{D}_\mathbb{Q}$. Observing that $\mathcal{D}_\mathbb{Q} \cap \mathcal{P} = \mathcal{D}$, completes the proof of Theorem 1.

A combinatorial characterization of the set Lie_n follows. Let $u \sqcup\!\sqcup v$ denote the usual shuffle product of $u = u_1 \cdots u_k$, $v = v_1 \cdots v_m \in \mathbb{N}^*$, that is

$$u \sqcup\!\sqcup v = \sum_w w,$$

where the sum ranges over all $w = w_1 \cdots w_{k+m} \in \mathbb{N}^*$ such that $u = w_{i_1} \cdots w_{i_k}$ and $v = w_{j_1} \cdots w_{j_m}$ for suitably chosen indices $i_1 < \cdots < i_k$, $j_1 < \cdots < j_m$ such that $[k+m] = \{i_1, \ldots, i_k, j_1, \ldots, j_m\}$. Furthermore, set

$$\bar{u} := u_k \cdots u_1$$

and denote by $\ell(u) := k$ the length of u.

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Proposition 2.3 *Let $n \in \mathbb{N}$ and $a \in [n]$, then $\{\sigma\omega_n \mid \sigma \in \mathcal{S}_n, \sigma(1) = a\}$ is a linear basis of* Lie_n.

Furthermore, for any choice of coefficients $c_\sigma \in \mathbb{Z}$ ($\sigma \in \mathcal{S}_n$, $\sigma(1) = a$), the coefficient of $\pi = uav \in \mathcal{S}_n$ in $\bigl(\sum_{\sigma(1)=a} c_\sigma \sigma\bigr)\omega_n$ is

$$(-1)^{\ell(u)} c_{a(\bar{u} \sqcup\!\sqcup v)}, \tag{1}$$

where $\sigma \mapsto c_\sigma$ has been extended linearly. In particular, the coefficient of $\sigma \in \mathcal{S}_n$ is c_σ whenever $\sigma(1) = a$.

This result is seemingly folklore; a proof follows for the reader's convenience.

Proof Let $\pi = uav \in \mathcal{S}_n$ and $\sigma = ax_2 \cdots x_n \in \mathcal{S}_n$, then the coefficient of π in $\sigma\omega_n$ is non-zero if and only if there is a valley permutation $\nu \in \mathcal{S}_n$ such that

$$uav = \pi = \sigma\nu = x_{\nu(1)} \cdots x_{\nu(k-1)} a x_{\nu(k+1)} \cdots x_{\nu(n)},$$

where $k := \nu^{-1}(1)$; that is, $u = x_{\nu(1)} \cdots x_{\nu(k-1)}$ and $v = x_{\nu(k+1)} \cdots x_{\nu(n)}$. Since $\nu(1) > \cdots > \nu(k-1)$ and $\nu(k+1) < \cdots < \nu(n)$, this is equivalent to saying that $x_2 \cdots x_n$ is a summand in the shuffle product of $\bar{u}$ and v; in this case, the coefficient of π in $\sigma\omega_n$ is $(-1)^{\nu^{-1}(1)-1} = (-1)^{\ell(u)}$. This proves (1). Since

$$\dim \mathrm{Lie}_n = (n-1)! = \#\{\sigma\omega_n \mid \sigma \in \mathcal{S}_n, \sigma(1) = a\}$$

and the coefficient of $\tilde{\sigma} = av \in \mathcal{S}_n$ in $\bigl(\sum_{\sigma(1)=a} c_\sigma \sigma\bigr)\omega_n$ is $c_{\tilde{\sigma}}$, the basis property follows. ■

Corollary 2.4 (Lie Relations) *Let $\varphi = \sum_{\pi \in \mathcal{S}_n} k_\pi \pi \in \mathbb{Z}\mathcal{S}_n$, then $\varphi \in$ Lie_n if and only if*

$$k_{uav} = (-1)^{\ell(u)} k_{a(\bar{u} \sqcup\!\sqcup v)} \tag{2}$$

for all $a \in [n]$, $u, v \in \mathbb{N}^*$ such that $\pi = uav \in \mathcal{S}_n$, where again, $\pi \mapsto k_\pi$ has been extended linearly.

Proof Let $\varphi \in \mathrm{Lie}_n$ and $a \in [n]$, then there are coefficients $c_\sigma \in \mathbb{Z}$ ($\sigma \in \mathcal{S}_n$, $\sigma(1) = a$) such that $\varphi = (\sum_{\sigma(1)=a} c_\sigma \sigma)\omega_n$, by Proposition 2.3, and

$$k_{uav} = (-1)^{\ell(u)} c_{a(\bar{u} \sqcup\!\sqcup v)} = (-1)^{\ell(u)} k_{a(\bar{u} \sqcup\!\sqcup v)},$$

by (1). Conversely, (2) implies $\varphi = (\sum_{\sigma(1)=a} k_\sigma \sigma)\omega_n \in \mathrm{Lie}_n$, by (1) again. ■

Proposition 2.2 and Corollary 2.4 may be restated as follows. Consider the scalar product on $\mathbb{Z}\mathcal{S}_n$ which turns $\mathcal{S}_n$ into an orthonormal basis. For all $T \subseteq \mathbb{Z}\mathcal{S}_n$, let $T^\perp$ be the space orthogonal to T with respect to this scalar product. For all $\varphi, \psi \in \mathbb{Z}\mathcal{S}_n$, write

$$\varphi \equiv_Q \psi \quad \text{(respectively, } \varphi \equiv_L \psi, \varphi \equiv_{\mathrm{LQ}} \psi\text{)},$$

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if $\varphi - \psi \in \mathcal{Q}_n^{\perp}$ (respectively, $\in \mathrm{Lie}_n^{\perp}$, $\in (\mathrm{Lie}_n \cap \mathcal{Q}_n)^{\perp}$). Now the necessity parts of Proposition 2.2 and Corollary 2.4 are

$$uaw(a+1)v \equiv_Q u(a+1)wav \tag{3}$$

for all $a \in [n-1]$, $u, v, w \in \mathbb{N}^*$ such that $uaw(a+1)v \in \mathcal{S}_n$ and w contains the letter $a-1$ or the letter $a+2$;

$$uav \equiv_L (-1)^{\ell(u)} a(\bar{u} \sqcup\!\sqcup v) \tag{4}$$

for all $a \in [n]$, $u, v \in \mathbb{N}^*$ such that $uav \in \mathcal{S}_n$. For later use, note that applying (4) twice gives

$$aubv \equiv_L (-1)^{n-1} \bar{v} b \bar{u} a \equiv_L (-1)^{n-1+\ell(v)} b(v \sqcup\!\sqcup \bar{u} a) \tag{5}$$

whenever $a, b \in [n]$ and $u, v \in \mathbb{N}^*$ such that $aubv \in \mathcal{S}_n$.

Remark The space $\mathrm{Lie}_n^{\perp}$ is linearly generated by all non-trivial shuffles $u \sqcup\!\sqcup v$, where $u, v \in \mathbb{N}^*$ such that $uv \in \mathcal{S}_n$ (see, for instance, [Duc91]). As a consequence of Corollary 2.4, for fixed $a \in [n]$, the elements

$$uav - (-1)^{\ell(u)} a(\bar{u} \sqcup\!\sqcup v),$$

where $u, v \in \mathbb{N}^*$ such that $uav \in \mathcal{S}_n$ and $u \neq \varnothing$, constitute a linear basis of $\mathrm{Lie}_n^{\perp}$. Another basis has been introduced by Duchamp (ibid.). This was pointed out to me by Christophe Reutenauer.

This section concludes with a helpful observation concerning the order reversing involution $\varrho_n = n(n-1)\cdots 1 \in \mathcal{S}_n$.

Proposition 2.5 *$\varrho_n \mathrm{Lie}_n + \mathrm{Lie}_n \varrho_n \subseteq \mathrm{Lie}_n$, and $\varrho_n \mathcal{Q}_n + \mathcal{Q}_n \varrho_n \subseteq \mathcal{Q}_n$.*

In particular, $\pi \equiv_{\mathrm{LQ}} \sigma$ implies $\pi \varrho_n \equiv_{\mathrm{LQ}} \sigma \varrho_n$ and $\varrho_n \pi \equiv_{\mathrm{LQ}} \varrho_n \sigma$, for all $\pi, \sigma \in \mathcal{S}_n$.

Proof First, $\omega_n \varrho_n = (-1)^{n-1} \omega_n$ yields $\mathrm{Lie}_n \varrho_n \subseteq \mathrm{Lie}_n$, while $\varrho_n \mathrm{Lie}_n \subseteq \mathrm{Lie}_n$ is obvious; and second, if $\sigma, \pi \in \mathcal{S}_n$ such that $\sigma \equiv_Q \pi$, then $\sigma \varrho_n \equiv_Q \pi \varrho_n$ and $\varrho_n \sigma \equiv_Q \varrho_n \pi$, as is readily seen from Proposition 2.2. This implies $\varrho_n \mathcal{Q}_n \subseteq \mathcal{Q}_n$ and $\mathcal{Q}_n \varrho_n \subseteq \mathcal{Q}_n$.

In particular, it follows that $\varrho_n (\mathrm{Lie}_n \cap \mathcal{Q}_n)^{\perp} + (\mathrm{Lie}_n \cap \mathcal{Q}_n)^{\perp} \varrho_n \subseteq (\mathrm{Lie}_n \cap \mathcal{Q}_n)^{\perp}$, since ϱ_n is an involution. ∎

3 Coplactic Lie Elements

The aim of this section is to show $\mathrm{Lie} \cap \mathcal{Q} \subseteq \mathcal{D}$, which implies Theorem 1, as was mentioned in the previous section. Throughout, $n \in \mathbb{N}$ is fixed. Bearing in mind Proposition 2.1, it suffices to show that

$$uaw(a+1)s \equiv_{\mathrm{LQ}} u(a+1)was \tag{6}$$

whenever $a \in [n-1]$, $u, s, w \in \mathbb{N}^*$ such that $uaw(a+1)s \in \mathcal{S}_n$ and $w \neq \varnothing$.

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Not surprisingly, the essential idea of the proof is to use proper coplactic and Lie relations on the left hand side of (6) to obtain an element $\varphi \in \mathbb{Z}\mathcal{S}_n$ such that $\varphi \equiv_Q \hat{\varphi}$, where $\hat{\varphi}$ is obtained by exchanging a and $a+1$ in φ, and to apply the same coplactic and Lie relations (in reverse order) to $\hat{\varphi}$ to obtain the right hand side of (6) (see Proposition 3.4). This concept is illustrated by the following:

Example 3.1 Let $a = 1$, $\pi = 15234$, $\sigma = 25134 \in \mathcal{S}_5$. Then π and σ are in descent, but not in coplactic relation. But applying (4) yields

$$\begin{aligned}\pi &\equiv_L -5(1 \sqcup\!\sqcup 234)\\ &= -51234 - 52134 - 52314 - 52341\\ &\equiv_Q -51234 - 52134 - 51324 - 51342\\ &= -5(2 \sqcup\!\sqcup 134)\\ &\equiv_L \sigma,\end{aligned}$$

hence $\pi \equiv_{\mathrm{LQ}} \sigma$.

In the general case, however, the proof has a quite intricate inductive structure. Some additional preparations are needed. $v \in \mathbb{N}^*$ is called a *sub-word* of $w = w_1 \cdots w_m \in \mathbb{N}^*$ if there exist $k \in [m]$ and $1 \le i_1 < \cdots < i_k \le m$ such that $v = w_{i_1} \cdots w_{i_k}$. For instance, 23 is a sub-word of 52143.

For $a \in [n-1]$, denote by $\tau_a = (a, a+1)$ the transposition in $\mathcal{S}_n$ swapping a and $a+1$. The word v *allows the a-switch in* $\mathcal{S}_n$ if $\pi \equiv_{\mathrm{LQ}} \tau_a\pi$ for all $\pi = uaw(a+1)s \in \mathcal{S}_n$ such that v is a sub-word of w. For instance, $v = a+2$ and $v = a-1$ allow the a-switch in $\mathcal{S}_n$, by (3). To save trouble, let it be said that, if v contains a letter twice or a letter $b > n$ or $b \in \{a, a+1\}$, then v allows the a-switch in $\mathcal{S}_n$; for in this case, there is no permutation $\pi = uaw(a+1)s \in \mathcal{S}_n$ such that v is a sub-word of w. Another way of stating (6) now is that $v \in \mathbb{N}^*$ allows the a-switch in $\mathcal{S}_n$ whenever $v \neq \varnothing$. The following three helpful observations will be applied frequently.

Proposition 3.2 *Let $v \in \mathbb{N}^*$ such that $\pi \equiv_{\mathrm{LQ}} \tau_a\pi$ for all*

$$\pi = aw(a+1)s \in \mathcal{S}_n$$

such that v is a sub-word of w, then v allows the a-switch in $\mathcal{S}_n$.

Proof Let $\pi = uaw(a+1)s \in \mathcal{S}_n$ such that v is a sub-word of w, then

$$\pi \equiv_L (-1)^{\ell(u)} a\big(\bar{u} \sqcup\!\sqcup w(a+1)s\big)$$

by (4). Each summand in this shuffle product is of the form $a\hat{w}(a+1)\hat{s}$ such that w (hence also v) is a sub-word of $\hat{w}$. It follows that

$$\pi \equiv_{\mathrm{LQ}} (-1)^{\ell(u)}(a+1)(\bar{u} \sqcup\!\sqcup was) \equiv_L u(a+1)was = \tau_a\pi,$$

hence v allows the a-switch in $\mathcal{S}_n$. ■

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Proposition 3.3 *Let $v = v_1 \cdots v_m \in \mathbb{N}^*$ and assume that v allows the a-switch in $\mathcal{S}_n$, then so does $\bar{v}$. Furthermore, if*

$$\tilde{v} := (n+1-v_1)\cdots(n+1-v_m) \in \mathbb{N}^*,$$

then $\tilde{v}$ allows the $(n-a)$-switch in $\mathcal{S}_n$.

This is an immediate consequence of Proposition 2.5.

Proposition 3.4 *Let $\pi \in \mathcal{S}_n$ and $\varphi_0, \ldots, \varphi_m \in \mathbb{Z}\mathcal{S}_n$ such that*

(i) $\varphi_0 = \pi$,
(ii) $\varphi_i \equiv_L \varphi_{i+1}$, *or* $\varphi_i \equiv_Q \varphi_{i+1}$ *and* $\tau_a\varphi_i \equiv_Q \tau_a\varphi_{i+1}$, *for all* $i \in [m-1] \cup \{0\}$,
(iii) $\varphi_m \equiv_{\mathrm{LQ}} \tau_a\varphi_m$,

then $\pi \equiv_{\mathrm{LQ}} \tau_a\pi$.

Proof $\varphi \equiv_L \psi$ implies $\tau_a\varphi \equiv_L \tau_a\psi$ for all $\varphi, \psi \in \mathbb{Z}\mathcal{S}_n$, since $\tau_a\mathrm{Lie}_n = \mathrm{Lie}_n$. Combined with (ii), this yields $\tau_a\varphi_i \equiv_{\mathrm{LQ}} \tau_a\varphi_{i+1}$ for all $i \in [m-1] \cup \{0\}$, hence

$$\pi = \varphi_0 \equiv_{\mathrm{LQ}} \varphi_1 \equiv_{\mathrm{LQ}} \cdots \equiv_{\mathrm{LQ}} \varphi_m \equiv_{\mathrm{LQ}} \tau_a\varphi_m \equiv_{\mathrm{LQ}} \cdots \equiv_{\mathrm{LQ}} \tau_a\varphi_1 \equiv_{\mathrm{LQ}} \tau_a\varphi_0 = \tau_a\pi,$$

by (i) and (iii). ■

We now show in four steps that each $v \in \mathbb{N}^* \setminus \{\varnothing\}$ allows the a-switch in $\mathcal{S}_n$. The first step is crucial and depends heavily on Lie relations. In Steps 2 and 3, proper coplactic relations are then used to deduce from Step 1 that $v \in \mathbb{N}^*$ allows the a-switch in $\mathcal{S}_n$ whenever $\ell(v) \geq 2$. As a final step, in Theorem 2, the general idea described at the beginning of this section is used once more to show that this already implies (6) as desired.

Step 1 Let $a, k \in [n-1]$ such that $k > a+1$, then $k(k+1)$ and $(k+1)k$ allow the a-switch in $\mathcal{S}_n$.

Proof Let $\pi = aw(a+1)s \in \mathcal{S}_n$ such that $k(k+1)$ or $(k+1)k$ is a sub-word of w. It suffices to prove $\pi \equiv_{\mathrm{LQ}} \tau_a\pi$, by Proposition 3.2.

If $k = a+2$, then this follows from (3). Let $k \geq a+3$, and proceed by induction on k.

If $k-1$ occurs in w, then $\pi \equiv_{\mathrm{LQ}} \tau_a\pi$, by induction. Let $k-1$ occur in s.

If $k = a+3$ and $\pi = au_1(a+3)u_2(a+4)u_3(a+1)u_4(a+2)u_5$, then

$$\pi \equiv_Q au_1(a+2)u_2(a+4)u_3(a+1)u_4(a+3)u_5.$$

Applying Proposition 3.4, yields $\pi \equiv_{\mathrm{LQ}} \tau_a\pi$ in this case. In particular, $(a+3)(a+4)$ allows the a-switch in $\mathcal{S}_n$, hence also $(a+4)(a+3)$, by Proposition 3.3.

Now let $k > a+3$. If $k-2$ occurs in w, then there are $u_i \in \mathbb{N}^*$ ($i \in [5]$) such that either

$$\pi = au_1ku_2(k-2)u_3(a+1)u_4(k-1)u_5$$
$$\equiv_Q au_1(k-1)u_2(k-2)u_3(a+1)u_4ku_5 =: \varphi$$

or

$$\pi = au_1(k-2)u_2ku_3(a+1)u_4(k-1)u_5$$
$$\equiv_Q au_1(k-1)u_2ku_3(a+1)u_4(k-2)u_5 =: \varphi.$$

In both cases, $\varphi \equiv_{\mathrm{LQ}} \tau_a\varphi$, by induction, hence $\pi \equiv_{\mathrm{LQ}} \tau_a\pi$, by Proposition 3.4.

Assume that $k-2$ occurs in s. Then there are $u_i \in \mathbb{N}^*$ ($i \in [6]$) such that one of the following four cases holds.

Case 1 $\pi = au_1(k+1)u_2ku_3(a+1)u_4(k-2)u_5(k-1)u_6$, then $\pi \equiv_Q \tau_{k-1}\pi \equiv_Q \tau_k\tau_{k-1}\pi$, and

$$\tau_k\tau_{k-1}\pi = au_1ku_2(k-1)u_3(a+1)u_4(k-2)u_5(k+1)u_6 \equiv_{\mathrm{LQ}} \tau_a(\tau_k\tau_{k-1}\pi),$$

by induction. Again Proposition 3.4 implies $\pi \equiv_{\mathrm{LQ}} \tau_a\pi$.

Case 2 $\pi = au_1ku_2(k+1)u_3(a+1)u_4(k-1)u_5(k-2)u_6$, then put $m := \ell(u_4) + \ell(u_5) + \ell(u_6) + 2$ and

$$\varphi := (-1)^{n-1+m}(a+1)\Big(u_4ku_5(k-2)u_6 \sqcup\!\sqcup \bar{u}_3(k+1)\bar{u}_2(k-1)\bar{u}_1a\Big)$$

to obtain $\pi \equiv_Q au_1(k-1)u_2(k+1)u_3(a+1)u_4ku_5(k-2)u_6 \equiv_L \varphi$, by (5). For all summands ν in φ with k to the left of a, that is $\nu^{-1}(k) < \nu^{-1}(a)$, $\nu \equiv_{\mathrm{LQ}} \tau_a\nu$, by induction, while each of the summands ν with k to the right of a is of the form

$$\nu = (a+1)v_1(k+1)v_2(k-1)v_3av_4kv_5 \equiv_Q \tau_k\nu,$$

hence $\nu \equiv_{\mathrm{LQ}} \tau_a\nu$, by induction and Proposition 3.4. Putting both parts together yields $\varphi \equiv_{\mathrm{LQ}} \tau_a\varphi$, hence $\pi \equiv_{\mathrm{LQ}} \tau_a\pi$, by Proposition 3.4.

Case 3 $\pi = au_1(k+1)u_2ku_3(a+1)u_4(k-1)u_5(k-2)u_6$, then putting $m := \ell(u_4) + \ell(u_5) + \ell(u_6) + 2$,

$$\pi \equiv_L (-1)^{n-1+m}(a+1)\Big(u_4(k-1)u_5(k-2)u_6 \sqcup\!\sqcup \bar{u}_3k\bar{u}_2(k+1)\bar{u}_1a\Big),$$

by (5). For each of the summands, swapping a and $a+1$ yields an LK-equivalent permutation, since either $k-1$ stands to the left of a and the induction hypothesis may be applied, or $k-1$ stands to the right of a and Case 2 may be applied. Thus $\pi \equiv_{\mathrm{LQ}} \tau_a\pi$, by Proposition 3.4.

Combining cases 1 and 3 shows that $(k+1)k$ allows the a-switch in $\mathcal{S}_n$, hence also $k(k+1)$, by Proposition 3.3. This, in particular, yields the assertion in the remaining case:

Case 4 $\pi = au_1ku_2(k+1)u_3(a+1)u_4(k-2)u_5(k-1)u_6$. ■

For the proof of the second step, an auxiliary result is needed, which is based on the following two observations:

$$(k+1)1\,2\,3\cdots(k-1)\,k \equiv_Q 2\,1\,3\,4\cdots k(k+1), \tag{7}$$

and

$$1\,2\cdots(j-2)(j-1)(j+1)\,j \equiv_Q 2\,3\cdots(j-1)\,j(j+1)1 \tag{8}$$

for all $j,k\in\mathbb{N}$.

Proposition 3.5 *Let $k,x\in[n]$, $j\in[k]$, $v_1,\dots,v_{k+1},w\in\mathbb{N}^*$ and set $x_i := x+i$ for all $i\in[k+1]$. If*

$$\pi := \; v_1x_1v_2x_2\cdots v_{j-1}x_{j-1}\; v_{k+1}x_{k+1}v_j\,x_jv_{j+1}x_{j+1}\cdots v_kx_k \quad w$$

is contained in $\mathcal{S}_n$, then

$$\pi \equiv_Q v_1x_2v_2x_3\cdots v_{j-1}x_j \quad v_{k+1}x_{j+1}v_j\,x_1v_{j+1}x_{j+2}\cdots v_kx_{k+1}\,w.$$

Proof The proof is split into two parts, the first of which is (up to the constant summand x_{j-1}) covered by (7), applied to $x_{k+1}x_jx_{j+1}\cdots x_k$; while the second is (up to the constant summand x) immediate from (8), applied to $x_1x_2\cdots x_{j-1}x_{j+1}x_j$.

More formally, let $u := v_1x_1v_2x_2\cdots v_{j-1}x_{j-1}$ and $\hat{u} := v_{j+1}x_{j+2}\cdots v_kx_{k+1}w$. Then

$$\begin{aligned}
\pi =\;& uv_{k+1}\underline{x_{k+1}}\,v_jx_jv_{j+1}x_{j+1}\cdots v_{k-1}x_{k-1}v_k\underline{x_k} \quad w\\
\equiv_Q\;& uv_{k+1}\underline{x_k} \quad v_jx_jv_{j+1}x_{j+1}\cdots v_{k-1}\underline{x_{k-1}}v_kx_{k+1}w\\
&\qquad\cdots\\
\equiv_Q\;& uv_{k+1}x_{j+1}v_jx_jv_{j+1}x_{j+2}\cdots v_{k-1}x_k \quad v_kx_{k+1}w\\
=\;& v_1x_1v_2x_2\cdots v_{j-2}x_{j-2}v_{j-1}\underline{x_{j-1}}v_{k+1}x_{j+1}v_j\underline{x_j} \quad \hat{u}\\
\equiv_Q\;& v_1x_1v_2x_2\cdots v_{j-2}\underline{x_{j-2}}v_{j-1}x_j \quad v_{k+1}x_{j+1}v_j\underline{x_{j-1}}\hat{u}\\
&\qquad\cdots\\
\equiv_Q\;& v_1x_2v_2x_3\cdots v_{j-2}x_{j-1}v_{j-1}x_j \quad v_{k+1}x_{j+1}v_jx_1 \quad \hat{u}
\end{aligned}$$

as asserted, where the letters in question are underlined in each step. ■

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Step 2 Let $a \in [n-1]$ and $x, y \in [n]$ such that $x, y > a+1$ or $x, y < a$, then xy allows the a-switch in $\mathcal{S}_n$.

In particular, if $v \in \mathbb{N}^*$ such that $\ell(v) \geq 3$, then v allows the a-switch in $\mathcal{S}_n$.

Proof If $x = y$, there is nothing to prove; let $x \neq y$. Let $\pi = aw(a+1)s$ such that xy is a sub-word of w.

Consider first the case where $x, y > a+1$. We may assume that $x < y$, by Proposition 3.3. The proof is done by induction on $m := y - x$.

If $m = 1$, then $\pi \equiv_{\mathrm{LQ}} \tau_a \pi$ follows from Step 1.

Let $m > 1$, and set $x_i := x + i$ for all $i \in \mathbb{N}$. Inductively, the case where x_i occurs in s for all $i \in [m-1]$ remains.

Choose $k \in [m-1]$ maximal such that $\pi^{-1}(x_1) < \pi^{-1}(x_2) < \cdots < \pi^{-1}(x_k)$, that is

$$\pi = a\, u_1\, x\, u_2\, y\, u_3\, (a+1)\, v_1\, x_1\, v_2\, x_2 \cdots v_k\, x_k\, v_{k+1}$$

for suitably chosen $u_i, v_i \in \mathbb{N}^*$.

If $x_k = y - 1$, then either $m = 2$ and $\pi \equiv_Q \tau_x \pi$, or $m > 2$ and $\pi \equiv_Q \tau_{y-1}\pi$; in both cases, $\pi \equiv_{\mathrm{LQ}} \tau_a \pi$, by induction and Proposition 3.4.

Let $x_k < y - 1$, then $y > x_2$; and there is an index $j \in [k]$ such that $\pi^{-1}(x_{j-1}) < \pi^{-1}(x_{k+1}) < \pi^{-1}(x_j)$ (where $x_0 := x$ if $j = 1$). Let $t := au_1xu_2yu_3(a+1)$, then

$$\begin{aligned}
\pi &= tv_1x_1v_2x_2 \cdots v_{j-1}x_{j-1}v_j^{(1)}x_{k+1}v_j^{(2)}x_jv_{j+1}x_{j+1} \cdots v_kx_kv_{k+1} \\
&\equiv_Q tv_1x_2v_2x_3 \cdots v_{j-1}x_jv_j^{(1)}x_{j+1}v_j^{(2)}x_1v_{j+1}x_{j+2} \cdots v_kx_{k+1}v_{k+1} \\
&=: \hat{\pi},
\end{aligned}$$

by Proposition 3.5. For example, consider

$$\pi = 1\,4\,12\,2\,5\,9\,10\,6\,11\,7\,8 \in \mathcal{S}_{12}$$

(and $a = 1$, $x = 4$, $y = 12$), then $k = 4$, since $x_5 = 9$ stands to the left of $x_4 = 8$. In this case,

$$\pi \equiv_Q \hat{\pi} = 1\,4\,12\,2\,6\,7\,10\,5\,11\,8\,9,$$

and there is the relation $\hat{\pi} \equiv_Q \tau_4\hat{\pi}$. Indeed, $\hat{\pi} \equiv_Q \tau_x\hat{\pi}$ holds in the general case, since $\hat{\pi}^{-1}(x) < \hat{\pi}^{-1}(x_2) < \hat{\pi}^{-1}(x_1)$. Now, by induction, $\tau_x\hat{\pi} \equiv_{\mathrm{LQ}} \tau_a(\tau_x\hat{\pi})$, hence also $\hat{\pi} \equiv_{\mathrm{LQ}} \tau_a\hat{\pi}$, by Proposition 3.4. Another application of Proposition 3.4 yields $\pi \equiv_{\mathrm{LQ}} \tau_a\pi$ and completes the proof in the case $x, y > a+1$.

Now assume that $x, y < a$, then $n+1-x, n+1-y > (n-a)+1$, hence $(n+1-x)(n+1-y)$ allows the $(n-a)$-switch in $\mathcal{S}_n$, by the part already proven. As a consequence, xy allows the a-switch in $\mathcal{S}_n$, by Proposition 3.3.

If $v \in \mathbb{N}^*$ such that there are three distinct letters $\neq a, a+1$ occurring in v, then at least two of these are $< a$ or $> a+1$. This completes the proof. ■

Step 3 Let $a \in [n-1]$ and $v \in \mathbb{N}^*$ such that $\ell(v) \geq 2$, then v allows the a-switch in $\mathcal{S}_n$.

Proof By (3) and Step 2, the case where $v = xy$ such that and $x < a-1$ and $y > a+2$, or $y < a - 1$ and $x > a + 2$, remains. By Proposition 3.3, it suffices to consider the case of $x < a - 1$ and $y > a + 2$. Let $\pi = aw(a + 1)s \in \mathcal{S}_n$ such that xy is a sub-word of w. If $\ell(w) \geq 3$, then $\pi \equiv_{\mathrm{LQ}} \tau_a\pi$ follows from Step 2.

Let $w = xy$, then each of the letters $x + 1, x + 2, \ldots, a - 1$ occurs in s, and $y > a + 1 > a > a - 1 \geq x + 1$. Applying Step 2 a number of times implies

$$\pi \equiv_{\mathrm{LQ}} \tau_x\pi \equiv_{\mathrm{LQ}} \tau_{x+1}\tau_x\pi \equiv_{\mathrm{LQ}} \cdots \equiv_{\mathrm{LQ}} \tau_{a-2}\cdots\tau_{x+1}\tau_x\pi = a(a-1)y(a+1)\hat{s}$$

for a properly chosen $\hat{s} \in \mathbb{N}^*$, hence $\pi \equiv_{\mathrm{LQ}} \tau_a\pi$ as asserted, by Proposition 3.4. ■

As the final step, we are now in a position to state and prove:

Theorem 2 $\mathrm{Lie} \cap \mathcal{Q} \subseteq \mathcal{D}$.

Proof It suffices to prove $\mathrm{Lie}_n \cap \mathcal{Q}_n \subseteq \mathcal{D}_n$, since $\mathrm{Lie} \cap \mathcal{Q} = \bigoplus_{n\geq 0} \mathrm{Lie}_n \cap \mathcal{Q}_n$. By Proposition 3.2, Step 3 and (6), it thus remains to be shown that $\pi \equiv_{\mathrm{LQ}} \tau_a\pi$ whenever $x, a \in [n]$, $s \in \mathbb{N}^*$ such that $\pi = ax(a + 1)s \in \mathcal{S}_n$.

If $n = 3$, this is immediate. Let $n > 3$ and choose $y \in \mathbb{N}$ and $u \in \mathbb{N}^*$ such that $s = yu$, then

$$\begin{aligned}
\pi &= ax(a+1)yu \\
&= a\big((a+1)\sqcup\!\sqcup xyu\big) - a(a+1)xyu - axy\big((a+1)\sqcup\!\sqcup u\big) \\
&\equiv_L -(a+1)axyu - a(a+1)xyu - axy\big((a+1)\sqcup\!\sqcup u\big), \quad \text{by (4)} \\
&\equiv_{\mathrm{LQ}} -(a+1)axyu - a(a+1)xyu - (a+1)xy(a\sqcup\!\sqcup u), \quad \text{by Step 3} \\
&\equiv_L (a+1)(a\sqcup\!\sqcup xyu) - (a+1)axyu - (a+1)xy(a\sqcup\!\sqcup u), \quad \text{by (4)} \\
&= (a+1)xayu \\
&= \tau_a\pi.
\end{aligned}$$

The theorem is proved. ■

Denote by Δ the coproduct in $\mathcal{P}$ (as in [MR95, p. 977]), and by $\epsilon \in \mathcal{S}_0$ the identity of $\mathcal{P}$. For any a Hopf subalgebra $\mathcal{A}$ of $\mathcal{P}$, let

$$\mathrm{Prim}(\mathcal{A}) = \{\alpha \in \mathcal{A} \mid \Delta(\alpha) = \alpha \otimes \epsilon + \epsilon \otimes \alpha\}$$

be the primitive Lie algebra of $\mathcal{A}$, and denote by $\mathrm{Prim}(\mathcal{A})_n$ its n-th homogeneous component. The subalgebra $\mathcal{C}$ of $\mathcal{P}$ generated by $\mathrm{Prim}(\mathcal{P})$ (the domain of co-commutativity of $\mathcal{P}$) contains $\mathcal{L}$. Furthermore, $\mathcal{C} \cap \mathcal{Q}$ (the domain of co-commutativity of $\mathcal{Q}$) is generated by $\mathrm{Prim}(\mathcal{Q})$ and contains $\mathcal{D} = \mathcal{L} \cap \mathcal{Q}$.

It turns out that $\mathcal{D}$ is *strictly* contained in $\mathcal{C} \cap \mathcal{Q}$. Indeed, $\varphi = 3412 + 2413 - (3142 + 2143) \in \mathcal{Q}_4 \setminus \mathcal{D}_4$, and $\Delta(\varphi) = \varphi \otimes \epsilon + \epsilon \otimes \varphi$. For $n = 4, 5, 6$, the dimension of $\mathrm{Prim}(\mathcal{D})_n$ is, respectively, 3, 6, and 9, while the dimension of $\mathrm{Prim}(\mathcal{Q})_n$ is, respectively, 4, 9, and 26. A description of the elements of $\mathrm{Prim}(\mathcal{D})_n$ as well as of its dimension is known in general ([BL93, 4.5], [BL96, 1.5]). It would be of interest if analogous results for $\mathcal{Q}$ were obtained.

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