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ON A CLASS OF POSITIVE LINEAR OPERATORS

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In a recent paper [3] Meir and Sharma introduced a generalization of the S_α-method of summability. The elements of their matrix, (a_{nk}), are defined by

$$\prod_{j=0}^{n} \frac{1-\alpha_j}{1-\alpha_j\theta} = \sum_{k=0}^{\infty} a_{nk}\theta^k \tag{1}$$

where $\{\alpha_j\}_j^\infty=0$ is a sequence of complex numbers. If $0<\alpha_j<1$ for each $j=0, 1, 2, \ldots$ then $a_{nk}\geq 0$ for each $n=0, 1, 2, \ldots$ and $k=0, 1, 2, \ldots$.

If we let

$$\alpha_j = \alpha_j(x) = h_j(x)/[1+h_j(x)]$$

$j=0, 1, 2, \ldots$ where $\{h_j(x)\}_{j=0}^\infty$ is a sequence of nonnegative real valued functions defined on $[0, +\infty)$, the identity
(1) takes the form

$$R_n(x;\theta) = \prod_{j=0}^{n} \frac{1}{1+h_j(x)-h_j(x)\theta} = \sum_{k=0}^{\infty} c_{nk}(x)\theta^k. \tag{2}$$

To each real valued function f, defined on $[0, +\infty)$ we associate the positive linear operator, L_n, defined by

$$L_n(f;x) = \sum_{k=0}^{\infty} c_{nk}(x) f\left(\frac{k}{n}\right). \tag{3}$$

The special case $h_j(x)=x$ for $j=0, 1, 2, \ldots$ results in an operator introduced by Baskakov [1]. In this note we prove the following convergence theorem for the sequence $\{L_n\}_{n=0}^\infty$ defined by [3].

THEOREM. *Suppose* $\{h_j(x)\}_{j=0}^\infty$ *is a sequence of continuous, nonnegative real valued functions defined on* $[0, +\infty)$. *Suppose that on each interval* $[0, a]$ *there is a constant* M *which depends only on* a *such that* $h_j(x)\leq M$ *for* $j=0, 1, 2, \ldots, x\in[0, a]$. *Let* f *be continuous on* $[0, \infty)$ *and satisfy* $|f(x)|\leq e^{Ax}$ *for some constant* $A>0$. *The sequence* $\{L_n(f)\}_{n=0}^\infty$ *defined by* [3] *converges to* f *uniformly on* $[0, a]$ *if* $\{h_j\}_{j=0}^\infty$ *is uniformly* $(C, 1)$ *summable to* x *on* $[0, a]$.

King [2] established an analogous result for a class of operators associated with the generalized Lototsky transform.

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Proof of the Theorem. We note that each series defined by [2] has radius of convergence bigger than or equal to $1+1/M$ and that $L_n(f; x)$ is defined for $n>A\{\ln[1+(1/M)]\}^{-1}$.

Let $\varepsilon>0$ be given. Choose $\delta>0$ so that $|f(t)-f(x)|<\varepsilon$ if $|t-x|<\delta$ for any t, x, $\varepsilon[0, 2a]$. Assume that $\delta<a$. Then

$$\begin{aligned}|L_n(f; x)-f(x)| &\le \sum_{|(k/n)-x|\le\delta} c_{nk}(x)\left|f\left(\frac{k}{n}\right)-f(x)\right|+\sum_{|(k/n)-x|\ge\delta} c_{nk}(x)\left|f\left(\frac{k}{n}\right)-f(x)\right| \\ &\le \varepsilon+\sum_{|(k/n)-x|\ge\delta} c_{nk}(x)\left|f\left(\frac{k}{n}\right)-f(x)\right| \\ &\le \varepsilon+\frac{1}{n^2\delta^2}\sum_{k=0}^{\infty}(k-nx)^2c_{nk}(x)\left|f\left(\frac{k}{n}\right)-f(x)\right| \\ &\le \varepsilon+\frac{e^{Ax}}{n^2\delta^2}\sum_{k=0}^{\infty}(k-nx)^2c_{nk}(x)+\frac{1}{n^2\delta^2}\sum_{k=0}^{\infty}(k-nx)^2c_{nk}(x)\left|f\left(\frac{k}{n}\right)\right| . \\ &\le \varepsilon+\frac{e^{Ax}}{n^2\delta^2}\sum_{k=0}^{\infty}(k-nx)^2c_{nk}(x)+\frac{1}{n^2\delta^2}\sum_{k=0}^{\infty}(k-nx)^2c_{nk}(x)(e^{A/n})^k.\end{aligned}$$

Then

$$\begin{aligned}(4)\quad \sum_{k=0}^{\infty}(k-nx)^2c_{nk}(x) &= n^2x^2-2nx\frac{\partial R_n(x;1)}{\partial\theta}+\frac{\partial R_n(x;1)}{\partial\theta}+\frac{\partial^2R_n(x;1)}{\partial\theta^2} \\ &= n^2x^2+(1-2nx)H_n(x;1)+\sum_{j=0}^{n}(h_j(x))^2+(H_n(x;1))^2\end{aligned}$$

and

$$\begin{aligned}(5)\quad &\sum_{k=0}^{\infty}(k-nx)^2c_{nk}(x)(e^{A/n})^k \\ &= n^2x^2R_n(x;e^{A/n})-2nxe^{A/n}\frac{\partial R_n(x;e^{A/n})}{\partial\theta}+e^{2A/n}\frac{\partial R_n(x;e^{A/n})}{\partial\theta}+e^{2A/n}\frac{\partial^2R_n(x;e^{A/n})}{\partial\theta} \\ &= e_{n^2x^2R_n(x;e^{A/n})}-2nxe^{A/n}R_n(x;e^{A/n})H_n(x;e^{A/n})+e^{2A/n}R_n(x;e^{A/n})H_n(x;e^{A/n}) \\ &\quad+e^{2A/n}R_n(x;e^{A/n})(H_n(x;e^{A/n}))^2+e^{2A/n}R_n(x;e^{A/n})\sum_{j=0}^{n}\frac{(h_j(x))^2}{(1+h_j(x)-h_j(x)e^{A/n})^2}\end{aligned}$$

where

$$H_n(x, y)=\sum_{j=0}^{n}\frac{h_j(x)}{1+h_j(x)-h_j(x)y}.$$

Using the representation (4), the inequality $(h_j(x))^2\le Mh_j(x)$ and the hypotheses, we see that

$$\frac{e^{Ax}}{n^2\delta^2}\sum_{k=0}^{\infty}(k-nx)^2c_{nk}(x)$$

approaches 0 uniformly on $[0, a]$ as $n\to\infty$.

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Since $1+h_j(x)-h_j(x)e^{A/n}>1+M(1-e^{A/n})$, it follows that $R_n(x;\, e^{A/n})$ is bounded as $n\to\infty$.

We can write

$$\frac{h_j}{1+h_j-h_je^{A/n}}=h_j-\frac{h_j^2(1-e^{A/n})}{1+h_j-h_je^{A/n}}$$

and

$$\left|\frac{h_j^2(1-e^{A/n})}{1+h_j-h_je^{A/n}}\right|\le\frac{M^2\,|1-e^{A/n}|}{|1+M(1-e^{A/n})|}.$$

Therefore,

$$\frac{1}{n}\sum_{j=0}^{n}\frac{h_j(x)}{1+h_j(x)-h_j(x)e^{A/n}}$$

converges to x uniformly on $[0, a]$.

Thus from the representation (5), we can conclude that

$$\frac{1}{n^2\delta^2}\sum_{k=0}^{\infty}(k-nx)^2c_{nk}(x)(e^{A/n})^k$$

converges to 0 uniformly on $[0, a]$.

This completes the proof.

The referee has pointed out that a weaker version of the result can be found in Pethe's Ph.D. dissertation [4].

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