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THE K-THEORY OF SOME HIGHER RANK EXEL–LACA ALGEBRAS

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Abstract

Let $\mathcal{O}$ be a higher rank Exel–Laca algebra generated by an alphabet $\mathcal{A}$. If $\mathcal{A}$ contains d commuting isometries corresponding to rank d and the transition matrices do not have finite rows, then $K_1(\mathcal{O})$ is trivial and $K_0(\mathcal{O})$ is isomorphic to K_0 of the abelian subalgebra of $\mathcal{O}$ generated by the source projections of $\mathcal{A}$.

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1. Introduction

Several authors have considered the K-theory of generalized Cuntz–Krieger algebras. For instance, the K-theory of the classical Cuntz–Krieger algebras yields a famous invariant by Bowen and Franks [2] for shifts of finite type. Another important application of K-theory of generalized Cuntz–Krieger algebras is their classification. In the best case they may be completely classified by the theorem of Kirchberg [17] and Phillips [20]: generalized Cuntz–Krieger algebras are often purely infinite and thus determine candidates in advance. The computation of the K-theory of rank-one graph C^*-algebras, see [11, 16, 19, 25, 30], and Cuntz–Krieger and Exel–Laca algebras, see [9, 10, 14], is completed.

In return, the computation of the K-theory of higher rank graph C^*-algebras [18, 23, 24] and Cuntz–Krieger algebras [27, 28] is extremely scanty. Explicit results exist only for rank two and rank three, see [1, 4, 12, 29]. Evans [12] proves that the K-groups of finitely generated higher rank graph C^*-algebras are finitely generated. A duality theorem was proved by Popescu and Zacharias [22].

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In this article we compute the K-theory of higher rank Exel–Laca algebras having the properties (I) and (II) for *all* ranks, see Theorem 5.6. The K_1-group is always trivial in this case, and the K_0-group is always torsion free. We use a crossed product representation $A \rtimes_\alpha \mathbb{Z}^d$, where A is an AF-algebra, and repeatedly apply the Pimsner–Voiculescu sequence. The computation, however, is not easy, and a blindfold approach would quickly collapse by the complexity of A and α. The property (II) is designed in such a way that in each step the K_1-group stays trivial. Without that assumption, that is, in the general case, the computation may only work if in each step the Pimsner–Voiculescu sequence would split naturally. This, however, is unsettled. The same problem arises for Evans in [12] who uses a theorem of Kasparov [15]: the spectral sequences that appear may not split naturally.

This paper is organized as follows. In Section 2 we recall the definition of higher rank Exel–Laca algebras and define the properties (I) and (II). In Sections 3–5 we prove the main result, Theorem 5.6. In Section 6 we use this theorem to compute the K-theory of some rank-two Cuntz–Krieger algebras inspired by shifts of finite type in dimension two [5].

2. Higher rank Exel–Laca algebras

A *triple* $(\mathcal{A}, \mathbb{F}, \mathbb{I})$ *of generators and relations* consists of an alphabet $\mathcal{A}$, the free nonunital $*$-algebra $\mathbb{F}$ over the field $\mathbb{C}$ generated by $\mathcal{A}$, and a two-sided self-adjoint ideal $\mathbb{I}$ in $\mathbb{F}$.

DEFINITION 2.1 (Higher rank Exel–Laca algebra [6]). Let $(\mathcal{A}, \mathbb{F}, \mathbb{I})$ be a triple of generators and relations, and let X be the quotient $*$-algebra $\mathbb{F}/\mathbb{I}$. It is convenient by an abuse of notation to denote the equivalence class $x + \mathbb{I}$ in X by x for $x \in \mathcal{A}$ or $x \in \mathbb{F}$. Throughout we use the notation $P_a = aa^* \in X$ and $Q_a = a^*a \in X$ when $a \in \mathcal{A}$.

Assume that $\mathcal{A}$ is endowed with a partition $\mathcal{A} = \bigsqcup_{v \in V} v$ such that the following six properties hold.

Rank-one Cuntz–Krieger relations. There exists a family $(s_v)_{v\in V}$ of maps $s_v : v \times v \to \{0, 1\}$, each of which is called a *transition matrix*, such that for all $v \in V$ and all $a, b \in v$ the identities $aa^*a = a$, $Q_a Q_b = Q_b Q_a$, $P_a P_b = \delta_{a,b} P_a$ and $Q_a P_b = s_v(a, b) P_b$ hold in X.

Permutation rules. For all $v_1, v_2 \in V$ such that $v_1 \neq v_2$, and for all $a_1 \in v_1, a_2 \in v_2$ and $\epsilon_1, \epsilon_2 \in \{1, *\}$, the product $a_1^{\epsilon_1} a_2^{\epsilon_2}$ vanishes in X, or there exist $b_1 \in v_1, b_2 \in v_2$ such that both identities

$$a_1^{\epsilon_1} a_2^{\epsilon_2} = b_2^{\epsilon_2} b_1^{\epsilon_1} \quad \text{and} \quad b_1^{\epsilon_1} (a_2^{\epsilon_2})^* = (b_2^{\epsilon_2})^* a_1^{\epsilon_1},$$

hold in X.

The next condition ensures the existence of certain gauge actions on X.

Invariance of the ideal. The ideal $\mathbb{I}$ is invariant under the automorphisms $t_\lambda : \mathbb{F} \to \mathbb{F}$ for all $\lambda \in \mathbb{T}^V$, where $t_\lambda(a) = \lambda_v a$ for all $a \in v \in V$.

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The following property ensures that the norm closure of the fixed point algebra $\mathbb{A}$ of certain gauge actions on X is an AF-algebra.

Stronger finiteness property. For an integer $N \geq 0$ and a subset $w \subseteq \mathcal{A}$, define

$$F_{w,N} = \{a_1 \dots a_n Q_{c_1} \dots Q_{c_m} b_n^* \dots b_1^* \in X \mid \\ 0 \leq n \leq N,\ 1 \leq m,\ a_i, c_i, b_i \in w\} \cup \{0\}.$$

For all finite subsets $\{v_1, \dots, v_n\} \subseteq V$ and all finite subsets $u_i \subseteq v_i$, where $1 \leq i \leq n$, we require that for all $1 \leq i \leq n$ there exist *finite* subsets $w_i \subseteq v_i$ containing u_i, such that for all $1 \leq i, j \leq n$ we have

$$F_{w_i,N} F_{w_j,N} \subseteq \operatorname{span}(F_{w_j,N} F_{w_i,N}),$$

($F_{w_i,N} F_{w_j,N}$ denotes the set of products in X) and

$$\{P_a \mid a \in w_i\} F_{w_j,N} \subseteq \operatorname{span}(F_{w_j,N} \{P_a \mid a \in w_i\}). \tag{1}$$

The next property is the counterpart to certain aperiodicity conditions for graphs.

Projections property. For all nonzero words $x = x_1 \dots x_m \in X$ in the letters $x_i \in \mathcal{A}$, all $v \in V$, and all sequences $(a_n)_{n \geq 1} \subseteq v$ there exists $N \geq 1$ such that $xx^* a_1 \dots a_N a_N^* \dots a_1^* \neq xx^*$ in X.

Finally we require a nontrivial representation π of X on a Hilbert space H as follows. (Throughout Alg and Alg* denote the generated algebra and $*$-algebra (without topology) respectively.)

Saturating $\mathbb{A}_{00}$-faithful representation. There exists a representation $\pi : X \to B(H)$ which is injective on

$$\mathbb{A}_{00} = \operatorname{Alg}\{aa^* \in X \mid a \in \mathcal{A} \cup \mathcal{A}^*\},$$

and such that for all $v \in V$ the strong operator sum $\sum_{a \in v} \pi(P_a)$ is a unit for $\pi(b)$ for all $b \in \mathcal{A}$.

The norm closure $\overline{\pi(X)}$ is denoted by $\mathcal{O}_{\mathcal{A},\mathbb{F},\mathbb{I}}$ and called the *higher rank Exel–Laca algebra* associated to $(\mathcal{A}, \mathbb{F}, \mathbb{I})$. The cardinality card$(V)$ is regarded as the *rank* of the Exel–Laca algebra (which, however, is not unique in general).

The 'stronger finiteness property' of Definition 2.1 is slightly sharper than the 'finiteness property' in [6]. However, the difference is minor and the extra amount required (namely (1)) seems 'natural' (owing to the permutation rules).

REMARK 2.2. If the alphabet $\mathcal{A}$ is finite then there exists a higher rank graph Λ such that $\mathcal{O}_{\mathcal{A},\mathbb{F},\mathbb{I}}$ is $*$-isomorphic to the higher rank graph C^*-algebra $C^*(\Lambda)$ defined in [18]. To be precise, the object set of Λ is defined by

$$\Lambda^0 = \{(a_1, a_2, \dots, a_k) \in v_1 \times v_2 \times \dots \times v_k \mid \pi(P_{a_1} P_{a_2} \dots P_{a_k}) \neq 0\},$$

where $V = \{v_1, \dots, v_k\}$. We introduce exactly one morphism $\theta_{a,b}^i$ in Λ^{e_i} with range $r(\theta_{a,b}^i) = a = (a_1, \dots, a_k)$ and source $s(\theta_{a,b}^i) = b = (b_1, \dots, b_k)$ if and only if

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$$0 \neq \pi(P_{a_1} P_{a_2} \dots P_{a_k} a_i P_{b_1} P_{b_2} \dots P_{b_k}).$$

Then

$$\begin{aligned} s_a &= \pi(P_{a_1} P_{a_2} \dots P_{a_k}) \quad \text{for } a = (a_1, \dots, a_k) \in \Lambda^0, \\ s_{\theta^i_{a,b}} &= \pi(a_i P_{b_1} P_{b_2} \dots P_{b_k}) \quad \text{for } a, b \in \Lambda^0,\ 1 \leq i \leq k, \end{aligned}$$

is a Cuntz–Krieger family in $\pi(X)$ which generates $\mathcal{O}_{\mathcal{A},\mathbb{F},\mathbb{I}}$ (see [7]).

In the rest of this paper we will fix a triple $(\mathcal{A}, \mathbb{F}, \mathbb{I})$ of generators and relations satisfying Definition 2.1. We also fix a *finite* partition $V = \{v_1, \dots, v_d\}$ of $\mathcal{A}$, a family of transition matrices $(s_v)_{v \in V}$, and a saturating $\mathbb{A}_{00}$-faithful representation π as required in Definition 2.1. Moreover, we assume that the following two properties hold.

(I) The quotient $*$-algebra X has a unit I and there exists a family $(S_v)_{v \in V}$ of commuting isometries such that $S_v \in v$ for all $v \in V$.

(II) Denote by q the commutative algebra $\mathrm{Alg}\{Q_a \in X \mid a \in \mathcal{A}\}$. Then for all $1 \leq i \leq d$, all finite subsets $\mathcal{B} \subseteq v_i$, and all non-zero $z \in q$ we require that $P = \sum_{b \in \mathcal{B}} P_b$ is not a unit for z (that is, $Pz \neq z$ in X).

REMARK 2.3. Each row of s_v (for $v \in V$) contains either no or infinitely many zeros and contains either no or infinitely many ones. In particular, $\mathcal{A}$ is infinite and the translation to graph algebras as in Remark 2.2 does not work. Hence, the result in Theorem 5.6 is presumably disjoint from the K-theory results [1, 4, 12, 22, 29] for graph C^*-algebras [18]. To see the claim, assume that the set $F_g = \{b \in v \mid s_v(a, b) = g\}$ is nonempty and finite for some fixed $v \in V$, $a \in v$ and $g \in \{0, 1\}$. Since π is saturating,

$$\pi(Q_a) = \pi(Q_a) \sum_{c \in v} \pi(P_c) = \sum_{c \in v} s_v(a, c) \pi(P_c). \tag{2}$$

However, this is a finite sum if $g = 1$. Since π is faithful on $\mathbb{A}_{00}$ and $Q_a - \sum_{c \in v} s_v(a, c) P_c \in \mathbb{A}_{00}$, we have $Q_a = \sum_{c \in v} s_v(a, c) P_c$. However, this contradicts property (II).

3. Proof part 1

In this section we write the stable form of $\mathcal{O}_{\mathcal{A},\mathbb{I},\mathbb{F}}$ as a crossed product of an AF-algebra A by an action α of $\mathbb{Z}^d$.

Recall that we fixed a triple $(\mathcal{A}, \mathbb{F}, \mathbb{I})$ which generates a higher rank Exel–Laca algebra and satisfies the properties (I) and (II). Let $\mathbb{Z}^d_+$ be the elements of $\mathbb{Z}^d$ with nonnegative coordinates. If $n = (n_1, \dots, n_d) \in \mathbb{Z}^d_+$ then we write $S^n = S_1^{n_1} \dots S_d^{n_d}$. Let

$$W = \{x_1 x_2 \dots x_n \in X \mid n \geq 1, x_i \in \mathcal{A} \cup \mathcal{A}^*\},$$

be the set of words. Let $\delta_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{Z}^d$ (ith position) for $1 \le i \le d$. We call the involutive semigroup homomorphism

$$\mathrm{bal} : W \setminus \{0\} \to \mathbb{Z}^d : \mathrm{bal}(a) = \delta_i, \quad 1 \le i \le d, \; a \in v_i,$$

the *balance function* (see [6]). That means that we have $\mathrm{bal}(xy) = \mathrm{bal}(x) + \mathrm{bal}(y)$ and $\mathrm{bal}(x^*) = -\mathrm{bal}(x)$. A word $x \neq 0$ is called *zero-balanced* if $\mathrm{bal}(x) = 0$, otherwise we call it *non-zero-balanced*. The linear span of all zero-balanced words forms a self-adjoint subalgebra in X denoted by $\mathbb{A}$.

By [6, Corollary 4.11], $\mathbb{A}$ is the inductively ordered union of the family Γ of its finite-dimensional sub-C^*-algebras. Thus, the norm closure $\overline{\mathbb{A}}$ may be regarded as the C^*-direct limit $\overline{\mathbb{A}} = \varinjlim_{\mathcal{M} \in \Gamma} \mathcal{M}$. For all $n \le m \in \mathbb{Z}^d$ we put $A_n = \overline{\mathbb{A}}$, and define injections

$$\Psi_{m,n} : A_n \to A_m : \Psi_{m,n}(x) = S^{m-n} x (S^{m-n})^*.$$

Write A for the associated direct limit $\varinjlim_{n \in \mathbb{Z}^d} A_n$. Let $\Psi_n : A_n \to A$ denote the natural embedding. Throughout we will regard A_n as a subset of A (via Ψ_n), and spell out Ψ_n only if there is danger of confusion. Consequently we regard A as $\overline{\bigcup_{n \in \mathbb{Z}^d} A_n}$. For each $1 \le i \le d$ define an action $\alpha_i \in \mathrm{Aut}(A)$ by

$$\alpha_i(\Psi_n(x)) = \Psi_n(S_i x S_i^*) = \Psi_{n-\delta_i}(x), \quad n \in \mathbb{Z}^d, \; x \in \overline{\mathbb{A}}.$$

LEMMA 3.1. *For all $n \le m$, $K_0(\Psi_{m,n})$ is injective.*

PROOF. It is enough to show that $\psi_i : \overline{\mathbb{A}} \to \overline{\mathbb{A}}$, such that $\psi_i(x) = S_i x S_i^*$, is injective in K-theory for all $1 \le i \le d$. First of all notice that ψ_i is injective and has image $\psi_i(\overline{\mathbb{A}}) = P_i \overline{\mathbb{A}} P_i$, where $P_i = S_i S_i^*$. Clearly $K_0(\psi_i')$ is injective for the isomorphism $\psi_i' : \overline{\mathbb{A}} \to P_i \overline{\mathbb{A}} P_i$ given by $\psi_i'(x) = \psi_i(x)$. Thus, it remains to show that $K_0(j)$ is injective for the identity embedding $j : P_i \overline{\mathbb{A}} P_i \to \overline{\mathbb{A}}$.

Since finite-dimensional C^*-algebras have the cancellation property and $\overline{\mathbb{A}}$ is the inductive limit of finite-dimensional C^*-algebras, $\overline{\mathbb{A}}$ has the cancellation property. Consider projections $p, q \in M_N(P_i \overline{\mathbb{A}} P_i)$ and assume that $K_0(j)([p] - [q]) = 0$. Then there exists a partial isometry $T \in M_N(\overline{\mathbb{A}})$ such that $p = TT^*$ and $q = T^*T$. However, since $p, q \le (1_N \otimes P_i) \in M_N \otimes \overline{\mathbb{A}}$, we have $T = (1_N \otimes P_i) T (1_N \otimes P_i) \in M_N(P_i \overline{\mathbb{A}} P_i)$. Hence, $[p] - [q] = 0$ in $K_0(P_i \overline{\mathbb{A}} P_i)$. □

By the last lemma $K_0(\Psi_n)$ is injective, and we therefore regard $K_0(A_n)$ as a subgroup of $K_0(A)$, and regard $K_0(A)$ as $\bigcup_{n \in \mathbb{Z}^d} K_0(A_n)$. Write $\alpha = (\alpha_1, \ldots, \alpha_d)$.

PROPOSITION 3.2. *There exists an isomorphism $\kappa : P(A \rtimes_\alpha \mathbb{Z}^d)P \to \mathcal{O}_{\mathcal{A},\mathbb{F},\mathbb{I}}$, where P is the unit of A_0, which maps $\Psi_0(x) \in A_0$ to $\pi(x)$ for $x \in \mathbb{A}$.*

PROOF. Basically we apply [5, Theorem 3.6]. To see that this is possible we argue as follows. The proof of the uniqueness theorem [6, Theorem 2.3] relies on a uniqueness theorem in [3]. Although the class handled in [3] is more general than the class of [5],

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the crossed product representation [5, Theorem 3.6] also holds for the class in [3]. Indeed the proof of [5, Theorem 3.6] remains valid if one uses the uniqueness theorem of [3] instead of that of [5]. Hence, [5, Theorem 3.6] also holds for higher rank Exel–Laca algebras. In the statement of [5, Theorem 3.6] a closed subgroup $H \subseteq \mathbb{T}^{\mathcal{A}}$ appears, which in this case we choose as in [6], namely

$$H = \{\lambda \in \mathbb{T}^{\mathcal{A}} \mid \forall a, b \in \mathcal{A}:\ \lambda_a = \lambda_b \text{ whenever } a, b \in v_i\}.$$

The dual $\widehat{H}$ of H is isomorphic to $\mathbb{Z}^d$. We define the map $S : \widehat{H} \cong \mathbb{Z}^d_+ \to \mathbb{F}/\mathbb{I}$ required in [5, Theorem 3.6] by $S(n) = S^n$ for all $n \in \mathbb{Z}^d_+$. The claim, including the shape of A, can then be verified by an analysis of the proof of [5, Theorem 3.6]. □

LEMMA 3.3. *Let $\tau : P(A \rtimes_\alpha \mathbb{Z}^d)P \to A \rtimes_\alpha \mathbb{Z}^d$ be the identical embedding. Then $K_0(\tau)$ and $K_1(\tau)$ are isomorphisms.*

PROOF. Let P_n be the unit of A_n. Then $(P_n)_{n\in\mathbb{Z}^d}$ is an approximate unit in A and $\alpha_m(P_n) = P_{n-m}$. Using these facts we easily check that $A \rtimes_\alpha \mathbb{Z}^d = \overline{\bigcup_{n\in\mathbb{Z}^d} \mathcal{O}_n}$, where $\mathcal{O}_n = P_n(A \rtimes_\alpha \mathbb{Z}^d)P_n$. Denote by $\iota_{m,n} : \mathcal{O}_n \to \mathcal{O}_m$ the identical embedding. For $m \geq n$ we have a $*$-isomorphism $\psi_{n,m} : \mathcal{O}_m \to \mathcal{O}_n$ by $\psi_{n,m}(z) = U_{m-n} z U^*_{m-n}$, where U_k denotes the unitary inducing the action α_k. Therefore, we obtain the commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & \mathcal{O}_n \arrow[r, "\iota_{m,n}"] \arrow[d, equal] & \mathcal{O}_m \arrow[r] \arrow[d, "\psi_{n,m}"] & \cdots \\
\cdots \arrow[r] & \mathcal{O}_n \arrow[r] & \mathcal{O}_n \arrow[r] & \cdots
\end{tikzcd}$$

It remains to show that $K_i(\psi_{n,m}\iota_{m,n})$ is the identity map. Now for $z \in \mathcal{O}_n$,

$$\psi_{n,m}\iota_{m,n}(z) = U_{m-n} z U^*_{m-n} = U_{m-n} P_n z P_n U^*_{m-n} = T z T^*,$$

where $T = U_{m-n} P_n$ is an isometry of $\mathcal{O}_n$ (since $T = U_{m-n} P_n U^*_{m-n} U_{m-n} P_n = P_{n-(m-n)} U_{m-n} P_n \in \mathcal{O}_n$ since $P_{n-(m-n)} \leq P_n$). However, for a $*$-homomorphism $\varphi : C \to C : z \mapsto T z T^*$, T an isometry in some unital C^*-algebra C, it is an elementary computation that $K_i(\varphi)$ is the identity map for both $i = 0$ and $i = 1$ (for the case $i = 1$ the statement of [26, Exercise 8.9] is useful). □

Our computation of K-theory is based on Proposition 3.2, Lemma 3.3, and the following lemma, proved by successively using the Pimsner–Voiculescu exact sequence.

LEMMA 3.4. *Let $\alpha_1, \ldots, \alpha_d$ be commuting automorphisms of a C^*-algebra A and assume that $K_1(A) = 0$. Denote $f_i = K_0(\alpha_i) - \mathrm{id} \in \mathrm{End}(K_0(A))$ and $Y_i = \mathrm{Range}(f_1) + \cdots + \mathrm{Range}(f_i)$. Assume that $f_i^{-1}(Y_{i-1}) \subseteq Y_{i-1}$ for all $i = 1, \ldots, d-1$. Then*

$$K_0(A \rtimes_\alpha \mathbb{Z}^d) \cong K_0(A)/Y_d,$$
$$K_1(A \rtimes_\alpha \mathbb{Z}^d) \cong f_d^{-1}(Y_{d-1})/Y_{d-1}.$$

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PROOF. By induction on $i \in \{0, 1, \ldots, d-1\}$, suppose that $K_1(A \rtimes_{(\alpha_1,\ldots,\alpha_i)} \mathbb{Z}^i) = 0$ and $h_i : K_0(A \rtimes_{(\alpha_1,\ldots,\alpha_i)} \mathbb{Z}^i) \to K_0(A)/Y_i$, such that $h_i([a]) = [a] + Y_i$ for all projections $a \in M_\infty(\tilde{A})$, is an isomorphism. We regard α_{i+1} as an action on $A \rtimes_{(\alpha_1,\ldots,\alpha_i)} \mathbb{Z}^i$ in the canonical way. Then we consider the Pimsner–Voiculescu exact sequence [21] (also see [8] where A is not supposed to be unital)

$$\begin{tikzcd}
K_0(A \rtimes \mathbb{Z}^i) \arrow[r, "\mathrm{id}-K_0(\alpha_{i+1}^{-1})"] & K_0(A \rtimes \mathbb{Z}^i) \arrow[r, "K_0(r)"] & K_0(A \rtimes \mathbb{Z}^{i+1}) \arrow[d] \\
K_1(A \rtimes \mathbb{Z}^{i+1}) \arrow[u] & 0 \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

where r is the canonical embedding. Using the isomorphism h_i we obtain

$$\begin{tikzcd}
K_0(A)/Y_i \arrow[r, "f_{i+1}\circ\widetilde{K_0(\alpha_{i+1}^{-1})}"] & K_0(A)/Y_i \arrow[r] & K_0(A \rtimes \mathbb{Z}^{i+1}) \arrow[d] \\
K_1(A \rtimes \mathbb{Z}^{i+1}) \arrow[u] & 0 \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

where $g_{i+1} = f_{i+1} \circ \widetilde{K_0(\alpha_{i+1}^{-1})}$ denotes the quotient map of $f_{i+1} \circ K_0(\alpha_{i+1}^{-1})$. Thus,

$$K_0(A \rtimes \mathbb{Z}^{i+1}) \cong (K_0(A)/Y_i)/\operatorname{Range}(g_{i+1}) \cong K_0(A)/Y_{i+1}.$$

Since Y_i is invariant under $K_0(\alpha_{i+1}^{-1})$, and f_{i+1} and $K_0(\alpha_{i+1}^{-1})$ commute,

$$K_1(A \rtimes \mathbb{Z}^{i+1}) \cong \ker(g_{i+1}) = f_{i+1}^{-1} \circ K_0(\alpha_{i+1}^{-1})^{-1}(Y_i)/Y_i = f_{i+1}^{-1}(Y_i)/Y_i,$$

and the right-hand term vanishes if $i + 1 \le d - 1$ by our assumptions. □

4. Proof part 2

The aim of this section is to prove Lemmas 4.3, 4.4, and 4.5. The other lemmas are preliminary to these lemmas and will not be used later on. We put $w^* = \{a^* \in \mathbb{F} \mid a \in w\}$ and $w^{\circledast} = w \cup w^*$ for subsets $w \subseteq \mathcal{A}$. We define the subalgebras

$$\mathbb{A}_0 = \operatorname{span}\{xx^* \in \mathbb{A} \mid x \in W\},$$
$$q = \operatorname{Alg}\{Q_a \in \mathbb{A} \mid a \in \mathcal{A}\}.$$

Note that $q \subseteq \mathbb{A}_{00} \subseteq \mathbb{A}_0$ ($\mathbb{A}_{00}$ was introduced in Definition 2.1). It is important that $\mathbb{A}_0$ is an abelian algebra, see [6, Lemma 4.4].

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Recall that $\overline{\mathbb{A}}$ may be regarded as the C^*-direct limit $\overline{\mathbb{A}} \cong \varinjlim_{\mathcal{M}\in\Gamma} \mathcal{M}$. Hence, we have a representation $K_0(\overline{\mathbb{A}}) \cong \varinjlim_{\mathcal{M}\in\Gamma} K_0(\mathcal{M})$ and a surjection

$$\varphi : \bigsqcup_{\mathcal{M}\in\Gamma} K_0(\mathcal{M}) \to K_0(\overline{\mathbb{A}}),$$

mapping $[p]-[q] \in K_0(\mathcal{M})$ to $\varphi([p]-[q]) = [p]-[q] \in K_0(\overline{\mathbb{A}})$ for projections $p, q \in M_N(\mathcal{M})$. Hence, we often regard $K_0(\overline{\mathbb{A}})$ (and $K_0(A_n)$ for $n \in \mathbb{Z}^d$) as the union $\bigcup_{\mathcal{M}\in\Gamma} K_0(\mathcal{M})$, where $[p_1]-[q_1] \in K_0(\mathcal{M}_1)$ is identified with $[p_2]-[q_2] \in K_0(\mathcal{M}_2)$ for $\mathcal{M}_1, \mathcal{M}_2 \in \Gamma$ if and only if there exists a $\mathcal{N} \in \Gamma$ such that $\mathcal{M}_1 \cup \mathcal{M}_2 \subseteq \mathcal{N}$ and $[p_1]-[q_1] = [p_2]-[q_2]$ in $K_0(\mathcal{N})$.

Next we are going to introduce notation for some special subgroups of $K_0(A)$ that will play a central role in our further computations. For $1 \le i \le d$, $n \in \mathbb{Z}^d$, $k \in \mathbb{Z}^d_+$ we define

$$\begin{aligned} \phi_i &= K_0(\alpha_i), \\ B^{(n)} &= K_0(A_n) \subseteq K_0(A), \\ B^{(n)}_k &= \mathrm{Group}\{[\Psi_n(S^l Q S^{l^*})] \in K_0(A) \mid l \in \mathbb{Z}^d_+,\ 0 \le l \le k,\ Q \in q\}. \end{aligned}$$

Here Group denotes the generated subgroup in $K_0(A)$. Note that $B^{(n)}_k$ is a subgroup of $K_0(A_n)$. If $k \in \mathbb{Z}^d \setminus \mathbb{Z}^d_+$ then we let $B^{(n)}_k$ be the trivial group. Furthermore, we put $B = B^{(0)}$ and $B_k = B^{(0)}_k$ for $k \in \mathbb{Z}^d$.

In the next lemma we need the stronger finiteness property of Definition 2.1 and we use the notation $F_{w,N}$ of Definition 2.1.

LEMMA 4.1. *The union of certain finite-dimensional C^*-algebras $\mathbb{A}$ is of the form $\mathcal{M} = \mathrm{span}(F_{w_1,N} F_{w_2,N} \ldots F_{w_d,N})$ where $N \ge 1$, $w_j \subseteq v_j$ are finite sets such that $S_j \in w_j$, and such that for all $i = 1, \ldots, d$ the sum $\sum_{a\in w_i} P_a$ is in the center of $\mathcal{M}$.*

PROOF. By [6, Lemma 4.10], $\mathbb{A}$ is the union of finite-dimensional C^*-algebras of the form $\mathcal{M} = \mathrm{span}\, F_{w_1,N} \ldots F_{w_d,N}$ where $N \ge 1$ and each w_i is a finite subset of v_i. In the proof of [6, Lemma 4.10], the sets w_i are chosen according to the claim in the ‘finiteness property’ in [6]. We modify the proof of [6, Lemma 4.10] in that we use the ‘stronger finiteness property’ rather than the ‘finiteness property’ once where it is needed. (Furthermore we may assume that $F_{w_i,N}$ contains the unit $I = S_i^* S_i$ by requiring that $S_i \in w_i$ without loss of generality.) Consequently, the assertion in line (1) holds. Hence, for all $1 \le k \le d$, $x \in \mathcal{M}$ and $a \in w_k$ there exist $y_b \in \mathcal{M}$ such that $P_a x = \sum_{b\in w_k} y_b P_b$, see (1). Let $P = \sum_{a\in w_k} P_a$. Then $Px(I-P) = 0$, and consequently $(I-P)x^* P = 0$, for all $x \in \mathcal{M}$. Hence, $Px = xP$. □

The next lemma is the key lemma which encodes property (II). Actually we only use property (II) in the proof of this lemma.

LEMMA 4.2. *Let* $1 \le k \le d$ *and put*

$$\begin{aligned} q_k &= \mathrm{Alg}\{Q_a \in \mathbb{A} \mid a \in v_k\}, \\ C_k &= \mathrm{span}\{yz \in \mathbb{A} \mid y \in q_k,\ z \in W \textit{ is a zero-balanced word} \\ &\qquad\qquad \textit{containing no letter of } v_k^{\circledast}\}. \end{aligned}$$

Let $w_k \subseteq v_k$ *be a finite subset, let* $P = \sum_{a \in w_k} P_a$ *and let* $x \in C_k$. *Then* $(I - P)x = 0$ *implies* $x = 0$.

PROOF. So x has a representation $x = \sum_{j=1}^m Q_j x_j$ where $Q_j \in q_k$ and $x_j \in \mathbb{A}$ is a zero-balanced word containing just letters of $(\mathcal{A} \setminus v_k)^{\circledast}$. Recall that π is the $\mathbb{A}_{00}$-faithful saturating representation. We present

$$\pi(x_j) = \sum_{a,b,\mathrm{bal}(a)=\mathrm{bal}(b)=M_j} \lambda_{j,a,b}\pi(ab^*),$$

as in [6, Lemma 4.12], where $\lambda_{j,a,b}$ are scalars, $M_j \in \mathbb{Z}_+^d$, and a, b are words in the letters of $\mathcal{A} \setminus v_k$. It follows from the proof of [6, Lemma 4.12] that we may assume that $M_j = M$ for all j for some fixed $M \in \mathbb{Z}_+^d$. Now suppose that $(I - P)x = 0$. Then

$$\begin{aligned} \pi(I-P)\pi(x) &= \pi(I-P)\sum_{j=1}^m \pi(Q_j) \sum_{a,b,\mathrm{bal}(a)=\mathrm{bal}(b)=M} \lambda_{j,a,b}\pi(ab^*), \\ 0 &= \pi(I-P) \sum_{a,b,\mathrm{bal}(a)=\mathrm{bal}(b)=M} \pi(L_{a,b})\pi(ab^*), \end{aligned} \tag{3}$$

for $L_{a,b} = \sum_{j=1}^m \lambda_{j,a,b} Q_j \in q_k$. Fix a and b. By [6, Lemma 4.7] there exists a finite set $w_k' \subseteq v_k$ such that for $P' = \sum_{c \in w_k'} P_c$ we have $a^*(I - P) = (I - P')a^*$. By [6, Lemma 4.1] there exists $L'_{a,b} \in q_k$ such that $a^* L_{a,b} = L'_{a,b} a^*$. If we multiply (3) from the left by $\pi(a^*)$ and from the right by $\pi(b)$ then we obtain $0 = \pi(I - P')\pi(L'_{a,b})\pi(a^*ab^*b)$. Since π is injective on $\mathbb{A}$ by [6, Proposition 4.8] (and recall that π is supposed to be injective on $\mathbb{A}_{00}$) we get $0 = (I - P')L'_{a,b}a^*ab^*b$. By property (II) we obtain $a^*L_{a,b}ab^*b = 0$, and thus $\pi(L_{a,b}ab^*) = 0$. If we sum here over all a, b then we obtain $\pi(x) = 0$ (recall the representation (3)). Hence, $x = 0$ since π is faithful on $\mathbb{A}$. □

LEMMA 4.3. *We have* $B = \bigcup_{n \in \mathbb{Z}_+^d} B_n$.

PROOF. Let $x \in B = B^{(0)} = K_0(A_0)$. Then there exists a finite-dimensional C^*-algebra $\mathcal{M} \subseteq \mathbb{A} \subseteq A_0$ such that $x = K_0(j)(y)$ for $y \in K_0(\mathcal{M})$ and the identity embedding $j : \mathcal{M} \to A_0$. As explained in the proof of Proposition 3.2, the paper [6] relies on the paper [3], and the conditions of [3, Proposition 3.3] are satisfied. By [3, Proposition 3.3] and enlarging $\mathcal{M}$ if necessary, we may assume that $\mathcal{M}$ has the maximal abelian subalgebra $C = \mathcal{M} \cap \mathbb{A}_0$. Hence, $K_0(\mathcal{M})$ is generated by elements

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of the form $[p]$ where p runs through the minimal projections of C. We have a representation of p in $\mathbb{A}_0$ as $l_1 X_1 X_1^* + \cdots + l_m X_m X_m^*$, where X_i are words and $l_i \in \{+1, 0, -1\}$ (since $\mathbb{A}_0$ is a commutative algebra, see [6, Lemma 4.4]). Since y is a linear combination of such $[p]$ in $K_0(\mathcal{M})$, $K_0(j)(y)$ is a linear combination of such $K_0(j)([p]) = l_1[X_1 X_1^*] + \cdots + l_m[X_m X_m^*]$ in $K_0(A)$. So any $x \in K_0(A_0)$ is a linear combination of elements of the form $[XX^*]$ in $K_0(A_0)$. Each XX^* is of the form $XX^* = aQa^*$ for some $Q \in q$ and some (possibly empty) word a in the letters of $\mathcal{A}$ by [6, Lemmas 4.3 and 4.5]. Since $a^*a \in q$, we may assume that $Q = Qa^*a$. The proof is completed by observing that $[aQa^*] = [S^{\mathrm{bal}(a)} Q S^{\mathrm{bal}(a)*}]$, since $aQa^* = TT^*$ and $S^{\mathrm{bal}(a)} Q S^{\mathrm{bal}(a)*} = T^*T$ for $T = aQS^{\mathrm{bal}(a)*} \in A_0$. □

LEMMA 4.4. *For all $i = 1, \ldots, d$ and $n = (n_1, \ldots, n_d) \in \mathbb{Z}_+^d$ such that $n_i = 0$ we have that if $x \in B$ and $\phi_i(x) \in B_n$ then $x = 0$.*

PROOF. Let $\phi_i(x) = a \in B_n$. Then a is a $\mathbb{Z}$-linear combination of elements of the form $[\Psi_0(S^k Q S^{k*})] \in K_0(A_0)$ where $k_i = 0$ and $Q \in q$. By identifying $\overline{\mathbb{A}}$ and A_0 we may omit writing Ψ_0. By [6, Lemma 4.1] we can 'permute' all expressions Q_a, for $a \in v_i$, to the left in the expression $S^k Q S^{k*}$. Thus, we can achieve the identity

$$S^k Q S^{k*} = \sum_{l=1}^{m} Q'_l S^k Q''_l S^{k*} \in C_i,$$

for some $Q'_l \in q_i$ and $Q''_l \in q$ such that $S^k Q''_l S^{k*}$ does not contain a letter of $v_i^{\circledast}$, where we use the definition in Lemma 4.2 for q_i and C_i.

Hence, there exist projections $a_1, a_2 \in M_N(C_i) \subseteq M_N(A_0)$ such that $a = [a_1] - [a_2]$ in $K_0(A_0)$.

We choose a finite-dimensional C^*-algebra $\mathcal{M} \subseteq \mathbb{A}$ such that $C_i \subseteq \mathcal{M}$ and $x = [x_1] - [x_2]$ holds in $K_0(A_0)$ for some projections $x_1, x_2 \in M_N(\mathcal{M})$. We use the identity $M_N(\mathcal{M}) \cong M_N \otimes \mathcal{M}$ for notational purposes. We have $\phi_i(x) = [x'_1] - [x'_2]$ where $x'_k = (1_N \otimes S_i)x_k(1_N \otimes S_i^*)$.

We enlarge $\mathcal{M}$ such that $x'_1, x'_2 \in M_N(\mathcal{M})$ and the identity

$$[x'_1] - [x'_2] = [a_1] - [a_2], \tag{4}$$

holds in $K_0(\mathcal{M})$ (and consequently holds in $K_0(A_0)$). If necessary, we now enlarge $\mathcal{M}$ to have the form

$$\mathcal{M} = \mathrm{span}(F_{w_1,m} F_{w_2,m} \ldots F_{w_d,m}),$$

where $S_j \in w_j$ and $P = \sum_{a \in w_i} P_a$ is in the center of $\mathcal{M}$ by Lemma 4.1. Hence, $\mathcal{M}$ may be written as the direct sum $(I - P)\mathcal{M} \oplus P\mathcal{M}$ and

$$K_0(\mathcal{M}) \cong K_0((I - P)\mathcal{M}) \oplus K_0(P\mathcal{M}).$$

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If we use this decomposition in the identity (4) and consider the part $K_0((I-P)\mathcal{M})$, then we obtain

$$[a_1 1_N \otimes (I-P)] = [a_2 1_N \otimes (I-P)],$$

since $x'_k(1_N \otimes (I-P)) = 0$ since $S_i^*(I-P) = 0$. Thus, by the cancellation property of finite-dimensional C^*-algebras,

$$a_1(1_N \otimes (I-P)) = TT^* \sim T^*T = a_2(1_N \otimes (I-P)),$$

for some $T \in M_N((I-P)\mathcal{M})$, and $(a_1 - TT^*)(1_N \otimes (I-P)) = 0$.

Note that $(I-P)\mathcal{M} = (I-P)\mathcal{N}$ for the subalgebra

$$\mathcal{N} = \mathrm{span}(F_{w_1,m} \ldots F_{w_{i-1},m} F_{w_i,0} F_{w_{i-1},m} \ldots F_{w_d,m}),$$

of $\mathcal{M}$ (since P is in the center of $\mathcal{M}$ and commutes with the subalgebras $F_{w_j,m} \subseteq \mathcal{M}$, $I-P$ cancels the elements of $\bigcup_{1 \le r \le m} F_{w_i,r}$).

Hence, we may write T as $t(1_N \otimes (I-P))$ for some $t \in M_N(\mathcal{N})$. Observe that $\mathcal{N} \subseteq C_i$ for C_i of Lemma 4.2 (observe that $F_{w_i,0} \subseteq q_i$ and use the permutation rules). Then $a_1 - tt^* = 0$ by Lemma 4.2. Analogously we obtain $a_2 - t^*t = 0$. Hence, $[a_1] = [a_2]$ holds in $K_0(\mathcal{M})$ and thus also in $K_0(A_0)$. Therefore, $\phi_i(x) = a = 0$ and, thus, $x = 0$ by Lemma 3.1. □

LEMMA 4.5. *Let $j : \overline{q} \to \overline{\mathbb{A}}$ be the identity embedding where $\overline{q}$ denotes the C^*-norm closure of q. Then $K_0(j)$ is injective.*

PROOF. The norm closure $\overline{q}$ of $q = \mathrm{Alg}\{Q_a \in \mathbb{A} \mid a \in \mathcal{A}\}$ may be regarded as the direct limit of the finite-dimensional C^*-subalgebras of q. Thus, if $x \in K_0(\overline{q})$, then $x = [q_1] - [q_2]$ for some projections $q_i \in M_N(q)$. Assume that $0 = j(x) = [q_1] - [q_2]$ in $K_0(\overline{\mathbb{A}})$. Then $[q_1] - [q_2] = 0$ in $K_0(\mathcal{M})$ for some finite-dimensional C^*-subalgebra $\mathcal{M}$ of $\mathbb{A}$. We enlarge $\mathcal{M}$ such that $\mathcal{M} = \mathrm{span}\, F_{w_1,m} \ldots F_{w_d,m}$ and $P^{(i)} = \sum_{a \in w_i} P_a$ is in the center of $\mathcal{M}$ by Lemma 4.1. Write $P = (I - P^{(1)}) \ldots (I - P^{(d)})$ (which is in the abelian algebra $\mathbb{A}_{00}$). Then $\mathcal{M} = P\mathcal{M} \oplus (I-P)\mathcal{M}$ and $K_0(\mathcal{M}) = K_0(P\mathcal{M}) \oplus K_0((I-P)\mathcal{M})$. Thus, $[(1_N \otimes P)q_1] - [(1_N \otimes P)q_2] = 0$ in $K_0(P\mathcal{M})$. By the cancellation property of finite-dimensional C^*-algebras, there exists $T \in M_N(P\mathcal{M})$ such that

$$(1_N \otimes P)q_1 = TT^* \sim T^*T = (1_N \otimes P)q_2.$$

Note that $(I - P^{(i)})F_{w_i,m} = (I - P^{(i)})F_{w_i,0}$ where $1 \le i \le d$. Hence, $P\mathcal{M} = P\mathcal{N}$ for the subalgebra $\mathcal{N} = \mathrm{span}\, F_{w_1,0} \ldots F_{w_d,0}$ of $\mathcal{M}$. Thus, we may write $T = (1_N \otimes P)t$ where $t \in M_N(\mathcal{N})$. Now we have

$$(q_1 - tt^*)(1_N \otimes (I - P^{(1)}) \ldots (I - P^{(d-1)})(I - P^{(d)})) = 0.$$

Put

$$y = (q_1 - tt^*)(1_N \otimes (I - P^{(1)}) \ldots (I - P^{(d-1)})).$$

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Let y_{ij} denote the matrix entries of y. By using the permutation rules or [6, Lemma 4.1], 'permute' each expression Q_a, for $a \in v_d$, in y_{ij} to the left such that it becomes evident that y_{ij} is an element of the set C_d defined in Lemma 4.2. Hence, $y_{ij} = 0$ by Lemma 4.2 (for $k = d$) and so $y = 0$. Successively continuing this argument we end by showing that $q_1 - tt^* = 0$. Similarly $q_2 - t^* t = 0$. Consequently $[q_1] = [q_2]$ in $K_0(\overline{q})$ since $t \in M_N(q)$. □

5. Proof part 3

In this section we prove our main result, Theorem 5.6, by an application of Lemma 3.4. For $1 \le i \le d$ we put

$$\begin{aligned} f_i &= \phi_i - \mathrm{id} \in \mathrm{End}(K_0(A)),\\ Y_i &= \mathrm{Range}(f_1) + \cdots + \mathrm{Range}(f_i) \subseteq K_0(A), \end{aligned}$$

and $Y_0 = 0$. Note that $f_1, \ldots, f_d, \phi_1, \ldots, \phi_d$ commute. We write ϕ^n for $\phi_1^{n_1} \ldots \phi_d^{n_d}$ where $n = (n_1, \ldots, n_d) \in \mathbb{Z}_+^d$.

Recall that in the representation $A = \overline{\bigcup_{m \in \mathbb{Z}^d} A_m}$, the element $\Psi_0 x \in A_0$ is identified with the element $\alpha_n(\Psi_n x) \in A_n$ for $x \in \mathbb{A}$, $n \in \mathbb{Z}_+^d$. Hence, in the representation $K_0(A) \cong \bigcup_{m \in \mathbb{Z}^d} K_0(A_m)$, the element $K_0(\Psi_0)[x] \in K_0(A_0)$, where $[x] \in K_0(\mathbb{A})$, is identified with $\phi^n(K_0(\Psi_n)[x]) \in K_0(A_n)$.

Lemma 5.1. *If $x \in B$ and $\phi_i(x) \in B_n$ when $1 \le i \le d$ and $n \in \mathbb{Z}_+^d$, then $x \in B_{n-\delta_i}$.*

Proof. Let $z \in B_n$. By the definition of B_n and the identities

$$[\Psi_0(S_i S^l Q S^{l^*} S_i{}^*)] = \phi_i([\Psi_0(S^l Q S^{l^*})]),$$

for $l \in \mathbb{Z}_+^d$ and $Q \in q$, we have a representation $z = \phi_i(y) + a$ for some $y \in B_{n-\delta_i}$ and $a \in B_m$ where $m = (n_1, \ldots, n_{i-1}, 0, n_{i+1}, \ldots, n_d) \in \mathbb{Z}^d$. Applying this to $z = \phi_i(x)$ we obtain $\phi_i(x) = \phi_i(y) + a$. Then by Lemma 4.4, $x - y = 0$, and thus $x = y \in B_{n-\delta_i}$. □

Note that for fixed j in $\{1, \ldots, d\}$ each element x in B can be uniquely decomposed into $x = \phi_j(a) + r$ for $a, r \in B$ such that $r \in B_n$ for some $n = (n_1, \ldots, n_{j-1}, 0, n_{j+1}, \ldots, n_d) \in \mathbb{Z}^d$. The existence of a and r is clear by Lemma 4.3 (and the proof of Lemma 5.1), and the uniqueness of $a, r \in B$ follows from Lemma 5.1. We refer to this decomposition as the *ϕ_j-decomposition* in B. We make the ϕ_j-decomposition in $K_0(A_N)$ rather than in $K_0(A_0)$. Elements of the set B_n (or $B_n^{(N)}$) for $n = (n_1, \ldots, n_d) \in \mathbb{Z}_+^d$ are said to have *j-degree n_j* in B (or $B^{(N)}$). Note, however, that the j-degree is not a *unique* number.

Lemma 5.2. *If $x \in B$ and $f_i(x) \in B_n$ when $1 \le i \le d$ and $n \in \mathbb{Z}_+^d$, then $x \in B_{n-\delta_i}$.*

PROOF. Let $x \in B$ and $f_i(x) \in B_n$, but suppose that $x \notin B_{n-\delta_i}$. By Lemma 4.3 choose $m \in \mathbb{Z}_+^d$ such that $m \geq n$ and $x \in B_m \setminus B_{m-\delta_i}$. Since $f_i(x) = \phi_i(x) - x \in B_n \subseteq B_m$ and $x \in B_m$, we have $\phi_i(x) \in B_m$. By Lemma 5.1, $x \in B_{m-\delta_i}$, which is a contradiction. □

Analogously to the ϕ_j-decomposition we get a unique f_j*-decomposition* in B. More precisely, for all $x \in B$ there exist unique $a, r \in B$ such that $x = f_j(a) + r$ where r has j-degree zero. The existence can be seen as follows. Say $x = [\Psi_0(S^n Q S^{n*})] \in K_0(A_0)$ for $n \in \mathbb{Z}_+^d$ ($n_j \geq 1$) and $q \in Q$ by Lemma 4.3. Then

$$[\Psi_0(S^n Q S^{n*})] = f_j([\Psi_0(S^{n-\delta_j} Q S^{n-\delta_j *})]) + [\Psi_0(S^{n-\delta_j} Q S^{n-\delta_j *})].$$

In the same way we further decompose $[\Psi_0(S^{n-\delta_j} Q S^{n-\delta_j *})]$, and so on, until we end at $x = f_j(a) + r$. The uniqueness of a and r follows from Lemma 5.2.

Note that if $x \in B_n^{(N)}$ for $n_k = 0$, that is, x has k-degree zero in $B^{(N)}$, and if $i \neq k$, then $f_i(x) \in B_{n+\delta_i}^{(N)}$, that is, $f_i(x)$ has k-degree zero in $B^{(N)}$.

LEMMA 5.3. *We have* $f_i^{-1}(Y_{i-1}) \subseteq Y_{i-1}$ *for all* i *in* $\{1, \dots, d\}$.

PROOF. Take $y \in f_i^{-1}(Y_{i-1})$, where $1 \leq i \leq d$. Then there exist $x_1, \dots, x_{i-1} \in K_0(A)$ such that $f_i(y) = f_1(x_1) + \cdots + f_{i-1}(x_{i-1})$. We may suppose that $y, x_1, \dots, x_{i-1} \in K_0(A_N)$ for some $N \in \mathbb{Z}^d$. Using the f_i-decomposition in $B^{(N)} = K_0(A_N)$ we may write $x_k = f_i(a_k) + r_k$ for some $a_k, r_k \in K_0(A_N)$ such that r_k has i-degree zero in $K_0(A_N)$. Hence,

$$f_i(y - f_1(a_1) - \cdots - f_{i-1}(a_{i-1})) = f_1(r_1) + \cdots + f_{i-1}(r_{i-1}).$$

The right-hand side of this equality has i-degree zero and thus

$$y - f_1(a_1) - \cdots - f_{i-1}(a_{i-1}) = 0,$$

by Lemma 5.2, and we obtain $y \in Y_{i-1}$ as claimed. □

LEMMA 5.4. *We have* $Y_d \cap B_0 = 0$.

PROOF. We want to prove the lemma by induction, so assume that $k \in \{1, 2, \dots, d\}$ and that $Y_{k-1} \cap B_0 = 0$. Now let $z \in Y_k \cap B_0$. Then $z = f_1(x_1) + \cdots + f_k(x_k) \in B_0$ for some $x_i \in K_0(A_N)$ and $N \geq 0$. Let $x_i = f_k(a_i) + r_i$ be the f_k-decomposition of x_i in $B^{(N)} = K_0(A_N)$ for $i = 1, \dots, k-1$. Then, since $z \in B_0$, for some $y \in B_0^{(N)}$ we have

$$z = f_k\left(\sum_{i=1}^{k-1} f_i(a_i) + x_k\right) + \sum_{i=1}^{k-1} f_i(r_i) = \phi^N(y) \in B_0. \tag{5}$$

Case 1. First we suppose that the kth coordinate N_k of N is zero. Then, since $f_1(r_1) + \cdots + f_{k-1}(r_{k-1})$ and $\phi^N(y)$ have k-degree zero in $B^{(N)}$, the summand

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$f_k(\ldots)$ in (5) has k-degree zero in $B^{(N)}$ and must thus vanish by Lemma 5.2. Hence, $z = f_1(r_1) + \cdots + f_{k-1}(r_{k-1}) \in Y_{k-1} \cap B_0$, and by the induction hypothesis, $z = 0$.
Case 2. If $N_k > 0$ then $\phi^N(y) = \phi_k^{N_k}(w)$ for $w = \phi^{N - N_k\delta_k}(y)$. Now

$$\begin{aligned} \phi_k^{N_k}(w) &= f_k(\phi_k^{N_k-1}(w)) + \phi_k^{N_k-1}(w) \\ &= f_k(\phi_k^{N_k-1}(w)) + f_k(\phi_k^{N_k-2}(w)) + \cdots + f_k(w) + w. \end{aligned}$$

If we substitute this in identity (5) and isolate w then we obtain

$$z' = f_k(\ldots) + f_1(r_1) + \cdots + f_{k-1}(r_{k-1}) = w = \phi^{N-N_k\delta_k}(y) \in B_0^{(N_k\delta_k)}.$$

By using the argument of Case 1 we obtain $z' = 0$. (We remark that by the symmetric configuration of A we also have $Y_{k-1} \cap B_0^{(L)} = 0$ for all $L \in \mathbb{Z}^d$. This may be deduced from $Y_{k-1} \cap B_0 = 0$ by using the bijections ϕ_i (Lemma 3.1).) Hence, $z = \phi^N(y) = \phi^{N_k\delta_k}(z') = 0$. $\square$

Corollary 5.5. *We have $K_0(A \rtimes \mathbb{Z}^d) \cong B_0$ and $K_1(A \rtimes \mathbb{Z}^d) = 0$.*

Proof. Since A is the inductively ordered union of finite-dimensional C^*-algebras, we have $K_1(A) = 0$. Combining Lemmas 3.4 and 5.3, we obtain that $K_1(A \rtimes \mathbb{Z}^d)$ is trivial and $K_0(A \rtimes \mathbb{Z}^d) = K_0(A)/Y_d$. Now let $x \in K_0(A)$. Using the fact that $\phi_i(x) \equiv x \mod Y_d$ for all $i = 1, \ldots, d$, for each $x \in K_0(A)$ we find some $y \in B_0$ such that $x \equiv y \mod Y_d$ by Lemma 4.3. Hence, the quotient map $B_0 \to K_0(A)/Y_d$ such that $x \mapsto x + Y_d$, is a surjection. This yields $B_0 \cong B_0/(Y_d \cap B_0) \cong K_0(A)/Y_d$ by Lemma 5.4. $\square$

We denote by $\mathrm{Ring}(M)$ the subring generated by a subset M of a ring R.

Theorem 5.6. *Suppose that $(\mathcal{A}, \mathbb{F}, \mathbb{I})$ induces a higher rank Exel–Laca algebra (Definition 2.1) and satisfies the properties (I) and (II). Let φ be the continuous extension of the embedding $\pi|_q : q \to \mathcal{O}_{\mathcal{A},\mathbb{F},\mathbb{I}}$.*

Then $K_0(\varphi)$ is an isomorphism and $K_1(\mathcal{O}_{\mathcal{A},\mathbb{F},\mathbb{I}}) = 0$. Hence, we have an isomorphism of abelian groups

$$K_0(\mathcal{O}_{\mathcal{A},\mathbb{F},\mathbb{I}}) \cong \mathrm{Ring}\{Q_a \in \mathbb{A} \mid a \in \mathcal{A}\}.$$

Proof. Let $t : \overline{q} \to A$ be the continuous embedding mapping $x \in q \subseteq \mathbb{A}$ to $\Psi_0(x) \in A_0$. Collecting various results in this paper we obtain the following diagram:

$$\begin{tikzcd} K_0(\overline{q}) \arrow[r, "K_0(t)"] & B_0 \arrow[r, "x\mapsto x+Y_d"] & K_0(A)/Y_d \arrow[r, "{[a]+Y_d\mapsto[a]}"] & K_0(A \rtimes \mathbb{Z}^d) \\ & & K_0(\mathcal{O}_{\mathcal{A},\mathbb{F},\mathbb{I}}) & K_0(P(A \rtimes \mathbb{Z}^d)P) \arrow[l, "K_0(\kappa)"'] \arrow[u, "K_0(\tau)"] \end{tikzcd}$$

Starting at the top-left corner, the first map $K_0(t)$ is an isomorphism by Lemma 4.5. The second map is an isomorphism by the proof of Corollary 5.5. The third map is an

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isomorphism by the proof of Lemma 3.4. The fourth map $K_0(\tau)$ is an isomorphism by Lemma 3.3. The fifth map $K_0(\kappa)$ is an isomorphism by Proposition 3.2.

This sequence of isomorphisms maps an element $[a] \in K_0(\overline{q})$ to $[\pi(a)] \in K_0(\mathcal{O}_{\mathcal{A},\mathbb{F},\mathbb{I}})$ for a projection $a \in M_N(q)$. Hence, $K_0(\varphi)$ is an isomorphism. The isomorphism between $K_0(\overline{q})$ and $\mathrm{Ring}\{Q_a \in \mathbb{A} \mid a \in \mathcal{A}\}$ is well known. By Proposition 3.2, Lemmas 3.3, 3.4, and 5.3, $K_1(\mathcal{O}_{\mathcal{A},\mathbb{I},\mathbb{F}})$ is zero. □

If the 'rank', that is, $\mathrm{card}(V)$, is arbitrary then we may write $\mathcal{O}_{\mathcal{A},\mathbb{F},\mathbb{I}}$ as the direct limit of higher rank Exel–Laca (sub)algebras, where V is restricted to finite subsets. In this way the last theorem yields the following result.

COROLLARY 5.7. *The last theorem holds for any rank.*

6. Examples

EXAMPLE 6.1. Let $\mathcal{O}_A$ be an Exel–Laca algebra [13], where A denotes the transition matrix. By [14], $K_0(\mathcal{O}_A)$ may be regarded as the quotient $R/_{\equiv}$ where R is the ring $R \subseteq \mathbb{A}$ generated by Q_a, P_b for all $a, b \in \mathcal{A}$, now regarded as an abelian group, and $\equiv$ is the equivalence relation $Q_a \equiv P_a$ for all $a \in \mathcal{A}$. It is clear that the quotient map

$$\varphi : R' = \mathrm{Ring}\{Q_a \in \mathbb{A} \mid a \in \mathcal{A}\} \to R/_{\equiv},$$

is surjective. Using property (II) it is easy to compute that φ is also injective. Hence, $R' \cong R/_{\equiv}$. Moreover, the group

$$K_1(\mathcal{O}_A) \cong \ker(I - A^t) = \left\{ (x_a) \in \bigoplus_{a\in\mathcal{A}} \mathbb{Z} \,\middle|\, \sum_{a\in\mathcal{A}} x_a(P_a - Q_a) = 0 \right\},$$

(by [14]) is zero if property (II) holds. So, the K-theory result of Theorem 5.6 is consistent with the K-theory result of Exel and Laca, as it should be. For the Cuntz-algebra $\mathcal{O}_\infty$ we get $K_0(\mathcal{O}_\infty) \cong \mathrm{Ring}\{I\} = \mathbb{Z}$ and $K_1(\mathcal{O}_\infty) = 0$.

EXAMPLE 6.2. In this example we consider the rank-two Cuntz–Krieger algebras defined in [5, Section 5]. These algebras are inspired by one-sided shifts J of finite type in dimension two. More precisely, let Ω be a finite set and let s be a function $s : \Omega^4 \to \{0, 1\}$. Then let J be the shift space

$$\begin{aligned} J = \{x \in \Omega^{\mathbb{N}^2} \mid\ & s(x_{n,m}, x_{n,m+1}, x_{n+1,m}, x_{n+1,m+1}) = 1 \\ & \text{for all but finitely many pairs } (n, m) \in \mathbb{N}^2\}. \end{aligned}$$

In other words, J is a one-sided shift of finite type with the modification that finitely many failures with respect to the 'test function' s are allowed. Let $\mathcal{A}$ be the

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alphabet $v_1 \sqcup v_2$ with the partition V given by $\{v_1, v_2\}$ and parts $v_1 = v_2 = \Omega^{\mathbb{N}}$. Let $a = (a_1, a_2, a_3, \ldots) \in v_1$ ($a_j \in \Omega$), $x \in J$ and put

$$y = \begin{pmatrix} x \\ a_1 a_2 a_3 \ldots \end{pmatrix} \in \Omega^{\mathbb{N}^2}.$$

(y arises by shifting x one step upwards and filling the arising blank line with $a_1 a_2 a_3 \ldots$.) Then we define a partial isometry $S_a \in B(\ell^2(J))$ by $S_a(\delta_x) = \delta_y$ if $y \in J$ and $S_a(\delta_x) = 0$ otherwise. We similarly induce a partial isometry T_b for $b \in v_2$ by shifting x to the right (rather than upwards) and filling the arising gap with b.

Then one can show that the C^*-algebra generated by $\{S_a \mid a \in v_1\} \cup \{T_b \mid b \in v_2\}$ in $B(\ell^2(J))$ is a rank-two Exel–Laca algebra $\mathcal{O}_{\mathcal{A},\mathbb{F},\mathbb{I}}$ associated to a triple $(\mathcal{A}, \mathbb{F}, \mathbb{I})$ satisfying Definition 2.1. The representation $\pi : \mathbb{F}/\mathbb{I} \to \mathcal{O}_{\mathcal{A},\mathbb{F},\mathbb{I}}$ such that $\pi(a) = S_a$ and $\pi(b) = T_b$ for all $a \in v_1$ and $b \in v_2$ is an $\mathbb{A}_{00}$-faithful saturating representation with dense image (see [5]).

Next we prove property (II). If $x \in q = \mathrm{Alg}\{Q_a \in \mathbb{A} \mid a \in \mathcal{A}\}$ is non-zero then there exist $a_i \in \mathcal{A}$ and $\varepsilon_i \in \{1, \perp\}$ such that the carrier of x is larger than $Q = Q_{a_1}^{\varepsilon_1} \ldots Q_{a_m}^{\varepsilon_m} \neq 0$ where $Q_{a_i}^{\perp} = I - Q_{a_i}$. There exist $b_i \in v_i$ such that $0 \neq P_{b_1} P_{b_2} \leq Q$ (by using the saturating representation π as in (2)). However, in fact there exist infinitely many modifications $b_1' \in v_1$ of b_1 such that $0 \neq P_{b_1'} P_{b_2} \leq Q$. The reason is that since we allow finitely many failures in J with respect to the test function s, we may choose infinitely many modifications b_1' of b_1 by modifying $b_1 \in \Omega^{\mathbb{N}}$ at single entries. We skip the details. Consequently, for any finite subset $\mathcal{B} \subseteq v_1$, $P = \sum_{c \in \mathcal{B}} P_c$ cannot be a unit for Q, and thus it cannot be a unit for x. This proves property (II).

If we fix some $z \in \Omega$ and suppose that

$$s \begin{pmatrix} x & y \\ z & z \end{pmatrix} = 1 \quad \text{and} \quad s \begin{pmatrix} z & x \\ z & y \end{pmatrix} = 1,$$

for all $x, y \in \Omega$, then $S_{(z,z,z,\ldots)}$ and $T_{(z,z,z,\ldots)}$ define commuting isometries as required in property (I). Hence, we can use Theorem 5.6 to obtain the K-theory of $\mathcal{O}_{\mathcal{A},\mathbb{F},\mathbb{I}}$.

For example, consider the full shift $J = \Omega^{\mathbb{N}^2}$. Then all operators S_a, T_b are isometries and we obtain $K_0(\mathcal{O}_{\mathcal{A},\mathbb{F},\mathbb{J}}) \cong \mathrm{Ring}\{I\} = \mathbb{Z}$.

Another example is this. Let $\Omega = \Omega_{\mathrm{red}} \sqcup \{z\} \sqcup \Omega_{\mathrm{green}}$ be a disjoint union; the red letters, the blue letter z, and the green letters. Let

$$s \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = 1,$$

if and only if all $x_{ij} \in \Omega_{\mathrm{red}}$, or if all $x_{ij} \in \Omega_{\mathrm{green}}$, or if $x_{21} = x_{22} = z$ and x_{11}, x_{12} are arbitrary, or if $x_{11} = x_{21} = z$ and x_{11}, x_{12} are arbitrary.

Let $a = (a_1 a_2 \ldots) \in \Omega^{\mathbb{N}} = v_1$. Note that $\pi(Q_a)$ is the projection onto the Hilbert space generated by $\{\delta_x \in \ell^2(J) \mid x \in J,\ S_a(\delta_x) \neq 0\}$. Thus, $Q_a \neq 0$ if and only if there exists $n_0 \in \mathbb{N}$ such that $a_n \in \Omega_{\mathrm{red}}$ for all $n \geq n_0$, or $a_n \in \Omega_{\mathrm{green}}$ for all $n \geq n_0$, or $a_n = z$

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for all $n \geq n_0$. We say that a is *red*, *green* or *blue*, respectively. Note that $Q_a = Q_b \neq 0$ for $a, b \in v_i$ for fixed $i = 1, 2$ if and only if the color of a coincides with the color of b. Hence, there exist only five different source projections Q_x for $x \in \mathcal{A}$. Namely an (blue) isometry I, and Q_r, Q_g, Q_R, Q_G where $r \in v_1$, $R \in v_2$ are any red letters, and $g \in v_1$, $G \in v_2$ are any green letters. Note that $Q_r \neq Q_R$ and $Q_g \neq Q_G$, and that $Q_x Q_y = 0$ when x is red and y is green. Hence, by Theorem 5.6 we have an isomorphism as abelian groups

$$\begin{aligned} K_0(\mathcal{O}_{\mathcal{A},\mathbb{F},\mathbb{I}}) &\cong \mathrm{Ring}\{Q_x \mid x \in \mathcal{A}\} \\ &= \mathbb{Z}(Q_r - Q_r Q_R) \oplus \mathbb{Z} Q_r Q_R \oplus \ldots = \mathbb{Z}^7. \end{aligned}$$

We emphasize that in this example $K_0(\mathcal{O}_{\mathcal{A},\mathbb{F},\mathbb{I}})$ is different from

$$\mathrm{Ring}\{Q_x \mid x \in v_1\} \otimes \mathrm{Ring}\{Q_y \mid y \in v_2\} = \mathbb{Z}^3 \otimes \mathbb{Z}^3 = \mathbb{Z}^9.$$

On the other hand, we have

$$K_0(C^*((S_a)_{a \in v_1}) \otimes C^*((T_b)_{b \in v_2})) = \mathbb{Z}^9,$$

by Theorem 5.6. This shows that $\mathcal{O}_{\mathcal{A},\mathbb{F},\mathbb{I}}$ is different from the tensor product $C^*((S_a)_a) \otimes C^*((T_b)_b)$. This is not very surprising since $S_a T_b = 0$ in $\mathcal{O}_{\mathcal{A},\mathbb{F},\mathbb{I}}$ for certain a and b.

COROLLARY 6.3. *The rank-two Cuntz–Krieger algebras associated to shifts of finite type* [5] *satisfy Definition 2.1 and property (II). Hence, Theorem 5.6 gives their K-theory if they also satisfy property (I).*

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