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CYCLIC COHOMOLOGY OF NON-COMMUTATIVE TORI

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1. Introduction. In this paper we shall compute the cyclic cohomology of a non-commutative torus, i.e., a certain algebra $\mathscr{A}$ associated with an antisymmetric bicharacter of a finite rank free abelian group G.

The main result is

$$(1.1) \quad H^n_\lambda(\mathscr{A}) = \operatorname{Ker} S \dot{+} \operatorname{Im} S \dot{+} V_n,$$

where

$$V_n \cong \Lambda^n(G \otimes \mathbf{C}).$$

The method of computation generalises the computation of the cyclic cohomology of the irrational rotation algebras given by Connes in [**3**]. (Our method works equally well also in the rational case, which was dealt with by a different method by Connes in [**3**].)

We first describe the Hochschild cohomology of $\mathscr{A}$ in an explicit way, and then combine this description with the exact sequence of [**3**]:

$$(1.2) \quad \dots \xrightarrow{S} H^n_\lambda(\mathscr{A}) \xrightarrow{I} H^n(\mathscr{A}, \mathscr{A}^*) \xrightarrow{B} H^{n-1}_\lambda(\mathscr{A}) \xrightarrow{S} \dots.$$

It turns out that the homology of the complex

$$(H^*(\mathscr{A}, \mathscr{A}^*), IB)$$

is isomorphic to the exterior algebra $\Lambda G \otimes \mathbf{C}$, with its natural grading. In the course of our computations we construct certain canonical representatives for the homology classes of this complex. These classes turn out to be the images under I of cyclic cocycles which survive under successive applications of S.

The main result follows from these facts.

It follows immediately from (1.1) that the periodic cyclic cohomology of $\mathscr{A}$ is given by

$$(1.3) \quad H(\mathscr{A}) \cong \Lambda G \otimes \mathbf{C}.$$

This result can also be obtained from the calculation of the periodic cyclic cohomology of arbitrary $\mathbf{Z}$-crossed products given in [**8**], a sequel to the present article. No analogue of (1.1) itself, however, appears in [**8**].

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$\mathscr{A}$, as defined below, is a dense subalgebra of the universal C^*-algebra A_ρ associated with G and an antisymmetric bicharacter ρ. As is easily seen by direct computation, $\mathscr{A}$ is closed under the holomorphic functional calculus inside A_ρ, and so, by [3], there is a natural coupling of $H^*_\lambda(\mathscr{A})$ to the K-theory of A_ρ. Combination of (1.3) with the description of the Chern character on $K_*(A_\rho)$ given by Elliott in [7] gives that after periodisation the coupling is nondegenerate, i.e.,

$$(K(A_\rho) \otimes \mathbf{C})^* \cong H(\mathscr{A}),$$

and allows one to go a long way towards the classification of non-commutative tori [6, 5, 2].

2. Notation. $\mathscr{A}$ is a Fréchet *-algebra defined as follows:

1. As a topological vector space $\mathscr{A}$ is the space $\mathscr{S}(\mathbf{Z}^N)$ of rapidly decreasing sequences indexed by $\mathbf{Z}^N$. We use the standard topology on $\mathscr{S}(\mathbf{Z}^N)$ given by the seminorms

$$\| (x_\alpha)_{\alpha \in \mathbf{Z}^N} \|_K = \sup_{\alpha \in \mathbf{Z}^N} (1 + |\alpha|^K)|x_\alpha|.$$

2. As a topological *-algebra, $\mathscr{A}$ is generated by unitaries $v_1, \ldots, v_N$, satisfying the commutation relations

$$(2.1) \quad v_i v_j = \lambda_{ij} v_j v_i, \quad i, j = 1, \ldots, N,$$

for a fixed family of scalars $\lambda_{ij} \in \mathbf{T}$ such that $\bar{\lambda}_{ij} = \lambda_{ji}$.

3. The correspondence between the two pictures is given by the map

$$\mathscr{S}(\mathbf{Z}^N) \to \mathscr{A}$$

$$(x_\alpha) \mapsto \sum_\alpha x_\alpha v^\alpha,$$

where we set

$$v^\alpha = v_1^{\alpha_1} \ldots v_N^{\alpha_N}, \quad v_i^0 = 1,$$

and note that the sum on the right hand side is convergent.

$\mathscr{A}$ has a canonical trace τ given by

$$\tau(1) = 1,$$

$$\tau(v^\alpha) = 0, \quad \alpha \neq (0, \ldots, 0),$$

and the map

$$\mathscr{S}^*(\mathbf{Z}^N) \to \mathscr{A}^*$$

$$(\phi_\alpha) \mapsto \sum_\alpha \phi_\alpha \tau(v^\alpha)$$

is a topological isomorphism from the space of tempered sequences $\mathcal{S}^*(\mathbf{Z}^N)$ to the dual $\mathcal{A}^*$ of $\mathcal{A}$.

We denote by $\mathcal{A}^e$ the enveloping algebra of $\mathcal{A}$; that is,

$$\mathcal{A}^e = \mathcal{A} \otimes \mathcal{A}^{\mathrm{op}},$$

where $\mathcal{A}^{\mathrm{op}}$ denotes the opposite algebra of $\mathcal{A}$.

The tensor products of locally convex vector spaces considered in this paper will always be the complete projective tensor products.

The elements of $\mathcal{A}^{\mathrm{op}}$ will be denoted by

$$\mathring{x}, \ x \in \mathcal{A},$$

and we shall write $x\mathring{y}(= \mathring{y}x)$ for $x \otimes \mathring{y} \in \mathcal{A}^e$.

The augmentation map ϵ is the linear map

$$\epsilon:\mathcal{A}^e \rightarrow \mathcal{A}$$

$$x\mathring{y} \mapsto xy.$$

We set

$$V = \mathbf{C}^N,$$

with the standard orthonormal basis $e_1, \ldots, e_N$ fixed. For $I = (i_1, \ldots, i_n)$, set

$$e_I = e_{i_1} \wedge \ldots \wedge e_{i_n}.$$

The exterior algebra ΛV will always be considered with its standard Euclidean structure given by

$$\|e_I\| = 1, \quad I = (i_1, \ldots, i_n) \quad \text{with } i_k \neq i_l \text{ for } k \neq l.$$

We shall denote by T_x for $x \in V$ the linear map

$$\Lambda V \rightarrow \Lambda V$$

$$w \mapsto x \wedge w.$$

T_x^* will denote the adjoint of T_x.

We shall use the following abbreviations:

$$v^\alpha|_{\leqq k} = v^\alpha|_{<k+1} = v_1^{\alpha_1} \ldots v_k^{\alpha_k},$$

$$v^\alpha|_{\geqq k} = v^\alpha|_{>k-1} = v_k^{\alpha_k} \ldots v_N^{\alpha_N}.$$

3. Projective resolutions. The standard projective resolution of $\mathcal{A}$ (see [3]) is given by the complex of $\mathcal{A}^e$-modules

$$\ldots \xrightarrow{b} \Lambda_n(b) \xrightarrow{b} \Lambda_{n-1}(\mathcal{A}) \rightarrow \ldots \xrightarrow{b} \mathcal{A}^e \xrightarrow{\epsilon} \mathcal{A},$$

where

$$\Lambda_n(\mathscr{A}) = \mathscr{A}^e \otimes \mathscr{A}^{\otimes n},$$

and

$$b:\Lambda_n(\mathscr{A}) \to \Lambda_{n-1}(\mathscr{A})$$

$$a_0 \otimes a_1 \otimes \ldots \otimes a_n \mapsto \sum_{i=0}^{n-1} (-1)^i a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_n$$

$$+ (-1)^n \mathring{a}_n a_0 \otimes a_1 \otimes \ldots \otimes a_{n-1}.$$

In what follows we will construct a finite length projective resolution of $\mathscr{A}$,

$$0 \to E_N \xrightarrow{d} E_{N-1} \xrightarrow{d} \ldots \xrightarrow{d} E_1 \xrightarrow{d} \mathscr{A}^e \xrightarrow{\epsilon} \mathscr{A},$$

and a comparison map

$$k:(\Lambda^*(\mathscr{A}), b) \to (E^*, d),$$

i.e., a degree zero map of complexes of $\mathscr{A}^e$-modules such that

$$k_0:\mathscr{A}^e \to \mathscr{A}^e$$

is the identity map.

To begin with, set

(3.1) $E_n = \mathscr{A}^e \otimes \Lambda^n V,$

and consider the $\mathscr{A}^e$-module map

(3.2) $h:E_n \to \Lambda_n(\mathscr{A})$

$$1 \otimes e_{i_1} \wedge \ldots \wedge e_{i_n}$$

$$\mapsto \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)(v_{\sigma(i_1)} \ldots v_{\sigma(i_n)})^{-1} \otimes v_{\sigma(i_1)} \otimes \ldots \otimes v_{\sigma(i_n)}.$$

Consider also the $\mathscr{A}^e$-module map

(3.3) $d:E_n \to E_{n-1}$

$$1 \otimes e_{i_1} \wedge \ldots \wedge e_{i_n}$$

$$\mapsto \sum_{k=1}^{n} (-1)^k (1 - \mathring{v}_{i_k} v_{i_k}^{-1}) \otimes e_{i_1} \wedge \ldots \wedge \hat{e}_{i_k} \wedge \ldots \wedge e_{i_n}.$$

It is easy to check that d and h are continuous, and a straightforward computation gives

LEMMA 3.1. *$bh = hd$.*

To facilitate the following computations, we will interpret the $\mathscr{A}^e$-module structure of E_n as an $\mathscr{A}$-bimodule structure:

$$E_n \cong \mathscr{A} \otimes \Lambda^n V \otimes \mathscr{A}.$$

We give the direct sum $E = \oplus_n E_n$ a graded associative $\mathscr{A}$-bialgebra structure by defining a multiplication, denoted by $\wedge$, as follows:

$$(1 \otimes x \otimes v^\alpha) \wedge (v^\beta \otimes y \otimes 1) = v^\alpha v^\beta (v^\alpha)^{-1} \otimes x \wedge y \otimes v^\alpha.$$

Write

$$\sum_{i=0}^{n}{}' = \begin{cases} \sum_{i=0}^{n} & \text{for } n \geqq 0, \\ 0 & \text{for } n = -1, \\ -\sum_{i=n-1}^{-1} & \text{for } n < -1, \end{cases}$$

and define $\mathscr{A}^e$-module maps

$$\rho_i : \Lambda_1(\mathscr{A}) \to E_1,$$
$$k : \Lambda_n(\mathscr{A}) \to E_n,$$

by the formulas

$$(3.4)\quad \rho_i((v^\alpha)^{-1} \otimes v^\alpha) = (v^\alpha|_{>i})^{-1} \left(\sum_{k=0}^{\alpha_i - 1}{}' v_i^{-k} \otimes e_i \otimes v_i^k \right) (v^\alpha|_{>i}),$$

$$(3.5)\quad k((v^{\alpha_1} \dots v^{\alpha_n})^{-1} \otimes v^{\alpha_1} \otimes \dots \otimes v^{\alpha_n})$$
$$= \sum_{i_1 > \dots > i_n} \rho_{i_1}((v^{\alpha_1})^{-1} \otimes v^{\alpha_1}) \wedge \dots \wedge \rho_{i_n}((v^{\alpha_n})^{-1} \otimes v^{\alpha_n}).$$

LEMMA 3.2. $kh = \text{id}$.

Proof. This can be seen by a straightforward computation.

LEMMA 3.3. $kb = dk$.

Proof. Note first the following identities:

$$(3.6)\quad d\rho_i((v^\alpha)^{-1} \otimes v^\alpha)$$
$$= (v^\alpha|_{>i})^{-1} \otimes 1 \otimes (v^\alpha|_{>i}) - (v^\alpha|_{\geqq i})^{-1} \otimes 1 \otimes (v^\alpha|_{\geqq i})$$

and

$$(3.7)\quad \rho_i((v^\alpha v^\beta)^{-1} \otimes v^\alpha v^\beta) = (v^\beta|_{>i})^{-1} \rho_i((v^\alpha)^{-1} \otimes v^\alpha)(v^\beta|_{>i})$$
$$+ (v^\alpha|_{>i-1})^{-1} \rho_i((v^\beta)^{-1} \otimes v^\beta)(v^\alpha|_{>i-1}).$$

These follow from the definitions of the respective maps and the commutation relations (2.1).

Now, given $i > j$, we get from (3.6) and (3.7) the identity

$$(3.8)\quad \rho_i((v^\alpha v^\beta)^{-1} \otimes v^\alpha v^\beta) \wedge \rho_j((v^\gamma)^{-1} \otimes v^\gamma)$$
$$-\rho_i((v^\alpha)^{-1} \otimes v^\alpha) \wedge \rho_j((v^\beta v^\gamma)^{-1} \otimes v^\beta v^\gamma)$$

$$= \sum_{i>l>j} \rho_i((v^\alpha)^{-1} \otimes v^\alpha) \wedge \rho_l((v^\beta)^{-1} \otimes v^\beta) \wedge \rho_j((v^\gamma)^{-1} \otimes v^\gamma).$$

Then (3.8) and the easily seen fact that d is a derivation of the bialgebra E lead us immediately to the identity $kb = dk$.

PROPOSITION 3.4. *(E_*, d) is a projective resolution of $\mathscr{A}$, and*

$$k:(\Lambda_*(\mathscr{A}), b) \rightarrow (E_*, d)$$

is a comparison map.

Proof. According to Lemma 3.1, (E_*, d) is a subcomplex of the complex $(\Lambda_*(\mathscr{A}), b)$, and in particular, $d^2 = 0$. Since there exists a contracting homotopy

$$I:\Lambda_*(\mathscr{A}) \rightarrow \Lambda_{*+1}(\mathscr{A}),$$

$$Ib + bI = \text{id},$$

we can set

$$\bar{I} = kIh.$$

Using Lemmas 3.1, 3.2, and 3.3 we get

$$d\bar{I} + \bar{I}d = dkIh + kIhd = k(bI + Ib)h = \text{id},$$

and hence (E_*, d) is acyclic. Since E_* is given by free $\mathscr{A}^e$-modules ($\mathscr{A}^e$ is unital), the result follows.

4. Hochschild cohomology of $\mathscr{A}$. According to Proposition 3.4, (E_*, d) is a projective resolution of $\mathscr{A}$, and hence the groups $H^n(\mathscr{A}, \mathscr{A}^*)$ can be computed as the cohomology groups of the complex

$$(\text{Hom}_{\mathscr{A}^e}(E_*, \mathscr{A}^*), {}^t d).$$

Let us fix the basis for ΛV consisting of the vectors e_I, $I = (i_1, \dots, i_n)$, $i_1 > i_2 > \dots > i_n$. We shall denote by $\hat{\otimes}$ the space of tempered sequences $\mathscr{S}^*(\mathbf{Z}^N, \cdot)$. The map

$$(4.1) \quad \text{Hom}_{\mathscr{A}^e}(E_n, \mathscr{A}^*) \rightarrow \hat{\otimes}\Lambda^n V$$

$$t \mapsto \left(\sum_I t(e_I)(v^\alpha)e_I\right)_{\alpha \in \mathbf{Z}^N}$$

is an isomorphism, and it is straightforward to see that ${}^t d$ is identified with the operator

$$(4.2) \quad \otimes T_x:\hat{\otimes}\Lambda^n V \rightarrow \hat{\otimes}\Lambda^{n+1} V$$

where

$$(4.2) \quad x_\alpha = \sum_{k=1}^{N} (1 - v_k v^\alpha v_k^{-1}(v^\alpha)^{-1})e_k.$$

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Thus $H^*(\mathscr{A}, \mathscr{A}^*)$ is identified with the cohomology of the complex

(4.3) $(\hat{\otimes}\Lambda^* V, \otimes T_x)$.

THEOREM 4.1. *The Hochschild cohomology of* $\mathscr{A}$ *with coefficients in* $\mathscr{A}^*$ *is isomorphic to the direct sum of the following two vector spaces*:

(1) $\hat{\otimes}\Lambda^* V$ *restricted to indices* $\alpha \in \mathbf{Z}^N$ *such that* $v^\alpha \in$ *centre* $\mathscr{A}$,

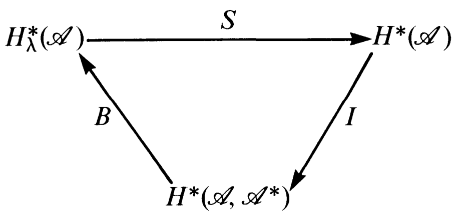
(2) *the quotient of the space of tempered sequences* $(x_\alpha \wedge z_\alpha)$, $z_\alpha \in \Lambda(V \ominus \mathbf{C}x_\alpha)$, *by the subspace consisting of those sequences for which* $(\|z_\alpha\|)$ *is tempered.*

Proof. The result follows from (4.2), (4.3), the identity

$$T_x T_x^* + T_x^* T_x = \|x\|^2 \text{ id},$$

and the fact that $\|x_\alpha\| = 0$ if and only if $v^\alpha \in$ centre $\mathscr{A}$.

5. De Rham homology of $\mathscr{A}$. Recall that the long exact sequence (1.2) can be written as an exact couple:

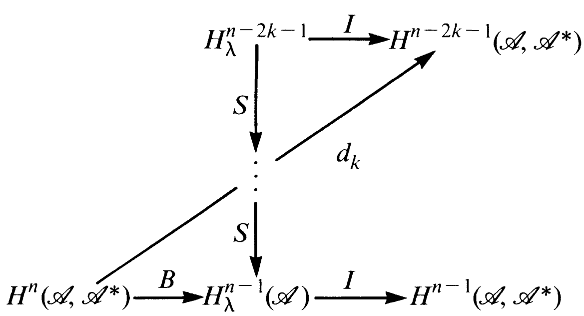


We will compute the limit of the corresponding spectral sequence $\mathbf{E}_n^*(\mathscr{A})$.

For the convenience of the reader, let us recall some definitions. $\mathbf{E}_n^*(\mathscr{A})$ is defined as the homology of the complex

$$(E_0^*(\mathscr{A}), d_0) = (H^*(\mathscr{A}, \mathscr{A}^*), IB).$$

$\mathbf{E}_{k+1}^*(\mathscr{A})$ is given inductively as the homology of the complex $(\mathbf{E}_k^*(\mathscr{A}), d_k)$, where the differential d_k, acting on $\phi \in H^n(\mathscr{A}, \mathscr{A}^*)$ with $d_i\phi = 0$ for $i < k$, is determined by the commutative diagram



using the fact that $d_1\phi = \ldots = d_{k-1}\phi = 0$ implies that $B\phi \in \text{Im } S^k$.

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$(\mathbf{E}_k^*(\mathcal{A}), d_k)$ converges to the graded vector space associated to the filtration by dimension of the periodic cyclic cohomology, and the limit, called $E_\infty^*(\mathcal{A})$, will be called the de Rham homology.

We will start by expressing the operator d_0 in terms of the description of the Hochschild cohomology of $\mathcal{A}$ obtained in Section 4.

For an element

$$\phi \in \mathrm{Hom}_{\mathcal{A}^e}(E, \mathcal{A}^*), \quad {}^t d\phi = 0,$$

the composition $\widetilde{\phi} = \phi \circ k$ gives a cocycle on $\mathcal{A}$ representing the same cohomology class as ϕ. Then

$$[d_0\phi] = [B\widetilde{\phi}] \quad \text{in } H^*(\mathcal{A}, \mathcal{A}^*).$$

According to Section 4, $[B\widetilde{\phi}]$ is given by the element of $\hat{\otimes}\Lambda V$ defined by

$$(B\widetilde{\phi})(\,(v^\alpha \otimes e_I)\,)$$
$$= \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma)\ B\widetilde{\phi}(v^\alpha(v_{\sigma(i_a)} \dots v_{\sigma(i_n)})^{-1}, v_{\sigma(i_1)}, \dots, v_{\sigma(i_n)})$$
$$= \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \cdot \text{cyclic antisymmetrisation of}$$
$$\phi k(1, v^\alpha(v_{\sigma(i_1)} \dots v_{\sigma(i_n)})^{-1}, v_{\sigma(i_1)}, \dots, v_{\sigma(i_n)})$$
$$+ (-1)^n \phi k(v^\alpha v_{\sigma(i_1)} \dots v_{\sigma(i_n)})^{-1}, v_{\sigma(i_1)}, \dots, v_{\sigma(i_n)}, 1).$$

Note that, by the definition of k, the second term in the above sum vanishes and only the cyclic permutations contribute to the first sum. Thus we end up with the sum

$$\sum_l (-1)^{\alpha_l} \phi k(1 \otimes v_{i_1} \otimes \dots \otimes v_{i_l} \otimes v^\alpha(v_{i_{l+1}} \dots v_{i_n} v_{i_1} \dots v_{i_l})^{-1}$$
$$\otimes v_{i_{l+1}} \otimes \dots \otimes v_{i_n}),$$

where

$$\alpha_l \equiv l(n+1) + ln \equiv l \pmod 2.$$

According to the definition of k, we can now write (see 3.5)

$$(d_0\phi)(v^\alpha \otimes e_I)$$
$$= \sum_l (-1)^l \sum_{i_l > m > i_{l+1}} \phi(v_{i_1} \dots v_{i_l} v^\alpha (v_{i_1} \dots v_{i_l})^{-1} e_{i_1} \wedge \dots \wedge e_{i_l}$$
$$\wedge \rho_m(v_{i_{l+1}} \dots v_{i_n} v_{i_1} \dots v_{i_l}(v^\alpha)^{-1} \otimes v^\alpha(v_{i_{l+1}} \dots v_{i_n} v_{i_1} \dots v_{i_l})^{-1})$$
$$\wedge e_{i_{l+1}} \wedge \dots \wedge e_{i_n}).$$

Using (3.4) we get d_0 represented as the map

(5.1) $\oplus T_y^*: \hat{\oplus}\Lambda V \to \hat{\oplus}\Lambda V,$

where

(5.2) $$y_\alpha = \sum_{l=1}^{n} \left(\sum_{k=0}^{\alpha_l - 1}{}' (v^\alpha)^{-1}(v^\alpha|_{>l}) v_l^k v^\alpha v_l^{-k} (v^\alpha|_{>l})^{-1} \right) e_l.$$

Note that, since $Bb = -bB$, $x_\alpha \perp y_\alpha$ for all $\alpha \in \mathbf{Z}^N$.

The following result is essentially the content of [**3**, Lemma 52].

LEMMA 5.1. *For* $\alpha \neq (0, \dots, 0)$,

$$\|x_\alpha\| + \|y_\alpha\| \geqq \frac{\pi^2}{8N}\left(\sum_{k=1}^{N} |\alpha_k|^2 \right)^{-1}.$$

Proof. Note first that, according to (4.2) and (5.2), we have

$$\|x_\alpha\|^2 = \sum_{k=1}^{N} |1 - \lambda_k|^2, \quad \lambda_k = \prod_{j=1}^{N} \lambda_{kj}^{\alpha_j}$$

and

$$\|y_\alpha\|^2 = \sum_{k=1}^{N} \left| \frac{1 - \lambda_k^{\alpha_k}}{1 - \lambda_k} \right|^2.$$

Hence

$$\|x_\alpha\| + \|y_\alpha\| \geqq \frac{1}{N}\left(\sum_{k=1}^{N} |1 - \lambda_k| + \left| \frac{1 - \lambda_k^{\alpha_k}}{1 - \lambda_k} \right| \right).$$

Looking at the kth term of the right hand sum, and setting $\lambda_k = \exp(i\theta_k)$, we have either

$$\left| \frac{1 - \lambda_k^{\alpha_k}}{1 - \lambda_k} \right| > 1 \quad \text{or} \quad \alpha_k \theta_k \notin \left] -\frac{\pi}{2}, \frac{\pi}{2} \right[.$$

Since in the second case

$$|\theta_k| > \frac{\pi}{2|\alpha_k|} \quad \text{and} \quad |1 - \lambda_k| > \frac{\pi^2}{8|\alpha_k|^2},$$

the stated inequality follows.

LEMMA 5.2. *Given* $\|x\| + \|y\| \neq 0$, *we have*

$$(T_y^* w = T_x w', \ T_x w = 0) \Rightarrow (w = T_x w_1 + T_y^* w_2).$$

Moreover, if w *and* w' *are tempered functions of* $\alpha \in \mathbf{Z}^N$, *and* x_α, y_α *are as above, then* w_1 *and* w_2 *can be chosen to be tempered functions of* α.

Proof. Choose an orthonormal basis $f_1, \ldots, f_N$ of V such that

$$x = \|x\| f_1, \quad y = \|y\| f_2.$$

Then

$$w = (\mu' + \mu'' \wedge f_2) \wedge f_1, \quad \mu', \mu'' \in \Lambda(V \ominus (\mathbf{C}f_1 \oplus \mathbf{C}f_2))$$

and we can reduce everything to the following two dimensional problem:

Given a, b, c, tempered sequences in $\mathcal{S}^*(\mathbf{Z}^N)$, such that $\|y\|b = \|x\|c$, find tempered sequences A, B, C such that

$$af_1 + bf_2 \wedge f_1 = T_y^*(Cf_2 \wedge f_1) + T_x(A + Bf_2).$$

According to Lemma 5.1, a solution of this is given by

$$A = C = \frac{a}{\|x\| + \|y\|}, \quad B = \frac{b + c}{\|x\| + \|y\|}.$$

THEOREM 5.3. (1) *S is injective on* Im S.

(2) $\quad \mathbf{E}^n_\infty(\mathcal{A}) \cong \mathbf{E}^n_1(\mathcal{A}) \cong \Lambda^n V.$

Proof. Note first that the formulas

$$\delta_i(v_j) = \delta_{ij} v_j, \quad i, j = 1, \ldots, N$$

define derivations

$$\delta_i : \mathcal{A} \to \mathcal{A},$$

such that

$$[\delta_i, \delta_j] = 0 \quad \text{and} \quad \tau\delta_i = 0.$$

Setting

$$(5.3) \quad \phi_{i_1,\ldots,i_n}(x_0, \ldots, x_n) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, \tau(x_0 \delta_{i_{\sigma(1)}}(x_1) \ldots \delta_{i_{\sigma(n)}}(x_n)),$$

we get cyclic cocycles on $\mathcal{A}$. (See [5].) Moreover, for $i_1 > \ldots > i_n$,

$$\phi_{i_1,\ldots,i_n}(v^\alpha h(e_I)) \neq 0 \Leftrightarrow I = (i_1, \ldots, i_n) \text{ and } \alpha = (0, \ldots, 0).$$

Thus, under the map (4.1), the linear space $\{\ \}^n_\lambda \subseteq H^n_\lambda(\mathcal{A})$ spanned by the cocycles (5.3) is mapped injectively onto the $(0, \ldots, 0)$-component of $\hat{\oplus}\Lambda^n V$. Hence, by Lemmas 5.1 and 5.2, the map

$$I:\{\ \}^n_\lambda \to \mathbf{E}^n_1(\mathcal{A})$$

is an isomorphism.

But $\{\ \}^n_\lambda$ is given by cyclic cocycles, and hence $d_1 = d_2 = \ldots = 0$. This gives in particular

$$\mathbf{E}_{\infty}^{n}(\mathscr{A}) = \mathbf{E}_{1}^{n}(\mathscr{A}) \cong \Lambda^{n} V,$$

i.e., (2). To prove (1) note that, for $k > 1$, we have

$$S^{k}\phi = 0 \Rightarrow S^{k-1}\phi \in \operatorname{Im} B \Rightarrow I\phi \in \operatorname{Im} d_{k-1}.$$

But $d_{k-1} = 0$ and thus $I\phi \in \operatorname{Im} d_{k-2}$. By induction, $I\phi \in \operatorname{Im} d_0$, i.e.,

$$\operatorname{Ker} S^{k} \subseteq S(\operatorname{Ker} S^{k+1}) + \operatorname{Im} B.$$

Iterating the above relation we get, for any $l > 0$,

$$\operatorname{Ker} S^{k} \subseteq S^{l}(\operatorname{Ker} S^{k+l}) + \operatorname{Im} B.$$

For dimensional reasons the first term converges to 0 as l tends to infinity, and hence we obtain (1):

$$\operatorname{Ker} S^{k} \subseteq \operatorname{Im} B = \operatorname{Ker} S.$$

COROLLARY 5.4 (cf. [**1**]). *Any continuous derivation D of $\mathscr{A}$ into $\mathscr{A}^*$ has a unique decomposition*

$$D = \bar{D} + \bar{\bar{D}},$$

where

$$\bar{D} = \sum_{i=1}^{N} A_i \delta_i,$$

A_i in the centraliser of $\mathscr{A}$ in $\mathscr{A}^*$ and

$$\bar{\bar{D}} = \lim_{M \to \infty} \operatorname{Ad} \sum_{|\alpha| < M} a_{\alpha} v^{\alpha}, \quad (\,|a_{\alpha}|\, \|x_{\alpha}\|\,) \in \mathscr{S}^*(\mathbf{Z}^N).$$

Moreover $\bar{\bar{D}}$ is inner if and only if (a_{α}) is a tempered sequence.

Proof. The existence of the decomposition is a consequence of Theorem 4.1, while the desired description of the summands is a consequence of the proof of Theorem 5.3.

Definition 5.5. $\{\ \}_{\lambda}^{n}$ denotes the linear space spanned by the cocycles (5.3). We use the same notation for the image of $\{\ \}_{\lambda}^{n}$ in both $H_{\lambda}^{n}(\mathscr{A})$ and $H^{n}(\mathscr{A}, \mathscr{A}^*)$.

6. Cyclic cohomology of $\mathscr{A}$.

THEOREM 6.1.

(1) $\quad H_{\lambda}^{n}(\mathscr{A}) = \operatorname{Im} B \dotplus \operatorname{Im} S \dotplus \{\ \}_{\lambda}^{n}.$

(2) $\quad S(H_{\lambda}^{n}(\mathscr{A})) \cong \bigoplus_{k \geqq 0} \Lambda^{n-2k} V.$

(3) $\quad \operatorname{Im} B \cong \operatorname{Ker} d_0 / \{\ \}_{\lambda}^{n},$

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where Ker d_0 *can be given an explicit description as* Ker $\oplus$ T_y^* *in the context of Section* 4.

Proof. By Theorem 5.3 (2), the natural map

$$\{\ \}_\lambda^n \rightarrow H_\lambda^n(\mathscr{A})/(\text{Im } B + \text{Ker } I)$$

is an isomorphism. This is equivalent to saying that

$$\text{Ker } d_0 = \text{Im } I \quad \text{and} \quad H_\lambda^n(\mathscr{A}) = (\text{Im } B + \text{Ker } I) \dot{+} \{\ \}_\lambda^n.$$

By Theorem 5.3 (1), Ker $S \cap$ Im $S = 0$. Since Im B = Ker S and Ker I = Im S, (1) follows.

(2) follows from (1) by induction.

(3) follows from (1) using Ker d_0 = Im I.

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