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SOME CONGRUENCES FOR GENERALIZED EULER NUMBERS

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1. Introduction. The generalized Euler numbers may be defined by

$$\sum_{n=0}^{\infty} E_n^{(m)} \frac{x^n}{n!} = \left[\sum_{n=0}^{\infty} \frac{x^{mn}}{(mn)!}\right]^{-1}.$$

Since $E_n^{(m)}$ is zero unless m divides n, we shall write $e_n^{(m)}$ for $E_{mn}^{(m)}$. Leeming and MacLeod [**12**] recently gave some congruences for these numbers. They found congruences (mod 16) for $e_n^{(m)}$ where $m = 3, 6, 8, 12$, and 16. Thus for $m = 3$, their congruence is

$$e_n^{(3)} = 3 - 4n + 8\binom{n+1}{3} \pmod{16}, \quad n \geqq 1.$$

They also proved that $e_n^{(3)} \equiv (-1)^n \pmod 9$, $e_n^{(5)} \equiv (-1)^n \pmod{125}$, and $e_n^{(6)} \equiv -1 \pmod 3$, and they made several conjectures which may be stated as follows:

(C1) $e_n^{(14)} \equiv e_n^{(7)} \equiv 7 - 8n \pmod{16}, \quad n \geqq 1$

(C2) $e_n^{(15)} \equiv 15 \pmod{16}, \quad n \geqq 1$

(C3) $e_n^{(p)} \equiv (-1)^n \pmod{p^3}, \quad p = 11 \text{ or } p = 13$

(C4) $e_n^{(2^k)} \equiv 1 - 2n + 8\binom{n}{2} \pmod{16}, \quad k \geqq 1.$

We shall develop here some properties of the generalized Euler numbers that enable us to go significantly beyond conjectures (C1) – (C4). In particular we shall prove:

$$(1.1) \quad e_n^{(7)} \equiv 7 - 8n + 2^4 \cdot 7\binom{n+1}{3} + 2^6\binom{n+1}{5} + 2^7\binom{n+1}{7} \pmod{2^8}, \; n \geqq 1$$

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(1.2) $$e_n^{(14)} \equiv 7 - 8n + 2^7 \cdot 3\binom{n}{2} - 2^9\binom{n}{3} \pmod{2^{10}},\ n \geqq 1$$

(1.3) $$e_n^{(15)} \equiv 15 - 16n - 32\binom{n+1}{3} \pmod{2^7}\ n \geqq 1$$

For all primes p, we have

(1.4) $$e_n^{(p^k m)} \equiv e_n^{(p^{k-1}m)} \pmod{p^{3k-\epsilon}}$$

where $\epsilon = 1$ for $p = 2$ or 3 and $\epsilon = 0$ for $p > 3$. Moreover, we have

(1.5) $$e_n^{(11)} \equiv (-1)^n\left[1 + 11^3 \cdot 530\binom{n}{2} + 11^5 \cdot 3\binom{n}{4}\right] \pmod{11^6}$$

(1.6) $$e_n^{(13)} \equiv (-1)^n\left[1 + 13^3 \cdot 2\binom{n}{2}\right] \pmod{13^5}$$

(1.7) $$e_n^{(2^k)} \equiv 1 - 2n + 8\binom{n}{2} - 16\binom{n}{3} \pmod{64},$$

for all $k \geqq 1$.

(Leeming and MacLeod stated that $e_n^{(7)} \equiv 4 \pmod{7^3}$ for $n = 11$ and 13. This is apparently a computational error.)

In the next section we prove a congruence of Jacobsthal [**1**] for binomial coefficients, and use it to prove (1.4). In the rest of the paper we show how the properties of the difference table of $e_n^{(m)}$ can be used to find congruences of the form of (1.1). A list of these congruences is given in Section 5. We shall show that for any m and any prime p, there is a j such that for any k and any i, $\pm e_{nj+i}^{(m)}$ is congruent (mod p^k) to a polynomial in n for n sufficiently large. In particular, if m is of the form p^a or $p^a - p^b$, we may take $j = 1$. This explains why Leeming and MacLeod found congruences (mod 16) for $m = 2^k$ and $m = 3, 6, 7, 12, 14$, and 15, but not for $m = 5$ or $m = 9$.

2. A congruence for binomial coefficients.

THEOREM 2.1. *Let p be a prime. Let $\epsilon = 1$ if p is 2 or 3, and $\epsilon = 0$ if p is greater than 3. Then*

$$e_n^{(p^k m)} \equiv e_n^{(p^{k-1}m)} \pmod{p^{3k-\epsilon}}.$$

We shall prove Theorem 2.1 with the help of the following lemma, to be proved later.

LEMMA A. *With p and ϵ as above,*

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$$\binom{p^k a}{p^k b} \equiv \binom{p^{k-1}a}{p^{k-1}b} \pmod{p^{3k-\epsilon}}.$$

Proof of Theorem 2.1. The assertion holds for $n = 0$. It follows from the definition of $e_i^{(m)}$ that for $n > 0$,

$$\sum_{i=0}^{n} \binom{p^k mn}{p^k mi} e_i^{(p^k m)} = 0,$$

so by Lemma A,

$$\sum_{i=0}^{n} \binom{p^{k-1}mn}{p^{k-1}mi} e_i^{(p^k m)} \equiv 0 \pmod{p^{3k-\epsilon}}.$$

Since

$$\sum_{i=0}^{n} \binom{p^{k-1}mn}{p^{k-1}mi} e_i^{(p^{k-1}m)} = 0,$$

the theorem follows by induction.

Theorem 2.1 does not always give a best possible congruence. For example, we shall see that $e_n^{(4)} \equiv e_n^{(2)} \pmod{64}$.

Lemma A follows from a congruence apparently found first by Jacobsthal [**1**], and later (in varying degrees of generality) by Kazandzidis [**10**] and Trakhtman [**17**]. Since these proofs may not be easily accessible, and since this congruence has other applications (see, for example, [**8**]), we give here a self-contained proof.

For the rest of this section, let p be a prime and let μ be 2, 1, or 0 according to whether p is 2, 3, or greater than 3. Let b be an integer divisible by p^β, $\beta \geqq 1$. Let S be the set of integers from 1 to b not divisible by p. We shall work in the ring of p-integral rational numbers; thus, $x \equiv y \pmod{p^k}$ means that $(x - y)/p^k$ is a rational number with denominator not divisible by p.

LEMMA 1. *$\sum_{i \in S} 1/i^2$ is divisible by $p^{\beta-\mu}$ for $p \neq 2$ and by $2^{\beta-1} = 2^{\beta-\mu+1}$ for $p = 2$.*

Proof. It is sufficient to consider the case $b = p^\beta$. Let

$$A = \sum_{i \in S} \frac{1}{i^2}.$$

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If $p \neq 2$ then

$$A \equiv \sum_{i \in s} \frac{1}{(2i)^2} = \frac{1}{4} A \pmod{p^\beta},$$

so $3/4\ A \equiv 0 \pmod{p^\beta}$ and the lemma follows.

If b is 2 or 4 the lemma is trivial. If $p = 2$ and $\beta \geqq 3$, then every odd square i^2 has four square roots (mod 2^β): i, $-i$, $2^{\beta-1} + i$, and $2^{\beta-1} - i$. Let T consist of one square root for each square (mod 2^β). Then

$$B = \sum_{i \in T} \frac{1}{i^2} \equiv \sum_{i \in T} \frac{1}{(3i)^2} \equiv \frac{1}{9} B \pmod{2^\beta},$$

so

$$\frac{8}{9} B \equiv 0 \pmod{2^\beta}.$$

Thus $2^{\beta-1}$ divides $4B \equiv A \pmod{2^\beta}$.

LEMMA 2. $\displaystyle\sum_{i \in S} \frac{1}{i} \equiv 0 \pmod{p^{2\beta-\mu}}$.

Proof. We have

$$2 \sum_{i \in S} \frac{1}{i} = \sum_{i \in S} \left(\frac{1}{i} + \frac{1}{b-i}\right) = \sum_{i \in S} \left[-\frac{b}{i^2} + \frac{b^2}{i^2(b-i)}\right]$$

$$\equiv -b \sum_{i \in S} \frac{1}{i^2} \pmod{p^{2\beta}},$$

and the result follows from Lemma 1.

We note that the case $b = p^\beta$ of Lemma 2 is a special case of Leudesdorf's Theorem [**9**, p. 101].

LEMMA 3. $\displaystyle\sum_{\substack{i,j \in S \\ i<j}} \frac{1}{ij} \equiv 0 \pmod{p^{\beta-\mu}}$.

Proof. If $p \neq 2$, the argument of Lemma 1 works. For $p = 2$ we have

$$2 \sum_{\substack{i,j \in S \\ i<j}} \frac{1}{ij} = \sum_{\substack{i,j \in S \\ i \neq j}} \frac{1}{ij} = \left[\sum_{i \in S} \frac{1}{i}\right]^2 - \sum_{i \in S} \frac{1}{i^2}.$$

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By Lemma 2 the first sum on the right is divisible by $2^{2\beta-\mu} = 2^{\beta-1}2^{\beta-\mu+1}$ and by Lemma 1 the second sum is divisible by $2^{\beta-\mu+1}$.

Now let $\nu(r) = \nu_p(r)$ denote the largest power of p dividing r.

THEOREM 2.2. *Let a and b be nonnegative integers divisible by p. Then unless $p = 2$ and $b \equiv a - b \equiv 2 \pmod 4$,*

$$\binom{a}{b} \equiv \binom{a/p}{b/p} \pmod{p^{\alpha+\beta+\gamma+\delta-\mu}},$$

where $\alpha = \nu(a)$, $\beta = \nu(b)$, $\gamma = \nu(a - b)$, $\delta = \nu\left(\binom{a/p}{b/p}\right)$, and μ is as before. In the exceptional case,

$$\binom{a}{b} \equiv -\binom{a/2}{b/2} \pmod{2^{\alpha+\beta+\gamma+\delta-2}}.$$

Proof. Let $c = a - b$ and let

$$F(x) = 1 + \sum_{j=1}^{b-b/p} F_j x^j = \prod_{i\in S}\left(1 + \frac{x}{i}\right).$$

By Lemma 2, $p^{2\beta-\mu}$ divides F_1 and by Lemma 3, $p^{\beta-\mu}$ divides F_2.

Without loss of generality, we may assume that $\gamma \geqq \beta$. Then it is easily seen that either $\gamma \geqq \alpha = \beta$ or $\alpha > \beta = \gamma$.

We first consider the case $\gamma \geqq \alpha = \beta$. Then

$$\binom{a}{b} = \frac{(c + 1)(c + 2)\dots(c + b)}{1\cdot 2\dots b}$$

and

$$\binom{a/p}{b/p} = \frac{(c/p + 1)\dots(c/p + b/p)}{1\dots(b/p)}$$
$$= \frac{(c + p)(c + 2p)\dots(c+ b)}{p\cdot 2p\dots b.}.$$

Thus

$$\binom{a}{b} = \binom{a/p}{b/p} F(c) \equiv \binom{a/p}{b/p}(1 + F_1c + F_2c^2) \pmod{p^{3\gamma+\delta}}.$$

Now F_1c is divisible by $p^{2\beta+\gamma-\mu} = p^{\alpha+\beta+\gamma-\mu}$ and F_2c^2 is divisible by $p^{\beta+2\gamma-\mu} = p^{\gamma-\alpha}p^{\alpha+\beta+\gamma-\mu}$, and the theorem follows.

Next we consider the case $\alpha > \beta = \gamma$. As before, we find that

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$$\binom{a}{b} = \binom{a/p}{b/p} \cdot (-1)^{b-b/p} F(-a).$$

Now

$$F(-a) \equiv 1 - F_1 a + F_2 a^2 \pmod{p^{3\alpha}}$$

and we see that $F_1 a$ is divisible by $p^{\alpha+2\beta-\mu} = p^{\alpha+\beta+\gamma-\mu}$, and $F_2 a^2$ is divisible by $p^{2\alpha+\beta-\mu} = p^{\alpha-\gamma}\, p^{\alpha+\beta+\gamma-\mu}$. Thus

$$\binom{a}{b} \equiv (-1)^{b-b/p} \binom{a/p}{b/p} \pmod{p^{\alpha+\beta+\gamma+\delta-\mu}}.$$

Now $b - b/p$ is even unless $p = 2$ and $b/2$ is odd. Since $\beta = \gamma$, this implies $b \equiv c \equiv 2 \pmod 4$. This completes the proof of Theorem 2.2.

The restriction that a is nonnegative may be removed by using the identity

$$\binom{-r}{b} = (-1)^b \frac{r}{r+b} \binom{r+b}{b}.$$

It may be noted that Jacobsthal's and Trakhtman's congruences are stronger than Theorem 2.2. They express the residue modulo a higher power of p in terms of Bernoulli numbers.

Combinatorial approaches to congruences like Theorem 2.2 have been taken by Rota and Sagan [**13**] and Smith [**14**]. However the moduli in their congruences are smaller powers of p.

Lemma A follows immediately from Theorem 2 for $p \neq 2$. For $p = 2$, we must show that

$$\binom{2^k a}{2^k b} \equiv \binom{2^{k-1} a}{2^{k-1} b} \pmod{2^{3k-1}}.$$

Since at least one of a, b, and $a - b$ is even, we have

$$\nu(2^k a) + \nu(2^k b) + \nu(2^k (a - b)) - 2 \geqq 3k - 1.$$

The exceptional case of Theorem 2.2 can arise only if $k = 1$, b is odd, and a is even. Here Theorem 2.2 yields

$$\binom{2a}{2b} \equiv - \binom{a}{b} \pmod 4.$$

But since $\binom{a}{b}$ is even in this case,

$$-\binom{a}{b} \equiv \binom{a}{b} \pmod 4.$$

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3. The difference table. The remaining congruences are obtained with the help of the following well-known fact:

LEMMA 3.1. *Suppose $f(n)$ and $g(n)$ are two functions defined on nonnegative integers. Then the following are equivalent*:

$$\text{(a)} \quad f(n) = \sum_{k=0}^{\infty} \binom{n}{k} g(k)$$

$$\text{(b)} \quad g(n) = \sum_{k=0}^{\infty} (-1)^{n-k} \binom{n}{k} f(k).$$

Note that the sums are really finite since $\binom{n}{k} = 0$ for $k > n$.

An easy proof of Lemma 3.1 follows from the fact that (a) and (b) are equivalent to

$$\text{(a}'\text{)} \quad \sum_{n=0}^{\infty} f(n) \frac{x^n}{n!} = e^x \sum_{k=0}^{\infty} g(k) \frac{x^k}{k!}$$

and

$$\text{(b}'\text{)} \quad \sum_{n=0}^{\infty} g(n) \frac{x^n}{n!} = e^{-x} \sum_{k=0}^{\infty} f(k) \frac{x^k}{k!}.$$

Now let Δ be the difference operator defined by

$$\Delta f(n) = f(n + 1) - f(n).$$

Then

$$\Delta^n f(j) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(j + k)$$

and Lemma 3.1 implies

$$f(n) = \sum_{k=0}^{\infty} \binom{n}{k} \Delta^k f(0),$$

and, more generally,

$$f(n + j) = \sum_{k=0}^{\infty} \binom{n}{k} \Delta^k f(j).$$

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The successive differences of f are easily computed in a difference table in which the values of f form the top row and each entry in a later row is the difference of the two entries above it. For example, the difference table for $f(n) = e_n^{(2)}$ begins as follows:

$$\begin{array}{ccccccccccccc}
1 & & -1 & & 5 & & -61 & & 1385 & & -50521 & & 2702765 \\
& -2 & & 6 & & -66 & & 1446 & & -51906 & & 2753286 & \\
& & 8 & & -72 & & 1512 & & -53352 & & 2805192 & & \\
& & & -80 & & 1584 & & -54864 & & 2858544 & & & \\
& & & & 1664 & & -56448 & & 2913408 & & & &
\end{array}$$

Thus

$$e_n^{(2)} = 1 - 2n + 8\binom{n}{2} - 80\binom{n}{3} + 1664\binom{n}{4} + \dots .$$

Each of these coefficients is divisible by a power of 2 at least as large as that dividing the previous coefficient. If this pattern holds in general we will have the congruences

$$e_n^{(2)} \equiv 1 - 2n + 8\binom{n}{2} \pmod{16},$$

$$e_n^{(2)} \equiv 1 - 2n + 8\binom{n}{2} - 80\binom{n}{3} \pmod{2^7},$$

and so on. If we look at the second diagonal we find that

$$e_{n+1}^{(2)} = -1 + 6n - 72\binom{n}{2} + 2^4 \cdot 3^2 \cdot 11\binom{n}{3}$$
$$+ 2^7 \cdot 3^2 \cdot 49\binom{n}{4} + \dots .$$

The last three coefficients are divisible by 9, but not by 27. Thus if this pattern continues we will have

$$e_{n+1}^{(2)} \equiv -1 + 6n \pmod{9}.$$

The powers of 3 in the second diagonal do not seem to get larger than 3^2. However, if we try the third diagonal we have

$$e_{n+2}^{(2)} = 5 - 2 \cdot 3 \cdot 11n + 2^3 \cdot 3^3 \cdot 7\binom{n}{2} - 2^4 \cdot 3^3 \cdot 127\binom{n}{3}$$
$$+ 2^7 \cdot 3^4 \cdot 281\binom{n}{4} + \dots ,$$

and we are led to expect the congruences

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$$e_{n+2}^{(2)} = 5 - 12n \pmod{27}$$

and

$$e_{n+2}^{(2)} = 5 + 15n - 27\binom{n}{2} - 27\binom{n}{3} \pmod{81}.$$

There are no primes other than 2 and 3 prominent in this table. But let us now look at the difference table for $e_{2n}^{(2)}$:

1		5		1385		2702765		19391512145
	4		1380		2701380		19388809380	
		1376		2700000		19386108000		

The entries in this table are divisible by 2, 3, and 5, and we are led as before to the congruence

$$e_{2n+2}^{(2)} \equiv 5 + 5n \pmod{125}.$$

We note that the powers of 2 are larger than in the previous table; thus the table suggests that

$$e_{2n}^{(2)} \equiv 1 + 4n \pmod{32}.$$

Similarly, if we look at the difference table for $e_{2n+1}^{(2)}$ we are led to expect the congruences

$$e_{2n+1}^{(2)} \equiv -1 - 60n \pmod{32},$$

$$e_{2n+1}^{(2)} \equiv -1 + 15n \pmod{25},$$

and

$$e_{2n+3}^{(2)} \equiv 64 + 40n \pmod{125}.$$

To prove these congruences we need theorems which guarantee that the divisibility patterns we have observed will continue. In the next section we prove these theorems.

4. Some Kummer congruences. Congruences of the form

$$\sum_{k=0}^{n} (-1)^{n-k}\binom{n}{k} a_{km+i} \equiv 0 \pmod{p^l}$$

are often called *Kummer congruences* after Kummer's well-known congruences for Bernoulli and Euler numbers [**11**]. These congruences have been studied by Carlitz [**2**], [**3**], [**4**], [**5**], Stevens [**15**], [**16**], and others. We shall prove some very general theorems on Kummer congruences for sequences defined by exponential generating functions.

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THEOREM 4.1. *Suppose that*

$$\sum_{n=0}^{\infty} B_n \frac{x^n}{n!} = \left[\sum_{n=0}^{\infty} A_n \frac{x^n}{n!} \right]^{-1},$$

where $A_0 = B_0 = 1$ *and the* A_n *are integers satisfying*

$$\sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} A_{j(p^s - p^r)+i} \equiv 0 \pmod{p^k} \tag{4.1}$$

where $s > r$, *for all* $k \geqq 0$ *and* $i \geqq kp^r$. *Then*

$$\sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} B_{j(p^s - p^r)+i} \equiv 0 \pmod{p^k} \tag{4.2}$$

for all $k \geqq 0$ *and* $i \geqq kp^r$.

Proof. We use the "symbolic" or "umbral" notation in which, after expansion, each term $\mathbf{A}^m \mathbf{B}^n$ in an expression is replaced by $A_m B_n$. Then (4.1) and (4.2) may be written

$$(\mathbf{A}^{p^s} - \mathbf{A}^{p^r})^k \, \mathbf{A}^l \equiv 0 \pmod{p^k} \tag{4.3}$$

and

$$(\mathbf{B}^{p^s} - \mathbf{B}^{p^r})^k \, \mathbf{B}^l \equiv 0 \pmod{p^k}, \tag{4.4}$$

where $l = i - kp^r \geqq 0$.

We shall prove (4.4) by induction on k and l. If $k = 0$, there is nothing to prove. From the formula

$$(x + y)^{p^s} - (x + y)^{p^r} = (x^{p^s} - x^{p^r}) + (y^{p^s} - y^{p^r}) + pv(x, y),$$

where $v(x, y)$ is a polynomial with integral coefficients, we have for $k > 0$,

$$\begin{aligned} 0 &= [\,(\mathbf{A} + \mathbf{B})^{p^s} - (\mathbf{A} + \mathbf{B})^{p^r}]^k \, (\mathbf{A} + \mathbf{B})^l \\ &= [\,(\mathbf{A}^{p^s} - \mathbf{A}^{p^r}) + (\mathbf{B}^{p^s} - \mathbf{B}^{p^r}) + pv(\mathbf{A}, \mathbf{B})\,]^k \, (\mathbf{A} + \mathbf{B})^l \\ &= \sum_{f+g+h=k} \sum_{i=0}^{l} \binom{k}{f,\, g,\, h} \binom{l}{i} (\mathbf{A}^{p^s} - \mathbf{A}^{p^r})^f \, (\mathbf{B}^{p^s} - \mathbf{B}^{p^r})^g \\ &\qquad \times p^h v^h(\mathbf{A}, \mathbf{B}) \mathbf{A}^i \mathbf{B}^{l-i}. \end{aligned}$$

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By induction all terms except $(\mathbf{B}^{p^s} - \mathbf{B}^{p^r})^k \mathbf{B}^l$ are divisible by p^k, hence it must be also.

By same technique we can prove the following theorem, which generalizes a result of Carlitz [**3**].

THEOREM 4.2. *Suppose A_n and B_n satisfy* (4.1) *and* (4.2). *Then C_n satisfies an analogous congruence, where*

$$\sum_{n=0}^{\infty} C_n \frac{x^n}{n!} = \left[\sum_{n=0}^{\infty} A_n \frac{x^n}{n!}\right]\left[\sum_{n=0}^{\infty} B_n \frac{x^n}{n!}\right],$$

i.e.,

$$C_n = \sum_{k=0}^{n} \binom{n}{k}(A_k B_{n-k}).$$

Instead of (4.3) we could have considered the slightly more general congruence

$$(\mathbf{A}^{p^s} - c\,\mathbf{A}^{p^r})^k \mathbf{A}^l \equiv 0 \pmod{p^k}$$

for some integer c.

Our next theorem describes how p-secting a sequence increases the power of p in its Kummer congruence.

THEOREM 4.3. *Let A_n be a sequence of integers satisfying* (4.1). *Then for any positive integer d, and any $i \geqq 0$,*

$$\text{(4.5)}\qquad \sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j} A_{jp^{d-1}(p^s-p^r)+i} \equiv 0 \pmod{p^M},$$

where $M = \min(dk, [i/p^r])$.

Proof. Let us set $q = p^s - p^r$. Then we may write (4.5) symbolically as

$$\text{(4.6)}\qquad (\mathbf{A}^{qp^{d-1}} - 1)^k \mathbf{A}^i \equiv 0 \pmod{p^M}.$$

It is not difficult to show that for $0 < j \leqq d$, $\binom{p^{d-1}}{j}$ is divisible by p^{d-j}. Then we have

$$\begin{aligned} x^{qp^{d-1}} - 1 &= (x^q - 1 + 1)^{p^{d-1}} - 1 \\ &= \sum_{j=1}^{p^{d-1}} \binom{p^{d-1}}{j}(x^q - 1)^j \end{aligned}$$

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$$= \sum_{j=1}^{d} a_j\, p^{d-j}(x^q - 1)^j + \sum_{j=d+1}^{p^{d-1}} b_j(x^q - 1)^j,$$

for some integers a_j, b_j. Thus

$$(4.7) \qquad (x^{qp^{d-1}} - 1)^k = \sum_{j=k}^{dk} f_j\, p^{dk-j}(x^q - 1)^j + \sum_{j=dk+1}^{kp^{d-1}} g_j(x^q - 1)^j$$

for some integers f_j, g_j.

We may now write

$$(\mathbf{A}^q - 1)^j \mathbf{A}^i =$$

$$\begin{cases} (\mathbf{A}^{p^s} - \mathbf{A}^{p^r})^j\, \mathbf{A}^{i-jp^r} & \text{if } i \geqq jp^r \\ (\mathbf{A}^{p^s} - \mathbf{A}^{p^r})^{[i/p^r]}(\mathbf{A}^q - 1)^{j-[i/p^r]}\mathbf{A}^{i-p^r[i/p^r]} & \text{if } i \leqq jp^r. \end{cases}$$

Thus $(\mathbf{A}^q - 1)^j\, \mathbf{A}^i$ is divisible by $p^{\min(j,[i/p^r])}$. It follows from (4.7) that to prove (4.6) we need only prove that

(i) for $k \leqq j \leqq dk$, $dk - j + \min(j, [i/p^r]) \geqq M$

and

(ii) for $dk < j \leqq k\, p^{d-1}$, $\min(j, [i/p^r]) \geqq M$.

The second inequality is immediate as is the first if $j \leqq [i/p^r]$. If $j > [i/p^r]$ then for $j \leqq dk$,

$$dk - j + [i/p^r] \geqq [i/p^r] \geqq M.$$

COROLLARY 4.4. *With the notation of Theorem* 4.3, *if* $p^{d-1}(p^s - p^r)$ *divides* $w > 0$, *then*

$$(4.8) \qquad \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} A_{jw+i} \equiv 0 \pmod{p^M}.$$

Proof. Let $w = tp^{d-1}q$, where $q = p^s - p^r$. Then (4.8) may be written symbolically as

$$(4.9) \qquad (\mathbf{A}^{tqp^{d-1}} - 1)^k\, \mathbf{A}^i \equiv 0 \pmod{M}.$$

But

$$(\mathbf{A}^{tqp^{d-1}} - 1)^k\, \mathbf{A}^i = (\mathbf{A}^{qp^{d-1}} - 1)^k f(\mathbf{A})\, \mathbf{A}^i,$$

where $f(\mathbf{A})$ is a polynomial in $\mathbf{A}$, and hence (4.9) follows from (4.6).

We now give the analogs of Theorems 4.1, 4.2, and 4.3 in which $p^s - p^r$

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is replaced by p^s. We omit the proofs of these theorems, which are similar to the proofs just given (but a little easier).

THEOREM 4.5. *Suppose that*

$$\sum_{n=0}^{\infty} B_n \frac{x^n}{n!} = \left[\sum_{n=0}^{\infty} A_n \frac{x^n}{n!}\right]^{-1},$$

where $A_0 = B_0 = 1$ *and the* A_n *are integers satisfying*

$$(4.10) \quad \sum_{j=0}^{k} a^{k-j} \binom{k}{j} A_{jp^s+i} \equiv 0 \pmod{p^k}$$

for some integer a *and all* $i, k \geqq 0$. *Then*

$$(4.11) \quad \sum_{j=0}^{k} (-a)^{k-j} \binom{k}{j} B_{jp^s+i} \equiv 0 \pmod{p^k}$$

for all $i, k \geqq 0$.

THEOREM 4.6. *Suppose that* A_n *and* B_n *satisfy*

$$\sum_{j=0}^{k} a^{k-j} \binom{k}{j} A_{jp^s+i} \equiv 0 \pmod{p^k}$$

and

$$\sum_{j=0}^{k} b^{k-j} \binom{k}{j} B_{jp^s+i} \equiv 0 \pmod{p^k}$$

for $i, k \geqq 0$. *Let*

$$C_k = \sum_{j=0}^{k} \binom{k}{j} A_j B_{k-j}.$$

Then

$$\sum_{j=0}^{k} (a+b)^{k-j} \binom{k}{j} C_{jp^s+i} \equiv 0 \pmod{p^k}$$

for $i, k, \geqq 0$. *In particular, if* $a + b = 0$ *then* $C_n \equiv 0 \pmod{p^k}$ *for* $n \geqq kp^s$.

THEOREM 4.7. *If* A_n *satisfies* (4.10) *then for any positive integer* d *and*

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any w *divisible by* p^{s+d-1},

$$\sum_{j=0}^{k} \alpha^{k-j} \binom{k}{j} A_{jw+i} \equiv 0 \pmod{p^{dk}},$$

where $\alpha = a^{w/p^s}$, *for* $i, k \geqq 0$.

The theorems we have proved can be applied to a number of generating functions of combinatorial and number-theoretic interest, such as

$$\left[\sum_{n=0}^{\infty} \frac{x^{mn+r}}{(mn+r)!}\right] \Big/ \left[\sum_{n=0}^{\infty} \frac{x^{mn}}{(mn)!}\right], \quad \left[\sum_{n=0}^{\infty} \frac{x^{mn}}{(mn)!}\right]^k, \quad \text{and}$$

$$\left[\sum_{n=0}^{\infty} \frac{x^{mn}}{(mn)!}\right] \Big/ \left[\sum_{n=0}^{\infty} (-1)^n \frac{x^{mn}}{(mn)!}\right].$$

However, we shall consider here only their application to generalized Euler numbers.

5. Applications to generalized Euler numbers. Applying the theorems of Section 4 to the generalized Euler numbers, we obtain the following results.

THEOREM 5.1. *Suppose that* m *is not a power of* p. *Choose* s *and* r *so that* m *divides* $p^s - p^r$, *with* $s > r$. (*We may take* r *to be* $\nu_p(m)$ *and then take* $s - r$ *to be the order of* p (mod m/p^r).) *Let* d *be a positive integer, and let* t *be divisible by* $p^{d-1}(p^s - p^r)/m$. *Then for* $k, i \geqq 0$,

$$\sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} e_{tj+i}^{(m)} \equiv 0 \pmod{p^{\min(dk, [im/p^r])}}.$$

The case $r = 0$ of Theorem 5.1 was found by Carlitz [**2**], who used an explicit (but complicated) formula for $e_n^{(m)}$ in his proof.

THEOREM 5.2. *Let* $m = p^s$. *Let* d *be a positive integer and let* t *be divisible by* p^{d-1}. *Then for* $k, i \geqq 0$,

$$\sum_{j=0}^{k} \binom{k}{j} e_{tj+i}^{(m)} \equiv 0 \pmod{p^{dk}}.$$

THEOREM 5.3. *Let* $m = 2^s$. *Let* d *be a positive integer and let* t *be divisible by* 2^{d-1}. *Then for* $k, i \geqq 0$,

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$$\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}e_{tj+i}^{(m)} \equiv 0 \pmod{2^{dk}}.$$

By Theorems 5.2 and 5.3 we may find congruences for $e_n^{(2^s)}$ of both forms

$$e_n^{(2^s)} \equiv (-1)^n \sum_{j=0}^{k} c_j \binom{n}{j} \pmod{2^k}$$

and

$$e_n^{(2^s)} \equiv \sum_{j=0}^{k} c_j' \binom{n}{j} \pmod{2^k}.$$

It turns out that those of the second form seem to be better in that c_j' is divisible by a higher power of 2 than the 2^j guaranteed by Theorem 5.3.

In our applications of Theorem 5.1 we will sometimes find empirically that

$$\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}e_{tj+i}^{(m)}$$

is divisible by p to a power greater than guaranteed by our theorems. In order to prove the congruences suggested by the data, we need to ascertain that the observed divisibility holds for all sufficiently large k. In these situations the following lemma is useful:

LEMMA 5.4. *Let f be an integer-valued function on the nonnegative integers. Suppose that $\Delta^i f(n) \equiv 0 \pmod{M}$ for $n = a, a + 1, \ldots, b$, and $\Delta^j f(b) \equiv 0 \pmod{M}$ for all $j \geqq i$. Then*

$$\Delta^j f(a) \equiv 0 \pmod{M} \quad \text{for all } j \geqq i.$$

Proof. Although a formal proof by induction on j and $b - a$ can be given, the lemma is best understood by observing that the entries of the difference table $\Delta^j f(n)$ with $a \leqq n \leqq b$ and $j \geqq i$ are determined by the entries $\Delta^i f(n)$, $a \leqq n \leqq b$ and the entries $\Delta^j f(b)$, $j \geqq i$. Thus all these entries are divisible by M.

From Theorems 5.1, 5.2, and 5.3, and Lemma 5.4 we find the following congruences (after some computation):

$$(5.1) \quad (-1)^n e_n^{(2)} \equiv 1 + 4\binom{n}{2} + 3 \cdot 2^4\binom{n}{3} - 7 \cdot 2^4\binom{n}{4} + 2^7\binom{n}{5} + 2^6\binom{n}{6} \pmod{2^8}$$

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$$(5.2)\qquad e_n^{(2)} \equiv 1 - 2n + 8\binom{n}{2} - 2^4\cdot 5\binom{n}{3} - 2^7\cdot 3\binom{n}{4} - 2^8\cdot 3\binom{n}{5} + 2^{10}\binom{n}{6} \pmod{2^{11}}.$$

Congruence (5.2) was given by Frobenius [**7**, p. 449].

$$(5.3)\qquad e_{2n}^{(2)} \equiv 1 + 2^2 n + 2^5\cdot 43\binom{n}{2} + 2^7\cdot 91\binom{n}{3} + 2^{11}\cdot 5\binom{n}{4} - 2^{13}\binom{n}{5} \pmod{2^{15}}$$

$$(5.4)\qquad e_{2n+1}^{(2)} \equiv -1 - 2^2\cdot 15n + 2^5\cdot 473\binom{n}{2} - 2^7\cdot 101\binom{n}{3} + 2^{11}\cdot 7\binom{n}{4} - 2^{13}\binom{n}{5} \pmod{2^{15}}$$

$$(5.5)\qquad e_{4n}^{(2)} \equiv 1 + 2^3\cdot 173n - 2^7\cdot 25\binom{n}{2} + 2^{10}\cdot 3\binom{n}{3} \pmod{2^{15}}$$

$$(5.6)\qquad e_{4n+1}^{(2)} \equiv -1 + 2^3\cdot 1877n - 2^7\cdot 75\binom{n}{2} - 2^{10}\cdot 13\binom{n}{3} \pmod{2^{15}}$$

$$(5.7)\qquad e_{4n+2}^{(2)} \equiv 5 + 2^3\cdot 1973n + 2^7\cdot 83\binom{n}{2} - 2^{10}\cdot 13\binom{n}{3} \pmod{2^{15}}$$

$$(5.8)\qquad e_{4n+3}^{(2)} \equiv -61 - 2^3\cdot 51n - 2^7\cdot 95\binom{n}{2} + 2^{10}\cdot 3\binom{n}{3} \pmod{2^{15}}$$

$$(5.9)\qquad e_{n+1}^{(3)} \equiv -1 - 12n - 16\binom{n}{2} - 8\binom{n}{3} \pmod{32}.$$

This congruence implies Theorem 4.1(i) of [**12**].

$$(5.10)\qquad e_{n+2}^{(3)} \equiv 19 + 4n + 2^3\cdot 13\binom{n}{2} + 2^3\cdot 15\binom{n}{3} + 2^5\cdot 3\binom{n}{4} + 2^6\binom{n}{6} \pmod{2^8}$$

$$(5.11)\qquad (-1)^n e_n^{(4)} \equiv 1 + 2^2\cdot 17\binom{n}{2} - 2^4\cdot 5\binom{n}{3} - 2^4\cdot 7\binom{n}{4} + 2^7\binom{n}{5} + 2^6\binom{n}{6} \pmod{2^8}$$

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(5.12) $$e_n^{(4)} \equiv 1 - 2n + 2^3 \cdot 9\binom{n}{2} - 2^4 \cdot 5\binom{n}{3} - 2^7 \cdot 3\binom{n}{4} + 2^8\binom{n}{5} \pmod{2^{10}}.$$

Carlitz [**6**] showed that

$$e_n^{(4)} \equiv e_n^{(2)} \equiv 1 - 2n + 8\binom{n}{2} \pmod{16},$$

and asked for the largest power of 2 dividing $e_n^{(4)} - e_n^{(2)}$ for all n. It is clear from (5.2) and (5.12) that the largest power of 2 is $2^6 = 64$; moreover, $e_n^{(4)} - e_n^{(2)} \equiv 2^6\binom{n}{2} \pmod{2^{10}}$.

(5.13) $$e_{3n}^{(5)} \equiv 1 \pmod 8$$

(5.14) $$e_{3n+1}^{(5)} \equiv -1 + 4n - 2^5\binom{n}{5} \pmod{2^6}$$

(5.15) $$e_{3n+2}^{(5)} \equiv -5 + 2^4 n - 2^3\binom{n}{2} - 2^5\binom{n}{5} \pmod{2^6}$$

(5.16) $$e_{3n+3}^{(5)} \equiv -31 + 2^3 \cdot 3n - 2^4\binom{n}{2} + 2^4\binom{n}{3} - 2^5\binom{n}{4} + 2^5\binom{n}{5} \pmod{2^6}$$

(5.17) $$e_{n+1}^{(6)} \equiv -1 - 4n \pmod{2^5}.$$

This implies Theorem 4.1(ii) of [**12**].

(5.18) $$e_{n+2}^{(6)} \equiv -101 + 2^2 \cdot 63n - 2^5 \cdot 3\binom{n}{2} - 2^5\binom{n}{3} - 2^8\binom{n}{4} + 2^9\binom{n}{5} \pmod{2^{10}}$$

(5.19) $$e_{n+1}^{(7)} \equiv -1 + 2^3 \cdot 13n - 2^5\binom{n}{2} - 2^4 \cdot 5\binom{n}{3} + 2^7\binom{n}{4} - 2^6\binom{n}{5} + 2^7\binom{n}{7} \pmod{2^8}.$$

This implies (1.1).

(5.20) $$e_{n+2}^{(7)} \equiv 359 + 2^3 \cdot 9n + 2^4 \cdot 25\binom{n}{2} + 2^4 \cdot 3\binom{n}{3} - 2^6 \cdot 7\binom{n}{4} - 2^6 \cdot 5\binom{n}{5} - 2^7\binom{n}{6} + 2^7\binom{n}{7} \pmod{2^{10}}$$

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$$(5.21)\quad e_n^{(8)} \equiv 1 - 2n - 2^3 \cdot 55\binom{n}{2} - 2^4 \cdot 5\binom{n}{3} - 2^7 \cdot 3\binom{n}{4} + 2^8\binom{n}{5} \pmod{2^{10}}$$

$$(5.22)\quad e_{n+1}^{(12)} \equiv -1 - 4n \pmod{2^5}.$$

This implies Theorem 4.1(v) of [**12**].

$$(5.23)\quad e_{n+2}^{(12)} \equiv -229 + 2^2 \cdot 31n + 2^5 \cdot 9\binom{n}{2} - 2^5 \cdot 3\binom{n}{3} + 2^8\binom{n}{4} + 2^9\binom{n}{5} \pmod{2^{10}}$$

$$(5.24)\quad e_{n+1}^{(14)} \equiv -1 + 2^3 \cdot 1071n - 2^7 \cdot 89\binom{n}{2} + 2^9 \cdot 43\binom{n}{3} + 2^{10} \cdot 15\binom{n}{4} - 2^{11}\binom{n}{5} - 2^{13} \cdot 3\binom{n}{7} \pmod{2^{16}}$$

$$(5.25)\quad e_n^{(15)} \equiv -1 - 2^4 \cdot 3n - 2^6\binom{n}{2} - 2^5 \cdot 13\binom{n}{3} + 2^8\binom{n}{4} + 2^7 \cdot 3\binom{n}{5} - 2^9\binom{n}{6} - 2^8\binom{n}{7} \pmod{2^{10}}$$

$$(5.26)\quad e_n^{(16)} \equiv 1 - 2n - 2^3 \cdot 55\binom{n}{2} - 2^4 \cdot 5\binom{n}{3} - 2^7 \cdot 3\binom{n}{4} + 2^8\binom{n}{5} \pmod{2^{10}}$$

$$(5.27)\quad e_{n+1}^{(2)} \equiv -1 + 6n \pmod{9}$$

$$(5.28)\quad e_{n+2}^{(2)} \equiv 5 + 3 \cdot 5n - 3^3\binom{n}{2} - 3^3\binom{n}{3} \pmod{3^4}$$

$$(5.29)\quad e_{n+3}^{(2)} \equiv -61 - 3 \cdot 4n - 3^3 \cdot 5\binom{n}{2} + 3^3 \cdot 5\binom{n}{3} - 3^5\binom{n}{4} - 3^5\binom{n}{5} \pmod{3^6}$$

$$(5.30)\quad (-1)^n e_n^{(3)} \equiv 1 + 3^2 \cdot 2\binom{n}{2} + 3^6 \cdot 2\binom{n}{3} + 3^5 \cdot 59\binom{n}{4} + 3^7 \cdot 4\binom{n}{5} + 3^8 \cdot 2\binom{n}{6} \pmod{3^{10}}$$

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$$(5.31)\quad e^{(4)}_{2n+1} \equiv -1 + 3^2 \cdot 4n + 3^2 \cdot 2\binom{n}{2} \pmod{3^4}$$

$$(5.32)\quad e^{(4)}_{2n+2} \equiv 3 \cdot 23 + 3 \cdot 77n + 3^3 \cdot 14\binom{n}{2} + 3^3 \cdot 11\binom{n}{3} - 3^4 \cdot 11\binom{n}{4} - 3^5 \cdot 4\binom{n}{5} \pmod{3^7}$$

$$(5.33)\quad e^{(4)}_{2n+3} \equiv -856 - 3^3 \cdot 19n + 3^2 \cdot 110\binom{n}{2} + 3^5 \cdot 4\binom{n}{3} - 3^4 \cdot 7\binom{n}{4} + 3^5 \cdot 2\binom{n}{5} - 3^6\binom{n}{6} \pmod{3^7}$$

$$(5.34)\quad e^{(6)}_{n+1} \equiv -1 + 3 \cdot 11n + 3^2 \cdot 4\binom{n}{2} \pmod{3^4}$$

$$(5.35)\quad e^{(6)}_{n+2} \equiv 923 + 3 \cdot 239n - 3^2 \cdot 77\binom{n}{2} - 3^4 \cdot 2\binom{n}{3} - 3^5 \cdot 2\binom{n}{4} - 3^6\binom{n}{5} \pmod{3^7}$$

$$(5.36)\quad e^{(8)}_{n+1} \equiv -1 - 3^2 \cdot 28n + 3^5 \cdot 4\binom{n}{2} - 3^4 \cdot 11\binom{n}{3} - 3^4 \cdot 2\binom{n}{4} - 3^5\binom{n}{5} \pmod{3^7}$$

$$(5.37)\quad (-1)^n e^{(9)}_n \equiv 1 + 3^2 \cdot 56\binom{n}{2} - 3^6\binom{n}{3} - 3^5 \cdot 4\binom{n}{4} \pmod{3^7}$$

$$(5.38)\quad e^{(18)}_{n+1} \equiv -1 + 3 \cdot 11n + 3^2 \cdot 4\binom{n}{2} \pmod{3^4}$$

$$(5.39)\quad e^{(2)}_{2n+1} \equiv -1 - 10n \pmod{5^2}$$

$$(5.40)\quad e^{(2)}_{2n+2} \equiv 5 + 5 \cdot 276n - 5^3 \cdot 11\binom{n}{3} \pmod{5^5}$$

$$(5.41)\quad e^{(2)}_{2n+3} \equiv -61 - 5 \cdot 92n - 5^4\binom{n}{2} - 5^3 \cdot 2\binom{n}{3} \pmod{5^5}$$

$$(5.42)\quad e^{(4)}_{n+1} \equiv -1 + 5 \cdot 14n - 5^2 \cdot 2\binom{n}{2} - 5^3\binom{n}{3} \pmod{5^4}$$

$$(5.43)\quad (-1)^n e^{(5)}_n \equiv 1 + 5^3 \cdot 2\binom{n}{2} + 5^6 \cdot 48\binom{n}{3} - 5^4 \cdot 2421\binom{n}{4} + 5^9 \cdot 2\binom{n}{5} + 5^8 \cdot 2\binom{n}{6} + 5^9 \cdot 2\binom{n}{7} - 5^9\binom{n}{8} \pmod{5^{10}}$$

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(5.44) $$e_{2n+1}^{(12)} \equiv -1 - 5^2 \cdot 7n + 5^3 \cdot 2\binom{n}{2} - 5^3 \cdot 2\binom{n}{3} \pmod{5^4}$$

(5.45) $$e_{2n+2}^{(12)} \equiv -5 \cdot 44 - 5 \cdot 14n - 5^2 \cdot 9\binom{n}{2} + 5^3 \cdot 2\binom{n}{3} \pmod{5^4}$$

(5.46) $$e_{n+1}^{(20)} \equiv -1 + 5 \cdot 139n - 5^2 \cdot 2\binom{n}{2} - 5^3\binom{n}{3} + 5^4\binom{n}{4} \pmod{5^5}$$

(5.47) $$e_{n+1}^{(24)} \equiv -1 + 5^2 \cdot 449n + 5^4 \cdot 53\binom{n}{2} + 5^5 \cdot 8\binom{n}{4} \pmod{5^6}$$

(5.48) $$e_{3n+1}^{(2)} \equiv -1 + 14n \pmod{7^2}$$

(5.49) $$e_{3n+2}^{(2)} \equiv 5 - 7 \cdot 15n - 7^2 \cdot 18\binom{n}{2} - 7^3\binom{n}{3} \pmod{7^4}$$

(5.50) $$e_{3n+3}^{(2)} \equiv -61 - 7 \cdot 100n + 7^2 \cdot 22\binom{n}{2} - 7^3 \cdot 2\binom{n}{3} \pmod{7^4}$$

(5.51) $$e_{3n+4}^{(2)} \equiv -1016 - 7 \cdot 19n + 7^2 \cdot 3\binom{n}{2} - 7^3 \cdot 2\binom{n}{3} \pmod{7^4}$$

(5.52) $$e_{2n+1}^{(3)} \equiv -1 + 7 \cdot 127n - 7^2 \cdot 11\binom{n}{2} + 7^3\binom{n}{3} \pmod{7^4}$$

(5.53) $$e_{2n+2}^{(3)} \equiv 19 + 7 \cdot 139n + 7^2 \cdot 2\binom{n}{2} + 7^3\binom{n}{3} \pmod{7^4}$$

(5.54) $$e_{n+1}^{(6)} \equiv -1 + 7 \cdot 132n - 7^2 \cdot 23\binom{n}{2} + 7^3\binom{n}{3} \pmod{7^4}$$

(5.55) $$(-1)^n e_n^{(7)} \equiv 1 + 7^3 \cdot 10\binom{n}{2} + 7^6 \cdot 9\binom{n}{3} - 7^5 \cdot 20\binom{n}{4} + 7^6 \cdot 6\binom{n}{6} \pmod{7^8}$$

(5.56) $$e_{n+1}^{(10)} \equiv -1 - 11 \cdot 507n + 11^2 \cdot 9\binom{n}{2} + 11^3 \cdot 5\binom{n}{3} \pmod{11^4}$$

(5.57) $$(-1)^n e_n^{(11)} \equiv 1 + 11^3 \cdot 530\binom{n}{2} - 11^6 \cdot 37\binom{n}{3} - 11^5 \cdot 492\binom{n}{4} - 11^7 \cdot 3\binom{n}{6} \pmod{11^8}$$

(5.58) $$e_{n+1}^{(12)} \equiv -1 - 13 \cdot 703n + 13^2 \cdot 37\binom{n}{2} - 13^3 \cdot 6\binom{n}{3} \pmod{13^4}$$

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$$(5.59) \quad (-1)^n e_n^{(13)} \equiv 1 + 13^3 \cdot 4734\binom{n}{2} + 13^6 \cdot 5\binom{n}{3} - 13^5 \cdot 55\binom{n}{4} \pmod{13^7}.$$

6. Observations. Although the theorems of Section 5 sometimes give best possible congruences, in many cases they do not. This is most evident for $p = 2$, but is also true for other primes. Frobenius [7] (see also [**4**]) proved that the power of 2 dividing

$$\Delta^n e_k^{(2)} = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} e_{j+k}^{(2)}$$

is $\nu_2(2^n n!)$. Theorem 5.3 implies that $\Delta^n e_k^{(2^s)}$ is divisible by 2^n, but empirical evidence suggests that

$$\nu_2(\Delta^n e_k^{(2^s)}) = \nu_2(2^n n!) \quad \text{for all } s \geqq 1.$$

Frobenius's proof for $s = 1$ does not seem to generalize, as it uses a relation between $e_n^{(2)}$ and the Bernoulli numbers. An even stronger result appears to be true: Let

$$f(n, s) = \nu_2(\Delta^n e_0^{(2^{s+1})} - \Delta^n e_0^{(2^s)}).$$

Then we have the following values for $f(n, s)$ compared with $\nu_2(2^n n!)$.

$s \backslash n$	2	3	4	5	6
1	6	10	12	14	17
2	9	13	18	19	
3	12	16	20		
$\nu_2(2^n n!)$	3	4	7	8	10

There are probably similar results for other primes, although we have numerical information only for $p = 3$. The power of 3 in $\Delta^n(-1)^k e_k^{(3)}$ seems to be at least $2[(n + 1)/2] + \nu_3(n!)$. Moreover, if

$$g(n) = \nu_3(\Delta^n(-1)^0 e_0^{(9)} - \Delta^n(-1)^0 e_0^{(3)}),$$

where $\Delta^n(-1)^0 e_0^{(m)}$ means $[\Delta^n(-1)^k e_k^{(m)}]_{k=0}$, then we have the following table:

n	2	3	4	5	6
$g(n)$	5	9	12	12	14

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Some of the irregularities in the prime powers in (5.1) – (5.59) disappear if instead of looking at $\nu_p(\Delta^n(\pm 1)^k e_k^{(m)})$ for specific values of k, we look at the minimum (or lim inf) over all k. Thus if we set

$$h(p, n) = \min_k \nu_p(\Delta^n(-1)^k e_k^{(p)})$$

we find the following values for $h(p, n)$:

p_n	1	2	3	4	5	6	7	8	9	10
3	2	2	5	5	7	8	10	10	14	14
5	3	3	4	4	8	8	9	9	$\geqq 12$	
7	3	3	5	5	6	6	$\geqq 10$			
11	3	3	5	5	7	7	$\geqq 8$			
13	3	3	5	5	$\geqq 7$					

Apparently $h(3, n) = 2[(n + 1)/2] + \nu_3(n!)$, but it is not clear what $h(p, n)$ is for $p > 3$.

Another curious fact turns out to be easy to prove:

THEOREM 6.1. *For* $p \geqq 3$, $\Delta^3(-1)^0 e_0^{(p^s)}$ *is divisible by* p^6.

Proof. We have

$$\Delta^3(-1)^0 e_0^{(m)} = -[e_3^{(m)} + 3e_2^{(m)} + 3e_1^{(m)} + e_0^{(m)}].$$

We easily find that

$$e_0^{(m)} = 1, \ e_1^{(m)} = -1, \ e_2^{(m)} = \binom{2m}{m} - 1, \quad \text{and}$$

$$e_3^{(m)} = -\binom{3m}{m}\binom{2m}{m} + 2\binom{3m}{m} - 1.$$

Thus

$$\Delta^3(-1)^0 e_0^{(m)} = \left[\binom{3m}{m} - 3\right]\left[\binom{2m}{m} - 2\right].$$

By Theorem 2.2, if $m = p^s$ with $p > 3$ then both factors are divisible by p^3, and if $m = 3^s$, $\binom{3m}{m} - 3$ is divisible by 3^4 and $\binom{2m}{m} - 2$ by 3^2.

For $p = 2$ and m of the form $2^s - 2^r$, the powers of 2 occurring in our congruences are generally greater than expected from the theorems of Section 4, although there seems to be no simple pattern. It is surprising that the most unexpectedly large powers of 2 appear for $m = 14$.

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