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A SIMPLE PROOF OF INTEGRATION BY PARTS FOR THE PERRON INTEGRAL

BY

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ABSTRACT. This note presents a very simple proof for the integration by parts formula for the Perron integral.

1. **Introduction**. It is a remarkable fact that the proof of such an elementary result as the integration by parts formula for an integral with such a simple definition was not given until fifty years after the integral was introduced. To this day the proofs are far from simple. The most elegant, due to Mařík [7], requires a non-standard, but reasonable, form for the definition of the integral; the most direct, that of Gordon and Lasher, [5], needs a special treatment for the case when 'the function to be differentiated' is not continuous; the earliest, McShane's proof, ([8], [9]), is cumbersome in its use of sixteen major and minor functions. The present note combines McShane's approach with some preliminary simplifications to give a short direct proof. A detailed history of this theorem has been given in Bullen [2].

2. **The integral and some simple properties**.

DEFINITION 1. *If* $f\colon [a, b] \to \bar{R}$ *then* f *is Perron integrable* (on $[a, b]$), $f \in P(a,b)$ or just $f \in P$ *if there is no ambiguity, iff*:

(I) $\exists\ m, M\colon [a, b] \to R$ *such that*

(a) m, M *are continuous;*

(b) $m(a) = M(a) = 0$;

(c) $\ell DM \geq f \geq uDm$, n.e.;

(d) $\ell DM > -\infty$, $uDm < \infty$ n.e.;

(ℓD *denotes the lower derivative*, uD *the upper derivative;* n.e. *means nearly everywhere, that is, except on a countable set*);

(II) inf $M(b)$ = sup $m(b)$, *where the* inf *is over all* M *in* (I), *and the* sup *over all* m.

Then the common value in (I) *is written* $P - \int_a^b f$, *the Perron integral of* f *over* $[a, b]$.

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LEMMA 2. (a) *If M, m are as in* (I) *then $M - m$ increasing and non-negative.*

(b) *If $f \epsilon P(a, b)$ then $f \epsilon P(a, x)$, $a \leq x \leq b$, and if $F(x) = P - \int_a^x f$, $a \leq x \leq b$, then $M - F$ and $F - m$ are increasing and non-negative;* (M, m as in (I)).

(c) $f \epsilon P(a, b)$ iff $\forall \epsilon > 0 \exists M$, m *as in* (I) *such that* $0 \leq M(b) - m(b) < \epsilon$.

The proof of Lemma 2 is almost immediate, but see, for instance Bruckner ([1], p. 174).

The functions M, m of (I) are called respectively major and minor functions of f. If in Definition 1 we had used unilateral derivatives we could have defined left, and right, major and minor functions of f. If now M a major function of f of any type, and m a minor function of f of any type, Lemma 2(a) still holds; the proof when M and m are of opposite types, one left and the other right, is a little deeper (see McShane [9], p. 313).

The following generalisation of Lemma 2(c) is due to McShane, ([8], [9]); but the fact that it is elementary to prove is due to Ridder [10].

LEMMA 3. $f \epsilon P(a, b)$ iff $\forall \epsilon > 0 \exists$ μ_1, μ_2, μ_3, μ_4 *being left major, right major, left minor, right minor functions of f respectively, such that* $|\mu_i(b) - \mu_j(b)| < \epsilon$, $1 \leq i, j \leq 4$; *and then*

$$P - \int_a^b f = \inf \{t;\ t = \mu_i(b)\ i = 1 \text{ or } 2\}.$$

$$= \sup \{t;\ t = \mu_i(b),\ i = 3 \text{ or } 4\}.$$

3. **The main result**.

THEOREM 4. *If $f \epsilon P(a, b)$, $F = P - \int f$, G of bounded variation then $fG \epsilon P(a, b)$ and*

$$P - \int_a^b fG = F(b)G(b) - F(a)G(a) - \int_a^b F dG, \tag{1}$$

(The right-hand side of (1) is to be interpreted as follows:

$$G(a) = G(a+); \quad G(b) = G(b-); \quad \int_a^b F dG = \lim_{\alpha \to a+,\, \beta \to b^-} \int_a^b F dG$$

the integrals being Riemann-Stieltjes' integrals.)

We remark that Theorem 4 is trivial if G is constant; further if Theorem 4 holds for G_1 and G_2 then it holds for all $\lambda G_1 + \mu G_2$, λ, $\mu \epsilon R$. Since, if G is of bounded variation, $G = G_1 - G_2$ where G_1 and G_2 are bounded increasing functions, to prove Theorem 4 it is sufficient to prove,

LEMMA 5. *If $f \epsilon P(a, b)$, $F = P - \int f$, G a bounded increasing function,* $G(a) = 0$, *then $fG \epsilon P(a, b)$ and*

$$P - \int_a^b fG = F(b)G(b) - \int_a^b F dG. \tag{2}$$

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PROOF. Let M be a major function of f, (as in (I)), and define

$$R(x) = M(x)G(x) - \int_a^x MdG, \quad a \leq x \leq b.$$

If then x and $x + h$ are in $[a, b]$, there is, by the mean value theorem for the Riemann-Stieljes integral, a y between x and $x + h$ such that

$$(3) \qquad R(x + h) - R(x) = \{M(x + h) - M(y)\}\{G(x + h) - G(x)\} + G(x)\{M(x + h) - M(x)\}$$

hence, since M is continuous, so is R.

From (3) we get on rewriting that

$$(4) \qquad \frac{R(x + h) - R(x)}{h} = G(x + h)\frac{M(x + h) - M(x)}{h} - \frac{G(x + h) - G(x)}{h}\{M(y) - M(x)\}.$$

Since $G \geq 0$, if x is a point at which $G'(x)$ is finite, and so a.e.,

$$\ell DR(x) = G(x)\ell DM(x);$$

hence by (c) of Definition 1

$$\ell DR \geq fG \text{ a.e.}$$

Finally, (4) can be rewritten as

$$\frac{R(x + h) - R(x)}{h} = G(x + h)\left\{\frac{M(x + h) - (x)}{h}\right\} - \{G(x + h) - G(x)\}\left\{\frac{M(y) - M(x)}{y - x}\right\}\left(\frac{y - x}{h}\right);$$

if then $h < 0$, and x is a point at which $\ell DM(x) > -\infty$, and so n.e.,

$$\ell D_{-}R(x) > -\infty;$$

(here ℓD_{-} denotes the left lower derivative); hence

$$\ell D_{-}R > -\infty, \text{ n.e.}$$

Since obviously $R(a) = 0$, we have that R in a left major function of fG.

In a similar way, if m is a minor function of f, then

$$r(x) = m(x)G(x) - \int_a^x mdG, \quad a \leq x \leq b,$$

is a left minor function of fG.

A consideration of the above discussion shows that if G had been a non-positive increasing function, rather than a non-negative one, then R would have been a right minor function of fG; similarly r would have been a right major function. If then $\gamma = \sup_{a \le x \le b} G(x)$, $G^*(x) = G(x) - \gamma$ and

$$R^*(x) = M(x)G^*(x) - \int_a^x MdG$$

we have that R^* is a right minor function of fG^*, and so

$$s(x) = \gamma m(x) + R^*(x)$$

is a right minor function of fG: similarly

$$S(x) = \gamma M(x) + r^*(x),$$

where

$$r^*(x) = mG^*(x) - \int_a^x mdG, \quad a \le x \le b,$$

is a right major function of fG.

If then $\epsilon > 0$ and M, m are chosen so that $M(b) - m(b) < \epsilon$, as is possible by Lemma 2(c), then the tetrad R,S,r,s can be taken as the μ_i, $1 \le i \le 4$, of Lemma 3, and so $fG \epsilon P(a, b)$; further, by Lemma 3, $P - \int_a^h f = \inf R(b)$, the inf being over all R defined above; this gives (2) and completes the proof of Lemma 5.

5. In [3] an integration by parts theorem for the Burkill approximately continuous integral, the P^*_{ap}-integral, was promised; this result has been recently proved by Chakrabarti and Mukhopadhyay [4].

THEOREM 6. *Let $f \epsilon P^*_{ap}(a, b)$, $F = P^*_{ap} - \int f$, g of bounded variation, $G = \int g$; if then $F \epsilon P(a, b)$ it follows that $fG \epsilon P^*_{ap}$ and*

$$P^*_{ap} \int_a^b fG = F(b)G(b) - F(a)G(a) - P - \int_a^b Fg. \tag{5}$$

For this result the difficulties that occur in trying to prove Theorem 4 do not occur; in particular a proof along the lines given above requires no appeal to Lemma 3. This is because $G' = g$ n.e. and so the analogously defined R, see the proof of Lemma 5 above, is a P^*_{ap}-major function, not, as in Lemma 5 merely a left major function. The proof following the lines of that of Theorem 4, but now much simpler, is different to that given in [4]. A further proof can be given using the descriptive definition of the P^*_{ap}-integral (see [2]). The right-hand side of (5) is, as a function of b, $[ACG^*_{ap}]$; its a.e. existing approximate derivative is the integrand on the left-hand side of (5).

The condition $F \epsilon P(a, b)$ ensures the existence of the right-hand side of (5), by Theorem 4; and the need for such a hypothesis is demonstrated in [4].

6. Professor K. Garg has brought to my attention that the above proof of Theorem 4 will provide an integration by parts theorem for two generalisations of the P-integral given by Ionescu-Tulcea [6], and Ridder [10]. If in Definition 1 right lower

derivates are used in the definition of the major functions M, but left upper derivates in the definition of the minor functions m, an integral more general than the P-integral is defined, the $P_{\pm}$-integral say. In an analogous way another generalisation the $P_{\mp}$-integral can be defined. While generalizing the P-integral they are not themselves related by one being more general than the other; both can be given descriptive definitions (see [6], [10]). Clearly the proof of Theorem 4 will also give a proof of a Theorem $4_{\pm}$, a theorem in which the hypothesis "$f \epsilon P, F = P - \int f$" of Theorem 4 is replaced by "$f \epsilon P_{\pm}, F = P_{\pm} - \int f$" and the conclusion is $fG \epsilon P_{\pm}$, with the integral on the left-hand side of (1) being $P_{\pm} - \int_a^b fG$; a similar Theorem $4_{\mp}$ can be obtained in the obvious way.

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