Canad. Math. Bull. Vol. **24** (2) 1981

ON THE RELATION BETWEEN THE LOGARITHMIC AND BOREL-TYPE SUMMABILITY METHODS

BY

D. BORWEIN AND B. WATSON

1. **Introduction.** Suppose throughout that $\{s_n\}$ is a sequence of real numbers, $\lambda > -1, \alpha > 0$, and β is real. Let N be any non-negative integer such that $\alpha N + \beta > 1$.

We are concerned primarily with the logarithmic summability method L and the Borel-type method (B, α, β). Some known results involve the Abel-type summability method A_λ. The methods are defined as follows. Let

$$L(x) = \frac{-1}{\log(1-x)} \sum_{n=0}^{\infty} \frac{s_n}{n+1} x^{n+1},$$

$$S(x) = \alpha e^{-x} \sum_{n=N}^{\infty} \frac{s_n x^{\alpha n + \beta - 1}}{\Gamma(\alpha n + \beta)},$$

$$\sigma(x) = (1-x)^{\lambda+1} \sum_{n=0}^{\infty} s_n \binom{n+\lambda}{n} x^n.$$

If $L(x)(\sigma(x))$ exists for $|x| < 1$ and tends to s as $x \to 1-$, then we say that $\{s_n\}$ is L-convergent (A_λ-convergent) to s and write $s_n \to s(L)(s_n \to s(A_\lambda))$.

If $S(x)$ exists for $x \geq 0$ and tends to s as $x \to \infty$, then we say that $\{s_n\}$ is (B, α, β)-convergent to s and write $s_n \to s(B, \alpha, \beta)$.

The methods A_0 and $(B, 1, 1)$ are the ordinary Abel and Borel exponential methods respectively.

A summability method P is said to be *regular* if $s_n \to s(P)$ whenever $s_n \to s$. The summability methods L, A_λ, and (B, α, β) are all regular. In addition, the following propositions are known.

PROPOSITION 1. *If $s_n \to s(B, \alpha, \beta)$ and $\sum_{n=0}^{\infty} s_n x^n$ converges for $|x| < 1$, then $s_n \to s(A_\lambda)$.*

PROPOSITION 2. *If $s_n \to s(A_\lambda)$, then $s_n \to s(L)$.*

The first of these propositions was proved by Shawyer and Yang in [6], and the second by Borwein in [1]. The converse of each of the above propositions is false.

Supported in part by the National Research Council of Canada, Grants A-2983 and A-4646
Received by the editors August 14, 1978.

https://doi.org/10.4153/CMB-1981-025-0 Published online by Cambridge University Press

Propositions 1 and 2 yield:

PROPOSITION 3. *If* $s_n \to s(B, \alpha, \beta)$ *and* $\sum_{n=0}^{\infty} s_n x^n$ *converges for* $|x| < 1$, *then* $s_n \to s(L)$.

The purpose of this paper is to investigate the reverse problem. That is, assuming the L-convergence of a sequence, what Tauberian condition will imply its (B, α, β)-convergence?

2. **The main theorem.** Suppose that ϕ is a continuous and unboundedly increasing function on $[a, \infty)$.

A real-valued function f on $[a, \infty)$ is said to be *slowly decreasing with respect to* ϕ if $\lim\inf(f(y) - f(x)) \geq 0$ as $y > x \to \infty$ and $\phi(y) - \phi(x) \to 0$, i.e. if, for each $\varepsilon > 0$, there exist positive numbers δ and M such that $f(y) - f(x) > -\varepsilon$ whenever $y > x \geq M$ and $\phi(y) - \phi(x) < \delta$.

For the A_λ and (B, α, β) methods, Shawyer and Yang established the following Tauberian result in [7].

PROPOSITION 4. *If* $s_n \to s(A_\lambda)$ *and* $S(x)$ *is slowly decreasing with respect to* $\log x$, *then* $s_n \to s(B, \alpha, \beta)$.

We established the following result in [4] for the L and A_λ methods.

PROPOSITION 5. *If* $s_n \to s(L)$ *and* $\sigma(x)$ *is slowly decreasing with respect to* $\log\log x$, *then* $s_n \to s(A_\lambda)$.

In the present paper we prove the following Tauberian theorem for the L and (B, α, β) methods.

THEOREM 1. *If* $s_n \to s(L)$ *and* $S(x)$ *is slowly decreasing with respect to* $\log\log x$, *then* $s_n \to s(B, \alpha, \beta)$.

3. **Preliminary results.**

LEMMA 1. $s_n \to s(L)$ *if and only if* $\dfrac{\alpha(n+1)}{\alpha n + \beta - 1} s_n \to s(L)$.

This result is a simple consequence of Lemma 1 in [2].

Let

$$J(t) = \frac{1}{\log t} \int_a^\infty \frac{e^{-u/t}}{u} S(u)\, du \quad \text{for} \quad t \geq a > 1.$$

LEMMA 2. (i) *If* $s_n \to s(L)$, *then* $J(t) \to s$ *as* $t \to \infty$.

(ii) *If* $L(x)$ *exists for* $|x| < 1$ *and* $J(t) \to s$ *as* $t \to \infty$, *then* $s_n \to s(L)$.

Proof. Suppose that $L(x)$ exists for $|x|\le 1$. *Then* $s_n=O(c^n)$ for $c>1$, and hence $S(x)$ exists for all $x\ge 0$.

Let

$$I(t)=\frac{1}{\log t}\int_0^a \frac{e^{-u/t}}{u}S(u)\,du \quad \text{for} \quad t\ge a.$$

Then

$$|I(t)|\le \frac{1}{\log t}\int_0^a \left|\frac{S(u)}{u}\right| du\to 0 \quad \text{as} \quad t\to\infty,$$

since $S(u)=O(u^{\alpha N+\beta-1})$ in $(0, a)$ and $\alpha N+\beta-1>0$.

Next we have, for $t\ge a$,

$$\begin{aligned}
I(t)+J(t)&=\frac{1}{\log t}\int_0^\infty \frac{e^{-u/t}}{u}\alpha e^{-u}\sum_{n=N}^\infty \frac{s_n u^{\alpha n+\beta-1}}{\Gamma(\alpha n+\beta)}du\\
&=\frac{1}{\log t}\sum_{n=N}^\infty \frac{\alpha s_n}{(\alpha n+\beta-1)\Gamma(\alpha n+\beta-1)}\\
&\quad\times\int_0^\infty e^{-u(1+t)/t}u^{\alpha n+\beta-2}\,du\\
&=\frac{1}{\log t}\sum_{n=N}^\infty \frac{\alpha(n+1)s_n}{(\alpha n+\beta-1)(n+1)}\left(\frac{t}{1+t}\right)^{\alpha n+\beta-1}\\
&=\left(\frac{t}{1+t}\right)^{\beta-1-\alpha}\frac{-\log(1-T)}{\log t}\cdot\frac{-1}{\log(1-T)}\\
&\quad\times\sum_{n=N}^\infty \frac{\alpha(n+1)}{\alpha n+\beta-1}\frac{s_n}{n+1}T^{n+1}
\end{aligned}$$

where $T=[t/(1+t)]^\alpha$, the inversion being justified since the final series is absolutely convergent. Also $(t/1+t)^{\beta-1-\alpha}$ and $-(\log(1-T)/\log t)$ tend to 1 as $t\to\infty$. In view of Lemma 1, the desired results follow.

LEMMA 3. *Let* $\gamma>1, t>1, a>0$. *Then*

(i) $$\frac{1}{\log t}\int_t^\infty \frac{e^{-u/t}}{u}du\to 0 \quad \textit{as} \quad t\to\infty,$$

(ii) $$\frac{1}{\log t}\int_a^t \frac{e^{-u/t}}{u}du\to 1 \quad \textit{as} \quad t\to\infty,$$

(iii) $$0<\int_a^t (e^{-u/t^\gamma}-e^{-u/t})\frac{du}{u}<1,$$

and

(iv) $$\frac{1}{\log t}\int_t^{t^\gamma}\frac{e^{-u/t^\gamma}}{u}du\to\gamma-1 \ \textit{as}\ t\to\infty.$$

https://doi.org/10.4153/CMB-1981-025-0 Published online by Cambridge University Press

Proof. (i) $0<\frac{1}{\log t}\int_t^\infty \frac{e^{-u/t}}{u}\,du=\frac{1}{\log t}\int_1^\infty \frac{e^{-v}}{v}\,dv\to 0$ as $t\to\infty$.

(ii) $$\frac{1}{\log t}\int_a^t \frac{e^{-u/t}}{u}\,du=\frac{1}{\log t}\int_{a/t}^1 \frac{e^{-v}}{v}\,dv=e^{-a/t}\left(1-\frac{\log a}{\log t}\right)+\frac{1}{\log t}\times\int_{a/t}^1 e^{-v}\log v\,dv\to 1 \quad \text{as} \quad t\to\infty.$$

(iii) By the mean-value theorem,

$$\begin{aligned}0<\int_a^t (e^{-u/t^\gamma}-e^{-u/t})\frac{du}{u}&\le\int_a^t\left(\frac{u}{t}-\frac{u}{t^\gamma}\right)\frac{du}{u}\\&<\left(\frac{1}{t}-\frac{1}{t^\gamma}\right)\int_0^t du\\&=1-t^{1-\gamma}<1.\end{aligned}$$

(iv) By parts (ii) and (iii),

$$\frac{1}{\log t}\int_t^{t^\gamma}\frac{e^{-u/t^\gamma}}{u}\,du=\frac{\gamma}{\log t^\gamma}\int_a^{t^\gamma}\frac{e^{-u/t^\gamma}}{u}\,du-\frac{1}{\log t}\int_a^t\frac{e^{-u/t}}{u}\,du-\frac{1}{\log t}\times\int_a^t(e^{-u/t^\gamma}-e^{-u/t})\frac{du}{u}\to\gamma-1 \quad \text{as} \quad t\to\infty.$$

4. **A general Tauberian result.**

THEOREM 2. *Suppose that the following conditions hold:*

(1) *$K(t, u)$ is defined, real-valued, and non-negative for $t>a$, $u\ge a$; moreover, $\int_a^\infty K(t, u)\,du$ exists in the sense of Lebesgue for each $t>a$,*

(2) *$\int_a^\infty K(t, u)\,du\to 1$ as $t\to\infty$,*

(3) *f is real-valued and continuous on $[a, \infty)$,*

(4) *$F(t)=\int_a^\infty K(t, u)f(u)\,du$ exists in the Cauchy-Lebesgue sense for each $t>a$,*

(5) *f is slowly decreasing with respect to ϕ,*

(6) *$\phi(t)-\phi(t-1)\to 0$ as $t\to\infty$,*

(7) *$\int_a^x K(t, u)\,du\to 0$ whenever $t\ge x\to\infty$ and $\phi(t)-\phi(x)\to\infty$,*

(8) *$\int_x^\infty K(t, u)(\phi(u)-\phi(x))\,du\to 0$ whenever $x\ge t\to\infty$ and $\phi(x)-\phi(t)\to\infty$, and*

(9) *$F(t)=O(1)$ for $t>a$.*

Then $f(u)=O(1)$ for $u>a$.

This result was established in [3].

https://doi.org/10.4153/CMB-1981-025-0 Published online by Cambridge University Press

5. **Proof of Theorem 1.** Set $a=1+e^e$,

$$K(t,u)=\begin{cases}\dfrac{1}{\log t}\dfrac{e^{-u/t}}{u} & \text{for} \quad t\ge a, u\ge a,\\ 0 & \text{otherwise,}\end{cases}$$

$$\phi(t)=\log\log t \qquad \text{for} \quad t\ge a,$$

$$f(u)=S(u) \qquad \text{for} \quad u\ge a.$$

Then

$$\int_a^\infty K(t,u)f(u)\,du=J(t) \quad \text{for} \quad t\ge a.$$

We first show that the conditions of Theorem 1 imply that $S(u)=O(1)$ for $u>a$.

Conditions (1), (3), (5), and (6) clearly hold, and $\int_a^\infty K(t,u)\,du\to 1$ as $t\to\infty$ by parts (i) and (ii) of Lemma 3. Furthermore, the L-convergence of $\{s_n\}$ and Lemma 2 guarantee that $F(t)$ exists and is bounded for $t>a$. In view of Theorem 2, to establish the boundedness of $S(u)$ in (a,∞) it suffices to prove that (7) and (8) hold.

To show that (7) holds, we observe that

$$\int_a^\infty K(t,u)\,du\le\frac{1}{\log t}\int_a^x\frac{du}{u}=\frac{\log x-\log a}{\log t}\to 0 \quad \text{as} \quad t\ge x\to\infty$$

and

$$\log\log t-\log\log x\to\infty.$$

To show that (8) holds, we note that

$$\begin{aligned}\int_x^\infty K(t,u)(\phi(u)-\phi(x))\,du&=\frac{1}{\log t}\int_x^\infty\frac{e^{-u/t}}{u}(\log\log u-\log\log x)\,du\\&\le\frac{1}{\log t}\int_x^\infty\frac{e^{-u/t}}{u}\left(\frac{u-x}{x\log x}\right)du\\&\le\frac{1}{x\log x\log t}\int_x^\infty e^{-u/t}\,du\\&=\frac{te^{-x/t}}{x\log x\log t}\to 0 \quad \text{as} \quad x\ge t\to\infty.\end{aligned}$$

Suppose, as we may without loss of generality, that $s_n\to 0(L)$. Then, by Lemma 2, $J(t)\to 0$ as $t\to\infty$. It remains to show that $S(u)\to 0$ as $u\to\infty$.

Assign $\varepsilon>0$. Since $S(u)$ is slowly decreasing with respect to ϕ, there exist numbers $x\ge a$ and $\delta>0$ such that $S(u)-S(t)>-\varepsilon$ whenever $u>t\ge x$ and $\log\log u-\log\log t<\delta$. Equivalently, setting $\gamma=e^\delta$,

$$S(t)-\varepsilon<S(u) \quad \text{whenever} \quad x<t<u<t^\gamma. \tag{10}$$

https://doi.org/10.4153/CMB-1981-025-0 Published online by Cambridge University Press

Relation (10) implies that, for $t > x$,

$$I_1 = \frac{1}{\log t} \int_t^{t^\gamma} \frac{e^{-u/t^\gamma}}{u} (S(t) - \varepsilon)\, du$$
$$\leq \frac{1}{\log t} \int_t^{t^\gamma} \frac{e^{-u/t^\gamma}}{u} S(u)\, du = I_2.$$

Now, by Lemma 3, and the fact that $S(u) = O(1)$,

$$I_2 = \gamma J(t^\gamma) - J(t) - \frac{\gamma}{\log t^\gamma} \int_{t^\gamma}^\infty \frac{e^{-u/t^\gamma}}{u} S(u)\, du$$
$$- \frac{1}{\log t} \int_a^t \frac{e^{-u/t^\gamma} - e^{-u/t}}{u} S(u)\, du + \frac{1}{\log t} \int_t^\infty \frac{e^{-u/t}}{u} S(u)\, du$$
$$= o(1) \quad \text{as} \quad t \to \infty.$$

Further, by part (iv) of Lemma 3,

$$I_1 = (S(t) - \varepsilon)(\gamma - 1 + o(1)).$$

Hence

$$S(t) - \varepsilon \leq \frac{I_2}{\gamma - 1 + o(1)} = o(1),$$

and therefore

(11) $$\limsup_{t \to \infty} S(t) \leq \varepsilon.$$

Rewriting (10) we get

(12) $$S(u) < S(t) + \varepsilon \quad \text{whenever} \quad x < t^{1/\gamma} < u < t.$$

Relation (12) implies that, for $t^{1/\gamma} \geq x$,

$$I_3 = \frac{1}{\log t} \int_{t^{1/\gamma}}^t \frac{e^{-u/t}}{u} S(u)\, du$$
$$\leq \frac{1}{\log t} \int_{t^{1/\gamma}}^t \frac{e^{-u/t}}{u} (S(t) + \varepsilon)\, du = I_4.$$

By Lemma 3 (with t replaced by $t^{1/\gamma}$) and the fact that $S(u) = O(1)$,

$$I_3 = J(t) - \frac{1}{\gamma} J(t^{1/\gamma}) - \frac{1}{\log t} \int_t^\infty \frac{e^{-u/t}}{u} S(u)\, du$$
$$- \frac{1}{\log t} \int_a^{t^{1/\gamma}} \frac{e^{-u/t} - e^{-u/t^{1/\gamma}}}{u} S(u)\, du + \frac{1}{\log t} \int_{t^{1/\gamma}}^\infty \frac{e^{-u/t^{1/\gamma}}}{u} S(u)\, du$$
$$= o(1) \quad \text{as} \quad t \to \infty.$$

https://doi.org/10.4153/CMB-1981-025-0 Published online by Cambridge University Press

Also

$$I_4 = (S(t)+\varepsilon)\left(1-\frac{1}{\gamma}+o(1)\right).$$

Hence

$$S(t)+\varepsilon \geq \frac{I_3}{1-1/\gamma+o(1)} = o(1),$$

and therefore

(13) $$\liminf_{t\to\infty} S(t) \geq -\varepsilon.$$

It follows from (11) and (13) that $S(t)\to 0$ as $t\to\infty$, and this completes the proof.

REFERENCES

1. D. Borwein, *On methods of summability based on power series*, Proc. Royal Soc. Edinburgh 64 (1957), 342–349.

2. ——, *A logarithmic method of summability*, Journal London Math. Soc. 33 (1958), 212–220.

3. —— and B. Watson, *Tauberian theorems on a scale of Abel-type summability methods*, Journal für die Reine und Angewandte Mathematik 298 (1978), 1–7.

4. ——, *Tauberian theorems between the logarithmic and Abel-type summability methods*, submitted for publication.

5. G. H. Hardy, *Divergent Series*, Oxford (1949).

6. B. L. R. Shawyer and G. S. Yang, *On the relation between the Abel-type and Borel-type methods of summability*, Proc. Amer. Math. Soc. 26 (1970), 323–328.

7. ——, *Tauberian relations between the Abel-type and the Borel-type methods of summability*, Manuscripta Math. 5 (1971), 341–357.

THE UNIVERSITY OF WESTERN ONTARIO
LONDON, ONTARIO, CANADA

MEMORIAL UNIVERSITY OF NEWFOUNDLAND
ST. JOHN'S, NEWFOUNDLAND, CANADA