NEIGHBOURHOOD AND THE EXISTENCE OF FRACTIONAL $k$-FACTORS OF GRAPHS

SIZHONG ZHOU, BINGYUAN PU and YANG XU

(Received 3 August 2009)

Abstract

Let $G$ be a graph, and $k$ a positive integer. Let $h : E(G) \to [0, 1]$ be a function. If $\sum_{e \ni x} h(e) = k$ holds for each $x \in V(G)$, then we call $G[F_h]$ a fractional $k$-factor of $G$ with indicator function $h$ where $F_h = \{ e \in E(G) \mid h(e) > 0 \}$. In this paper we use neighbourhoods to obtain a new sufficient condition for a graph to have a fractional $k$-factor. Furthermore, this result is shown to be best possible in some sense.

2000 Mathematics subject classification: primary 05C70.

Keywords and phrases: graph, neighbourhood, fractional $k$-factor.

1. Introduction

In this paper we consider only finite undirected graphs which have neither loops nor multiple edges. We refer the readers to [1] for the terminology not defined here. Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$. For each $x \in V(G)$, we use $d_G(x)$ to denote the degree of $x$ in $G$, and $N_G(x)$ to denote the neighbourhood of $x$ in $G$. For a subset $X$ of $V(G)$, we define the neighbourhood of $X$ as

$$N_G(X) = \bigcup_{x \in X} N_G(x).$$

Note that $N_G(x)$ does not contain $x$, but it may happen that $N_G(X) \supseteq X$. For any $S \subseteq V(G)$, we use $G[S]$ and $G - S$ to denote the subgraph of $G$ induced by $S$ and $V(G) - S$, respectively. A vertex set $S \subseteq V(G)$ is called independent if $G[S]$ has no edges. Let $S$ and $T$ be two disjoint subsets of $V(G)$; we denote by $E_G(S, T)$ the set of edges with one end in $S$ and the other end in $T$, and $e_G(S, T) = |E_G(S, T)|$. We denote the minimum degree of $G$ by $\delta(G)$. Let $r$ be a real number. Recall that $\lfloor r \rfloor$ is the greatest integer such that $\lfloor r \rfloor \leq r$.

Let $k$ be an integer such that $k \geq 1$. Then a spanning subgraph $F$ of $G$ is called a $k$-factor if $d_F(x) = k$ for each $x \in V(G)$. Let $h : E(G) \to [0, 1]$ be a function. If

This research was sponsored by Qing Lan Project of Jiangsu Province and was supported by Jiangsu Provincial Educational Department (07KJD110048) and Sichuan Provincial Educational Department (08zb068).

© 2010 Australian Mathematical Publishing Association Inc. 0004-9727/2010 $16.00
\[ \sum_{e \in x} h(e) = k \] holds for each \( x \in V(G) \), then we call \( G[F_h] \) a fractional \( k \)-factor of \( G \) with indicator function \( h \) where \( F_h = \{ e \in E(G) \mid h(e) > 0 \} \).

Many authors have investigated graph factors \([2, 5, 6, 8, 12]\). Liu and Zhang \([3]\) obtained a necessary and sufficient condition for a graph to have a fractional \( k \)-factor. Liu and Zhang \([4]\) gave a toughness condition for a graph to have a fractional \( k \)-factor. Zhou \([9–11]\) gave some other sufficient conditions for graphs to have fractional \( k \)-factors. Yu et al. \([7]\) obtained a degree condition for a graph to have a fractional \( k \)-factor.

The following results on fractional \( k \)-factors are known.

**Theorem 1.1** \([4]\). Let \( k \geq 2 \) be an integer. A graph \( G \) of order \( n \) with \( n \geq k + 1 \) has a fractional \( k \)-factor if its toughness \( t(G) \geq k - 1/k \).

**Theorem 1.2** \([7]\). Let \( k \) be an integer with \( k \geq 1 \), and let \( G \) be a connected graph of order \( n \) with \( n \geq 4k - 3, \delta(G) \geq k \). If

\[
\max\{d_G(x), d_G(y)\} \geq \frac{n}{2}
\]

for each pair of nonadjacent vertices \( x, y \) of \( G \), then \( G \) has a fractional \( k \)-factor.

**Theorem 1.3** \([10]\). Let \( k \) be an integer such that \( k \geq 1 \), and let \( G \) be a connected graph of order \( n \) such that \( n \geq 9k - 1 - 4\sqrt{2(k-1)^2 + 2} \), and the minimum degree \( \delta(G) \geq k \). If

\[
|N_G(x) \cup N_G(y)| \geq \max\left\{ \frac{n}{2}, \frac{1}{2}(n + k - 2) \right\}
\]

for each pair of nonadjacent vertices \( x, y \in V(G) \), then \( G \) has a fractional \( k \)-factor.

**Theorem 1.4** \([9]\). Let \( k \) be a positive integer and \( G \) a graph of order \( n \) with \( n \geq 4k - 6 \). Then:

(a) if \( k \) is even and

\[
|N_G(X)| \geq \frac{(k-1)n + |X| - 1}{2k - 1}
\]

for every nonempty independent subset \( X \) of \( V(G) \), and

\[
\delta(G) \geq \frac{k - 1}{2k - 1}(n + 2),
\]

then \( G \) has a fractional \( k \)-factor; and

(b) if \( k \) is odd, and

\[
|N_G(X)| > \frac{(k-1)n + |X| - 1}{2k - 1}
\]

for every nonempty independent subset \( X \) of \( V(G) \), and

\[
\delta(G) > \frac{k - 1}{2k - 1}(n + 2),
\]

then \( G \) has a fractional \( k \)-factor.
In this paper we use neighbourhoods to obtain a new sufficient condition for a graph to have a fractional $k$-factor. The main result is the following theorem.

**Theorem 1.5.** Let $k$ be an integer with $k \geq 1$, and let $G$ be a graph of order $n$ with $n \geq 6k - 12 + 6/k$. Suppose, for any subset $X \subset V(G)$, that

$$N_G(X) = V(G) \quad \text{if } |X| \geq \left\lceil \frac{kn}{2k-1} \right\rceil; \quad \text{or}$$

$$|N_G(X)| \geq \frac{2k-1}{k} |X| \quad \text{if } |X| < \left\lceil \frac{kn}{2k-1} \right\rceil.$$

Then $G$ has a fractional $k$-factor.

**2. The Proof of Theorem 1.5**

The proof of Theorem 1.5 relies heavily on the following lemmas.

**Lemma 2.1 [3].** Let $G$ be a graph. Then a graph $G$ has a fractional $k$-factor if and only if for every subset $S$ of $V(G)$,

$$\delta_G(S, T) = k|S| + d_{G-S}(T) - k|T| \geq 0,$$

where $T = \{x : x \in V(G)\setminus S, d_{G-S}(x) \leq k - 1\}$ and $d_{G-S}(T) = \sum_{x \in T} d_{G-S}(x)$.

**Lemma 2.2.** Let $G$ be a graph of order $n$ which satisfies the assumption of Theorem 1.5. Then $\delta(G) \geq ((k - 1)n + k)/(2k - 1)$.

**Proof.** Let $x$ be a vertex of $G$ with degree $\delta(G)$. Set $X = V(G)\setminus N_G(x)$. Obviously, $x \notin N_G(X) \text{ and } N_G(X) \neq V(G)$. Thus, we obtain

$$n - 1 \geq |N_G(X)| \geq \frac{2k-1}{k} |X|,$$

that is,

$$(2k - 1)|X| \leq k(n - 1). \quad (2.1)$$

Using (2.1) and $|X| = n - \delta(G)$,

$$(2k - 1)(n - \delta(G)) \leq k(n - 1).$$

Hence,

$$\delta(G) \geq \frac{(k - 1)n + k}{2k - 1}.$$  

This completes the proof of Lemma 2.2. \qed

**Proof of Theorem 1.5.** Let $G$ be a graph satisfying the hypotheses of Theorem 1.5, which has no fractional $k$-factor. Then by Lemma 2.1, there exists some $S \subseteq V(G)$ such that

$$\delta_G(S, T) = k|S| + d_{G-S}(T) - k|T| \leq -1 \quad (2.2)$$
where $T = \{x : x \in V(G) \setminus S, d_{G-S}(x) \leq k - 1\}$. Obviously, $T \neq \emptyset$ by (2.2). Define

$$h = \min\{d_{G-S}(t) \mid t \in T\}.$$ 

From the definition of $T$, we obtain

$$0 \leq h \leq k - 1.$$ 

**Case 1.** $2 \leq h \leq k - 1$.

In terms of Lemma 2.2 and the definition of $h$, we get

$$|S| + h \geq \delta(G) \geq \frac{(k - 1)n + k}{2k - 1}. \quad (2.3)$$

According to (2.2) and $|S| + |T| \leq n$, we obtain

$$-1 \geq \delta_G(S, T) = k|S| + d_{G-S}(T) - k|T|$$

$$\geq k|S| + h|T| - k|T|$$

$$= k|S| - (k - h)|T|$$

$$\geq k|S| - (k - h)(n - |S|)$$

$$= (2k - h)|S| - (k - h)n.$$  

This inequality implies that

$$|S| \leq \frac{(k - h)n - 1}{2k - h}. \quad (2.4)$$

From (2.3) and (2.4),

$$\frac{(k - 1)n + k}{2k - 1} \leq \delta(G) \leq |S| + h \leq \frac{(k - h)n - 1}{2k - h} + h. \quad (2.5)$$

If the left-hand and right-hand sides of (2.5) are denoted by $A$ and $B$ respectively, then (2.5) says that $A - B \leq 0$. But, after some rearranging, we find that

$$(2k - 1)(2k - h)(A - B) = (h - 1)(kn - (2k - 1)(2k - h) + k - 1)$$

$$- 2k^2 + 5k - 2. \quad (2.6)$$

Since $n \geq 6k - 12 + 6/k$, we obtain

$$kn - (2k - 1)(2k - 2) + k - 1 \geq 2k^2 - 5k + 3 \geq 0. \quad (2.7)$$

Using (2.6), (2.7), $2 \leq h \leq k - 1$ and $n \geq 6k - 12 + 6/k$, we get

$$(2k - 1)(2k - h)(A - B)$$

$$= (h - 1)(kn - (2k - 1)(2k - h) + k - 1) - 2k^2 + 5k - 2$$

$$\geq (h - 1)(kn - (2k - 1)(2k - 2) + k - 1) - 2k^2 + 5k - 2$$

$$\geq kn - (2k - 1)(2k - 2) + k - 1 - 2k^2 + 5k - 2$$

$$= kn - 6k^2 + 12k - 5 \geq 1.$$
This inequality implies that
\[ A - B > 0, \]
which contradicts \( A - B \leq 0. \)

**Case 2.** \( h = 1. \)

**Subcase 2.1.** \( |T| \geq \lfloor kn/(2k - 1) \rfloor + 1. \)

In terms of the definition of \( h \) and \( h = 1 \), there exists \( t \in T \) such that \( d_{G-S}(t) = h = 1. \) Thus, we obtain
\[ t \notin N_G(T \setminus N_G(t)), \]
which implies that
\[ N_G(T \setminus N_G(t)) \neq V(G). \]  
(2.8)

On the other hand, using \( |T| \geq \lfloor kn/(2k - 1) \rfloor + 1 \) and \( d_{G-S}(t) = 1, \)
\[ |T \setminus N_G(t)| \geq |T| - 1 \geq \left\lfloor \frac{kn}{2k - 1} \right\rfloor. \]

Combined with the condition of Theorem 1.5, the inequality above implies that
\[ N_G(T \setminus N_G(t)) = V(G), \]
which contradicts (2.8).

**Subcase 2.2.** \( |T| \leq \lfloor kn/(2k - 1) \rfloor. \)
Since \( h = 1 \), there exists \( u \in T \) such that \( d_{G-S}(u) = 1. \) Thus, from Lemma 2.2,
\[ |S| + 1 = |S| + d_{G-S}(u) \geq d_G(u) \geq \delta(G) \geq \frac{(k - 1)n + k}{2k - 1}, \]
that is,
\[ |S| \geq \frac{(k - 1)n + k}{2k - 1} - 1 = \frac{(k - 1)(n - 1)}{2k - 1}. \]  
(2.9)

**Subcase 2.2.1.** \( |T| > (k(n - 1))/(2k - 1). \)

In terms of (2.9) and \( |T| > (k(n - 1))/(2k - 1) \), we get
\[ |S| + |T| > \frac{(k - 1)(n - 1)}{2k - 1} + \frac{k(n - 1)}{2k - 1} = n - 1. \]

Combining this with \( |S| + |T| \leq n, \) we obtain
\[ |S| + |T| = n. \]  
(2.10)

According to (2.2), (2.10) and \( |T| \leq \lfloor kn/(2k - 1) \rfloor \leq kn/(2k - 1), \)
\[ -1 \geq \delta_G(S, T) = k|S| + d_{G-S}(T) - k|T| \geq k|S| + |T| - k|T| = k|S| - (k - 1)|T| = k(n - |T|) - (k - 1)|T| = kn - (2k - 1)|T| \]
\[ \geq kn - (2k - 1) \cdot \frac{kn}{2k - 1} = 0, \]

which is a contradiction.

**Subcase 2.2.2.** \(|T| \leq (k(n - 1))/(2k - 1)\).

Since \(k - 1 \geq h = 1\), we obtain \(k \geq 2\) in this case. Set

\[ p = |\{ t : t \in T, d_{G-S}(t) = 1 \}|. \]

Clearly, \(|T| \geq p\). Combining this with (2.9) and \(k \geq 2\) and \(|T| \leq (k(n - 1))/(2k - 1)\), we obtain

\[ \delta_G(S, T) = k|S| + d_{G-S}(T) - k|T| \]
\[ \geq k|S| + 2(|T| - p) + p - k|T| \]
\[ = k|S| - (k - 2)|T| - p \]
\[ \geq k \cdot \frac{(k - 1)(n - 1)}{2k - 1} - (k - 2) \cdot \frac{k(n - 1)}{2k - 1} - p \]
\[ = \frac{k(n - 1)}{2k - 1} - p \]
\[ \geq |T| - p \geq 0. \]

This contradicts (2.2).

**Case 3.** \(h = 0\).

Let \(m\) be the number of vertices \(x\) in \(T\) such that \(d_{G-S}(x) = 0\). Clearly, \(m \geq 1\) since \(h = 0\). Set \(Y = V(G) \setminus S\). Then \(N_G(Y) \neq V(G)\) since \(h = 0\).

**Claim 1.** \(|Y| < \lfloor kn/(2k - 1) \rfloor\).

If \(|Y| \geq \lfloor (kn/(2k - 1)) \rfloor\), then by the condition of Theorem 1.5 we have \(N_G(Y) = V(G)\). This contradicts \(N_G(Y) \neq V(G)\) and proves Claim 1.

In terms of Claim 1 and the condition of Theorem 1.5, we obtain

\[ n - m \geq |N_G(Y)| \geq \frac{2k - 1}{k} |Y| = \frac{2k - 1}{k} (n - |S|). \]

This inequality implies that

\[ |S| \geq \frac{(k - 1)n + km}{2k - 1}. \quad (2.11) \]

From (2.2), (2.11), \(m \geq 1\) and the fact that \(|S| + |T| \leq n\),

\[ -1 \geq \delta_G(S, T) = k|S| + d_{G-S}(T) - k|T| \]
\[ \geq k|S| + |T| - m - k|T| \]
\[ = k|S| - (k - 1)|T| - m \]
\[ \geq k|S| - (k - 1)(n - |S|) - m \]
\[ = (2k - 1)|S| - (k - 1)n - m \]
This is a contradiction.

In all the cases above, we deduced contradictions. Hence, $G$ has a fractional $k$-factor. This completes the proof of Theorem 1.5. $\square$

**Remark 2.3.** Let us show that the condition in Theorem 1.5 cannot be replaced by the condition that $N_G(X) = V(G)$ or $|N_G(X)| \geq ((2k-1)/k)|X|$ for all $X \subseteq V(G)$. Let $k$ be an odd integer with $k \geq 2$. Let $m$ be any odd positive integer. We construct a graph $G$ of order $n$ as follows. Let $V(G) = S \cup T$ (disjoint union), $|S| = (k-1)m$ and $|T| = km + 1$, and put $T = \{t_1, t_2, \ldots, t_{2l}\}$, where $2l = km + 1$. For each $s \in S$, define $N_G(s) = V(G) \setminus \{s\}$, and for any $t \in T$, define $N_G(t) = S \cup \{t_i\}$, where $\{t, t_i\} = \{t_{2i-1}, t_{2i}\}$ for some $i$, $1 \leq i \leq l$. Obviously, $n = (2k-1)m + 1$. We first show that the condition that $N_G(X) = V(G)$ or $|N_G(X)| \geq ((2k-1)/k)|X|$ for all $X \subseteq V(G)$ holds. Let any $X \subseteq V(G)$. It is obvious that if $|X \cap S| \geq 2$, or $|X \cap S| = 1$ and $|X \cap T| \geq 1$, then $N_G(X) = V(G)$. Of course, if $|X| = 1$ and $X \subseteq S$, then

$$|N_G(X)| = |V(G)| - 1 = n - 1 > \frac{n - 1}{km} = \frac{(2k-1)m}{km} = \frac{2k-1}{k}m = \frac{2k-1}{k}X.$$ 

Hence, we may assume that $X \subseteq T$. Since $|N_G(X)| = |S| + |X| = (k-1)m + |X|$, $|N_G(X)| \geq ((2k-1)/k)|X|$ holds if and only if $(k-1)m + |X| \geq ((2k-1)/k)|X|$. This inequality is equivalent to $|X| \leq km$. Thus if $X \neq T$ and $X \subset T$, then $|N_G(X)| \geq ((2k-1)/k)|X|$ holds for all $X \subseteq V(G)$. If $X = T$, then $N_G(X) = V(G)$. Consequently, $N_G(X) = V(G)$ or $|N_G(X)| \geq ((2k-1)/k)|X|$ for all $X \subseteq V(G)$ follows. In the following, we show that $G$ has no fractional $k$-factor. For above $S$ and $T$, obviously, $d_{G-S}(t) = 1$ for each $t \in T$. Thus, we obtain

$$\delta_G(S, T) = k|S| + d_{G-S}(T) - k|T|$$

$$= k|S| + |T| - k|T|$$

$$= k|S| - (k-1)|T|$$

$$= k(k-1)m - (k-1)(km + 1)$$

$$= -(k-1) \leq -1.$$ 

In terms of Lemma 2.1, $G$ has no fractional $k$-factor. In the above sense, the condition in Theorem 1.5 is best possible.

**References**


SIZHONG ZHOU, School of Mathematics and Physics, Jiangsu University of Science and Technology, Mengxi Road 2, Zhenjiang, Jiangsu 212003, PR China
e-mail: zsz_cumt@163.com

BINGYUAN PU, Department of Fundamental Course, Chengdu Textile College, Chengdu, Sichuan 611731, PR China

YANG XU, Department of Mathematics, Qingdao Agricultural University, Qingdao, Shandong 266109, PR China
e-mail: xuyang_825@126.com