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THE HAUSDORFF AND BOX DIMENSION OF FRACTALS WITH DISJOINT PROJECTIONS IN TWO DIMENSIONS

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Abstract. In this paper, we obtain an exact formula for the Hausdorff and box dimensions of a class of self-affine sets in two dimensions, namely those with disjoint projections. We prove, in particular, that fractals in this class have a Hausdorff and box dimension that is equal to the maximum Hausdorff and box dimension of one of their projections.

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1. Introduction. Measuring the length or volume of fractals (using the conventional definitions) often provides us with the dilemma of having either infinite length or zero volume. This consequently gives us no quantitative idea of the size of such objects. Alternative definitions of the dimension of objects have been given by several mathematicians with the aim of overcoming the problems posed by quantifying figures such as Sierpinski's Sieve. An ingenious definition of dimension was provided by Felix Hausdorff—it incorporates many of the characteristics of the conventional definition without adopting the problems posed by it. Another definition of dimension, which is sometimes equal to Hausdorff dimension (and is generally easier to calculate) is the box dimension.

In this paper we consider the box and Hausdorff dimension of self-affine fractals in $\mathbb{R}^2$ with disjoint projections.

The basic properties of attractors of Iterated Function Schemes were first set out in a paper by J. Hutchinson [**6**].

M. Pollicott and H. Weiss [**11**] calculated the box dimension and Hausdorff dimension for a dynamically constructed model in the plane with two distinct contraction coefficients.

The box and Hausdorff dimensions were also calculated by C. McMullen [**10**] for a family $\overline{R}$ of planar sets which are generalizations of the classical Cantor set. McMullen began his proof by reformulating the dimension of $\overline{R}$ in terms of coverings by a selected class of rectangles. The covering problem was then lifted to a sequence space through a map $\psi : S_r \to \overline{R}$. A probability measure was introduced on S_r and a sequence of functions f_k was defined as functions which measure the difference between this measure and the δ-dimensional Hausdorff measure on $\overline{R}$. By proving that $\overline{\lim} f_k \geq 1$ on all of S_r, it was shown that $\dim \overline{R} \leq \delta$ and, by proving that $\lim f_k = 1$ almost everywhere, it was shown that $\dim \overline{R} < \delta$.

K. J. Falconer [**5**] has used potential-theoretic methods to show that for almost all $(a_1, \ldots, a_k) \in \mathbb{R}^{nk}$, the Hausdorff dimension of self-affine fractals $F = \cup_{i=1}^{k} (T_i(F) + a_i)$

is $\min\{d, n\}$, where $T_1, \ldots, T_k$ are contractive linear transformations on $\mathbb{R}^n$ and d is defined in terms of the singular values of $T_{i_1} \ldots T_{i_r}$, $1 \leq i_j \leq k$.

Gatzouras and Lalley have considered a class of self-affine sets that are more general than those of McMullen but less general than Falconer's. Their hypotheses guarantee that the first iteration of rectangles are arranged in rows and have height exceeding width.

Unfortunately, the results mentioned above do not give explicit formulae for examples such as the one given below. In this paper, we shall give explicit formulae for the dimensions of sets C such as those in the following example.

Example 1. Let $R_0 = [0, 1/2] \times [3/4, 1]$ and $R_1 = [3/4, 1] \times [0, 1/2]$ and let $C_1 = \{R_0, R_1\}$. Iterate this construction with an affine copy of C_1 and let C be the limiting set obtained. (See Figure 2.)

We shall address the problem of finding the dimension of sets such as the one in Example 1 by proving a more general result in $\mathbb{R}^2$.

Definition 1. Let R_0 and R_1 be two rectangles in $[0, 1] \times [0, 1]$ and let $C_1 = \{R_0, R_1\}$. We say C_1 has *disjoint projections* if $\pi_1 R_0 \cap \pi_1 R_1 = \emptyset$ and $\pi_2 R_0 \cap \pi_2 R_1 = \emptyset$, where $\pi_1, \pi_2 : \mathbb{R}^2 \to \mathbb{R}$ are the horizontal and vertical projections. We can give a similar definition for m rectangles. (See Figure 2 for an illustration.)

Theorem 1. *Let C be a self-affine set in $\mathbb{R}^2$ with disjoint projections. Then the box dimension and Hausdorff dimension of the set C are given by*

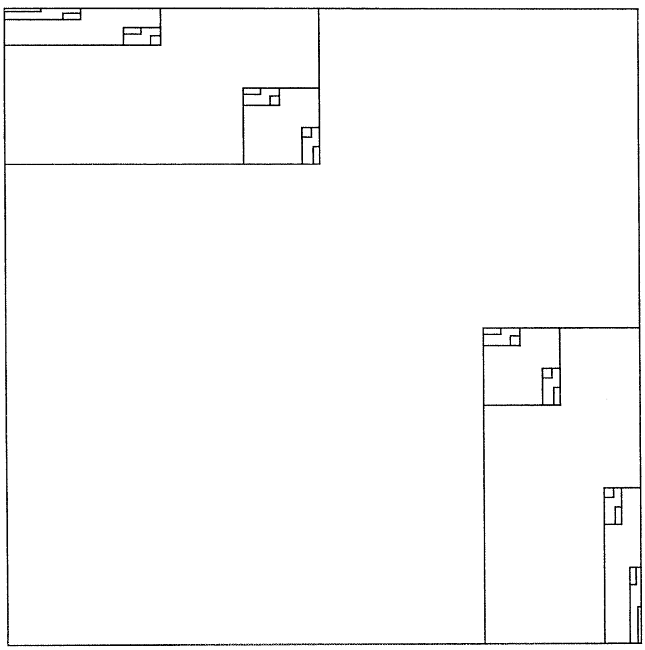

Figure 1. The first four iterations of a fractal with disjoint projections.

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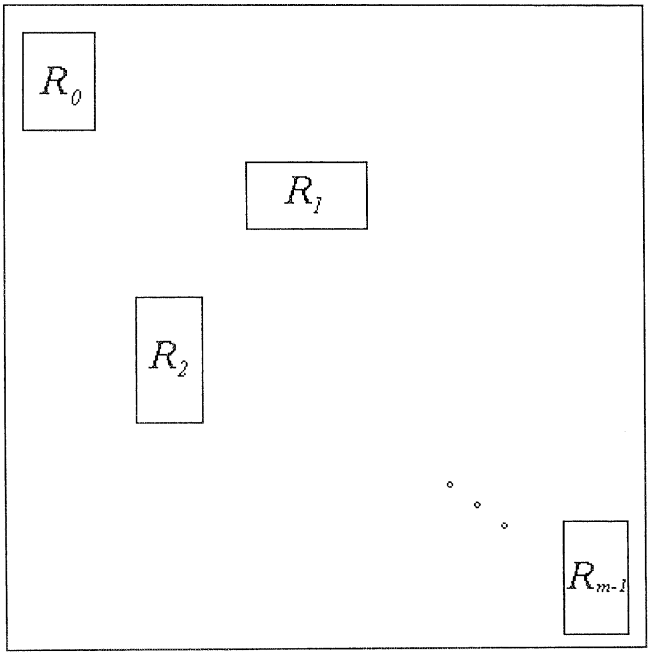

Figure 2. An example of a set with disjoint projections.

$$\begin{aligned}\dim_B(C) &= \max\{\dim_B(\pi_1 C), \dim_B(\pi_2 C)\}\\ &= \max\{\dim_H(\pi_1 C), \dim_H(\pi_2 C)\}\\ &= \dim_H(C),\end{aligned}$$

where $\pi_1, \pi_2 : \mathbb{R}^2 \to \mathbb{R}$ *are the horizontal and vertical projections.*

In the next section, we introduce some notation to help prove this result.

2. Notation. In order to prove the Theorem above we shall first introduce some notation. Suppose that $R_0, R_1, \ldots R_{m-1}$ are m rectangles in $[0, 1] \times [0, 1]$. The rectangle $R_{i_1 i_2}$ denotes the affine copy of R_{i_2} in R_{i_1}, $i_1, i_2 \in \{0, \ldots, m-1\}$. Inductively, $R_{i_1 i_2 \ldots i_n}$ denotes the affine copy of R_{i_n} in $R_{i_1, i_2 \ldots i_{n-1}}$, $i_j \in \{0, \ldots, m-1\}$, where $1 \leq j \leq n$.

Denote by C_1 the collection of rectangles $R_0, R_1, \ldots, R_{m-1}$; (see Figure 2). Iterate this construction with each rectangle replaced by an affine copy of C_1, and let C be the limiting set obtained. Note therefore that

$$C_n = \bigcup_{i_1 i_2 \ldots i_n} I_{i_1 i_2 \ldots i_n} \times J_{i_1 i_2 \ldots i_n},$$

where $i_1, i_2, \ldots, i_n \in \{0, \ldots, m - 1\}$ and

$$C = \bigcap_{n=0}^{\infty} C_n.$$

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Suppose also that the rectangles have disjoint projections. In other words $\pi_1 R_i \cap \pi_1 R_j = \emptyset$ and $\pi_2 R_i \cap \pi_2 R_j = \emptyset$, $i \neq j$, where $\pi_1, \pi_2 : \mathbb{R}^2 \to \mathbb{R}$ are the horizontal and vertical projections.

3. The box and Hausdorff dimension of C**.** We begin this section with several definitions.

Definition 2. If U is any non-empty subset of $\mathbb{R}^n$, the *diameter* of U is defined as

$$|U| = \sup\{|x - y| : x, y \in U\}.$$

If $\{U_i\}$ is a countable or finite collection of sets of diameter at most δ such that $F \subset \cup_{i=1}^{\infty} U_i$, with $0 < |U_i| \leq \delta$ for every i, we say that $\{U_i\}$ is a δ-*cover* of F.

Definition 3. Let F be any non-empty subset of $\mathbb{R}^n$ and let $N_\delta(F)$ be the smallest number of sets of diameter at most δ that can cover F. The *box dimension* of F is defined as

$$\dim_B F = \lim_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta}.$$

In Section 1, we stated a theorem relating to Hausdorff dimension that is defined as follows.

Definition 4. Let F be a non-empty bounded subset of $\mathbb{R}^n$. Then the *Hausdorff dimension* of F is defined as

$$\dim_H(F) = \inf\{s : \mathcal{H}^s(F) = 0\} = \sup\{s : \mathcal{H}^s(F) = \infty\},$$

where

$$\mathcal{H}^s(F) = \lim_{\delta \to 0} \inf\{\sum_{i=1}^{\infty} |U_i|^s : \{U_i\} \text{ is a } \delta\text{-cover of } F\}.$$

The following notation and theorem will be required in the proof of our main result.

Notation. Let λ_N be the length of the shortest side of the rectangles in the Nth iteration. We cover C_N with boxes of side λ_N. The process of covering C_N with these boxes is done in two stages—'horizontally' and 'vertically'.

Let H_N represent the collection of boxes partly covering C_N 'horizontally'. The horizontal partial covering is shown in Figure 3. Notice that, for each $x \in \pi_1 C_N$, there exists exactly one box $s_x \in S_N$ such that $x \in \pi_1 s_x$.

Let V_N represent the collection of boxes covering the remaining uncovered part of C_N 'vertically'. The vertical partial covering is shown in Figure 3. Notice that, for each $y \in \pi_2 C_N$, there exists at most one box $v_y \in V_N$ such that $y \in \pi_2 v_y$.

Clearly $H_N \cup V_N$ cover C_N.

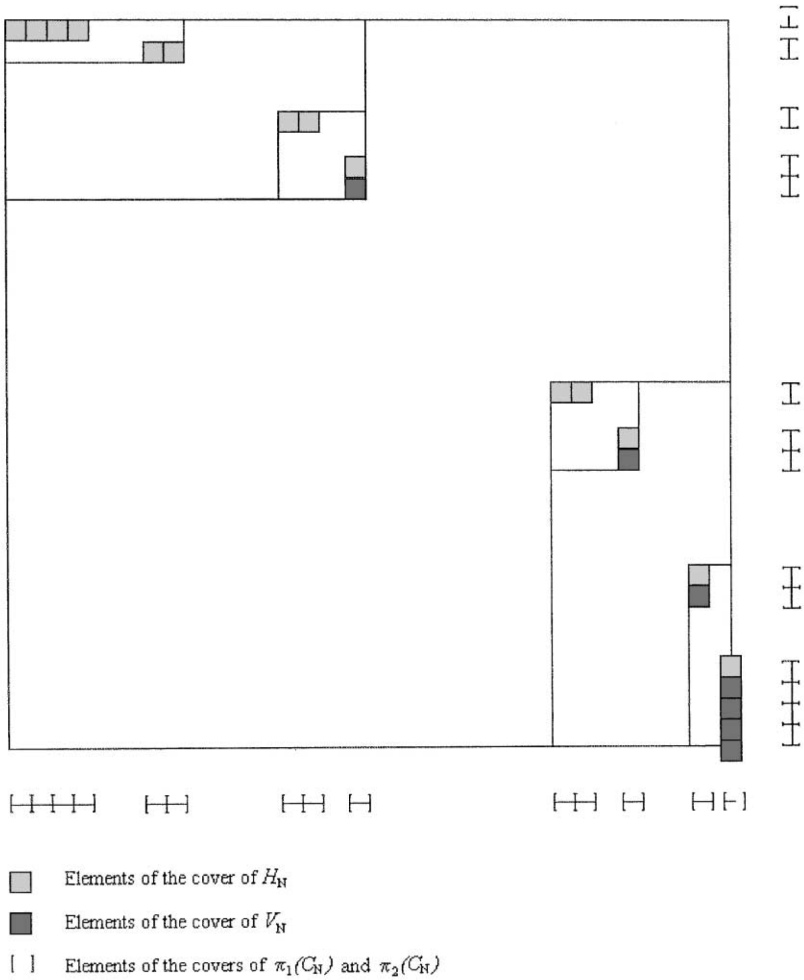

Figure 3.

THEOREM 2. *Let N be any natural number. Then if we iterate our fractal N times, we have*

$$N_{\lambda_N}(C_N) \leq 2N_{\lambda_N}(\pi_1 C_N) + 2N_{\lambda_N}(\pi_2 C_N).$$

Proof. We project C_N onto the x-axis. Recall that, for each $x \in \pi_1 C_N$, there exists exactly one box $s_x \in S_N$ such that $x \in \pi_1 s_x$. Therefore, when H_N, the 'horizontal partial covering' of C_N, is projected onto the x-axis, it covers $\pi_1 C_N$; that is $\pi_1 H_N \supset \pi_1 C_N$.

It may be possible to have a more economical covering of $\pi_1 C_N$. This can only occur when there is a gap between the rectangles. See, for example, Figure 4.

In $\mathbb{R}^2$, the section of the fractal shown is covered horizontally by four boxes. However, when projected onto the x-axis, the projection can be covered by three 'boxes'.

Now since the length λ_N of the boxes is equal to the shortest side of the rectangles, the number of gaps is less than half the number of boxes required to cover C_N. Hence

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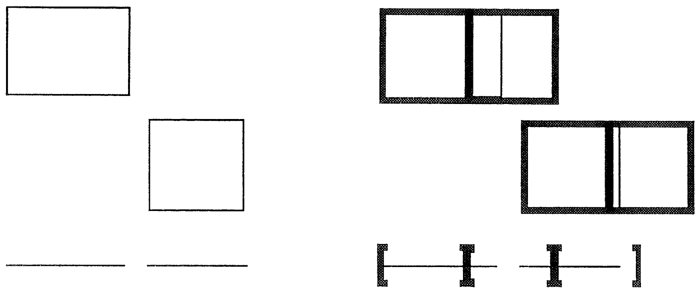

The fractal and its projection. The fractal and its projection with their respective coverings.

Figure 4.

$$N_{\lambda_N}(H_N) \leq 2N_{\lambda_N}(\pi_1 C_N).$$

Similarly, we have

$$N_{\lambda_N}(V_N) \leq 2N_{\lambda_N}(\pi_2 C_N).$$

Since $H_N \cup V_N \supset C_N$, we have $N_{\lambda_N}(H_N) + N_{\lambda_N}(V_N) \geq N_{\lambda_N}(C_N)$. Hence

$$N_{\lambda_N}(C_N) \leq 2N_{\lambda_N}(\pi_1 C_N) + 2N_{\lambda_N}(\pi_2 C_N).$$

□

Given this result, we prove the following theorem, which was stated in Section 1.

THEOREM 3. *Let C be a self-affine set in $\mathbb{R}^2$ with disjoint projections. Then the box dimension and Hausdorff dimension of the set C are given by*

$$\dim_B(C) = \max\{\dim_B(\pi_1 C), \dim_B(\pi_2 C)\},$$

where $\pi_1, \pi_2 : \mathbb{R}^2 \to \mathbb{R}$ are the horizontal and vertical projections.

Proof. By Theorem 2,

$$\begin{aligned} \dim_B(C) &= \lim_{N\to\infty} \frac{\log N_{\lambda_N}(C_N)}{-\log \lambda_N} \\ &\leq \lim_{N\to\infty} \frac{\log(2N_{\lambda_N}(\pi_1 C_N) + 2N_{\lambda_N}(\pi_2 C_N))}{-\log \lambda_N} \\ &= \max\{\dim_B(\pi_1 C), \dim_B(\pi_2 C)\}. \end{aligned}$$

But $\max\{\dim_B(\pi_1 C), \dim_B(\pi_2 C)\} \leq \dim_B(C)$. Therefore,

$$\dim_B(C) = \max \dim_B(\pi_1 C), \dim_B(\pi_2 C)\}.$$

□

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THEOREM 4. *The Hausdorff dimension of the set C is given by*

$$\dim_H(C) = \dim_B(C).$$

Proof. Since $\pi_1 C$ and $\pi_2 C$ are Cantor sets, we have that

$$\begin{aligned}\max\{\dim_B(\pi_1 C), \dim_B(\pi_2 C)\} &= \max\{\dim_H(\pi_1 C), \dim_H(\pi_2 C)\} \\ &\leq \dim_H(C) \text{ (See K. Falconer [5, pp. 43–44].)} \\ &\leq \dim_B(C) \text{ (See K. Falconer [5, p. 43].)} \\ &= \max\{\dim_B(\pi_1 C), \dim_B(\pi_2 C)\}\end{aligned}$$

by Theorem 3. Therefore

$$\dim_H(C) = \dim_B(C).$$

□

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