EXTENSIVE SUBCATEGORIES OF THE CATEGORY OF $T_{0}$-SPACES

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Introduction. It is well known that epimorphisms in the category Top (Top$_{1}$, respectively) of topological spaces ($T_{1}$-spaces, respectively) and continuous maps are precisely onto continuous maps. Since every mono-reflective subcategory of a category is also epi-reflective and every embedding in Top (Top$_{1}$, respectively) is a monomorphism, there is no nontrivial reflective subcategory of Top (Top$_{1}$, respectively) such that every reflection is an embedding. However, in the category Top$_{0}$ of $T_{0}$-spaces and continuous maps as well as in the category Haus of Hausdorff spaces and continuous maps, there are epimorphisms which are not onto. Moreover, every reflection of a reflective subcategory of Top$_{0}$ which contains a non $T_{1}$-space, is an embedding [16]. For an epi-reflective subcategory $\mathcal{B}$ of Haus, there is a hereditary subcategory $\mathcal{R}\mathcal{B}$ of Haus such that $\mathcal{B}$ is extensive in $\mathcal{R}\mathcal{B}$, i.e. every reflection is an extension. Using extensive operators, we have been able to characterize every extensive subcategory of a hereditary subcategory of Haus [11;13]. In this paper, we deal with all extensive subcategories of Top$_{0}$. We introduce idempotent semi-limit-operators. With these, we can also characterize all extensive subcategories of Top$_{0}$. In this approach, one of the main advantages is that by using the trace filters, in this case union filters, one can easily characterize extensive subcategories of Top$_{0}$ and every reflection is precisely given. Moreover, one can associate an extensive subcategory of Top$_{0}$ with a coreflective subcategory of Top. We obtain also some interesting results about the front closure operator which determines the coreflective subcategory of Top generated by indiscrete spaces.

All topological and categorical concepts will be used in the sense of N. Bourbaki [4] and H. Herrlich [7], respectively. In particular, we assume throughout this paper that a subcategory of a category is full and isomorphism-closed. The closure of a subset $A$ of a topological space $X$ will be written $\text{cl}_{X}A$ (cl $A$ when no confusion is possible) and for $x \in X$, $\text{cl}x$ means $\text{cl}\{x\}$, which will be called point closure.

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1. Front closure. The following definition is due to S. Baron [3] and L. Skula [16].

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1.1 Definition. Let $X$ be a topological space. For $A \subseteq X$, let

$$fcl_x A = \{x \in X\} \text{ for any neighborhood } V \text{ of } x, V \cap cl x \cap A \neq \emptyset.$$ 

Then the operator $fcl$ is called the front closure operator.

It is known that the front closure operator $fcl$ satisfies the Kuratowski axioms. We shall give the usual names preceded by front to the topological concepts with respect to the topology defined by the front closure operator. H. Herrlich [8] has defined limit-operators with which he has been able to construct every coreflective subcategory of Top. The second Proposition on p. 205 and the first statement of the Theorem on p. 206 [8], were incorrect (see the following proposition).

1.2 Proposition. The front closure operator $fcl$ is an idempotent limit-operator and the subcategory $\mathcal{C}(fcl)$ of Top determined by

$$\{X \in \text{Top} | \text{each subset } A \text{ of } X \text{ with } cl_x A = A \text{ is closed in } X\}$$

is the category of all coproducts of indiscrete spaces.

Proof. The first part is immediate from the fact that $g(fcl_x A) \subseteq fcl_y g(A)$ for any continuous map $g : X \to Y$ and $A \subseteq X$.

For the second part, it is enough to show that a topological space $X$ belongs to $\mathcal{C}(fcl)$ if and only if $cl x$ is open for every $x \in X$ (see [8, II, (2)]). Suppose $X$ belongs to $\mathcal{C}(fcl)$. For any $x \in X$ and any $y \in cl x$, $cl y \cap C cl x = \emptyset$, where $C A$ means the complement of $A$. Hence $y \notin fcl_x C cl x$, i.e., $C cl x$ is closed; $cl x$ is open. Noting that $fcl_x$ in a space $X$ is precisely the original closure operator provided that every point closure in $X$ is open, the converse is also true.

It is well known [5] that any morphism $f : X \to Z$ in the category Haus, whose restriction to a dense subset $Y$ of $X$ is a homeomorphism, carries $X - Y$ into $Z - f(Y)$. Obviously it is not so in the category Top.  

1.3 Theorem. Let $Y$ be a front-dense subspace of a $T_0$-space $X$ and let $f : X \to Z$ be a continuous map. If the restriction of $f$ to $Y$ is a homeomorphism, then $f(X - Y) \subseteq Z - f(Y)$.

Proof. Suppose, on the contrary, that $f(x) = f(y)$ for some $y \in Y$ and $y \neq x \in X$. Suppose $y \notin cl x$. Then there is an open neighborhood $W$ of $f(y)$ with $f(Y \cap C cl x) = W \cap f(Y)$. Since $Y$ is front-dense in $X$, every neighborhood of $x$ contains points of $Y \cap cl x$. Then the homeomorphism $f|Y$ takes all such points into $Z - W$; $f$ is not continuous at $x$. Hence $y \notin cl x$. Since $X$ is $T_0$-space, $x \notin cl y$. Since $C cl y$ is a neighborhood of $x$, $Y \cap cl x \cap C cl y \neq \emptyset$. Let $p$ be an element of the set. Then $f(p)$ does not belong to $cl_{f(Y)}f(y)$. On the other hand, $f(p)$ belongs to $cl_{f(Y)}f(x) = cl_{f(Y)}f(x) = cl_{f(Y)}f(y)$, which is a contradiction. This completes the proof.

By the same argument as in [5], one has:
1.4 Corollary. Every retract of a $T_0$-space $X$ is front-closed in $X$. In particular, the graph of a morphism $f: X \to Y$ in $\text{Top}_0$ is front-closed in $X \times Y$, and if a $T_0$-space $Y$ contains a product $X = \times X_i$ and each projection $p_i: X \to X_i$ has a continuous extension to $Y$, then $X$ is front-closed in $Y$.

2. Quasi-sobre spaces and sobre spaces.

2.1 Definition. A closed subset of a topological space is said to be irreducible if it cannot be expressed as the union of two proper closed subsets. A topological space is said to be quasi-sobre if every non-empty irreducible closed subset of the space is a point closure. A $T_0$ quasi-sobre space is called sobre.

We note that sobre spaces have been also called $pc$-spaces in [14], or spectral spaces in [9].

2.2 Proposition. A topological space is quasi-sobre if and only if its $\text{Top}_0$-reflection space is sobre.

Proof. For any topological space $X$, let $\tau_X: X \to \tau X$ be a $\text{Top}_0$-reflection of $X$. It is known that $\tau X$ is a quotient space $X/\sim$ of $X$, where $x \sim y$ if and only if $\text{cl} \ x = \text{cl} \ y$. Suppose $X$ is quasi-sobre. Let $F$ be a non-empty irreducible closed subset of $\tau X$. Since $\tau X$ is closed and onto, and every closed subset of $X$ is saturated with respect to the equivalence relation $\sim$, $\tau X^{-1}(F)$ is also a non-empty irreducible closed subset; $\tau X^{-1}(F) = \text{cl}_X x$ for some $x \in X$. Hence $F = \tau X(\tau X^{-1}(F)) = \tau X(\text{cl}_X x) = \tau X(\text{cl}_X x)$.

Conversely, let $G$ be a non-empty irreducible closed subset of $X$. Since $\tau X$ is closed, $\tau X(G)$ is also a non-empty irreducible closed subset of $\tau X$. Since $\tau X$ is sobre, there is a point $x$ of $G$ with $\tau X(G) = \text{cl}_X \tau X(x)$. Using the fact that $\tau X$ is open, it is easy to show that $G = \text{cl}_x x$. We omit the detail of the proof.

It is well known [6; 9; 14] that the subcategory $\text{Sob}$ of $\text{Top}_0$ determined by sobre spaces is epi-reflective in $\text{Top}_0$. Using the limit-operator $\text{fcl}$, we give here another proof.

For any set $X$, let $\mathcal{P}_0(X)$ denote the set of all non-empty subsets of $X$. Let $X$ be a non-empty topological space. We define a space $\bar{X}$ as follows: its underlying set is $\mathcal{P}_0(X)$ and its topology has $\{\mathcal{P}_0(F)|F$: closed subset of $X\}$ as a subbase for the closed subsets. It is then obvious that the map $x \mapsto \{x\}$ is an embedding of $X$ into $\bar{X}$ and for any continuous map $f: X \to Y$, the map $\bar{f}: \bar{X} \to \bar{Y}$ defined by $M \mapsto f(M)$ is also continuous (see [4, Ex. 7, §2]). We now show that $\bar{X}$ is a quasi-sobre space. Let $\Lambda$ be a non-empty irreducible closed subset of $\bar{X}$;

$$\Lambda = \bigcap_{i \in I} \left\{ \bigcup_{j=1}^{n_i} \mathcal{P}_0(F_{i,j})|n_i: \text{natural number and} \right.\bigg\{ F_{i,j}: \text{non-empty closed subset of } X \bigg\}.$$

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Since $\Lambda$ is irreducible, we may assume that

$$\Lambda = \bigcap_{i \in I} \mathcal{P}_0(F_i) = \mathcal{P}_0\left(\bigcap_{i \in I} F_i\right).$$

Since $\text{cl}_X M = \mathcal{P}_0(\text{cl}_X M)$ for every $M \in \mathcal{X}$, $\Lambda = \text{cl}_X \{\bigcap_{i \in I} F_i\}$. We note that $\mathcal{X}$ is $T_0$ if and only if $X$ is discrete.

Let $\mathcal{C}(X)$ be the $\text{Top}_0$-reflection space of $\mathcal{X}$. By Lemma 1 in [2], the map $\mathcal{C}_X: X \to \mathcal{C}(X) = X \to \mathcal{X} \to \mathcal{C}(X)$ is also an embedding if $X$ is $T_0$. Since we may assume that $\mathcal{C}_X$ is the natural embedding, one has by Theorem 3.2 in [14] and Proposition 2.2 the following.

2.3 PROPOSITION. A $T_0$-space $X$ is sober if and only if $X$ is front-closed in $\mathcal{C}(X)$.

2.4 THEOREM. The subcategory $\text{Sob}$ of all sober spaces is epi-reflective in the category $\text{Top}_0$.

Proof. For any $X \in \text{Top}_0$, let $\pi X$ be the subspace of $\mathcal{C}(X)$ whose underlying set is $\text{fcl}_{\mathcal{C}(X)} X$ and $\pi X: X \to \pi X$ be the natural embedding of $X$ into $\pi X$. Since $\pi X$ is a front-closed subspace of the sober space $\mathcal{C}(X)$, $\pi X$ is sober. Moreover, the map $\pi X: X \to \pi X$ is an epimorphism (see [3]). For any $Y \in \text{Sob}$ and any $g: X \to Y$ in $\text{Top}_0$, we have the following commutative diagram:

$$
\begin{array}{ccc}
X & \xrightarrow{\pi X} & \pi X \subseteq \mathcal{C}(X) \\
g \downarrow & & \downarrow \tilde{g}^- \\
Y & \xleftarrow{\text{fcl}_{\mathcal{C}(Y)}} & \text{fcl}_{\mathcal{C}(Y)} Y \subseteq \mathcal{C}(Y),
\end{array}
$$

where $\tilde{g}^-$ is determined by $\tilde{g}$ and the reflection property. Since fcl is a limit-operator, $\tilde{g}^-(\pi X) = \tilde{g}^-(\text{fcl}_{\mathcal{C}(X)} X) \subseteq \text{fcl}_{\mathcal{C}(Y)} \tilde{g}^-(X) \subseteq \text{fcl}_{\mathcal{C}(Y)} Y = Y$. Hence there is a unique morphism $g^*: \pi X \to Y$ such that $\tilde{g}^- j = g^*$ followed by the natural embedding $Y \to \mathcal{C}(Y)$. It is obvious that $g^* \pi X = g$.

2.4 Remark. (1) The space $\mathcal{C}(X)$ is actually the subspace of $\mathcal{X}$ with the set of all non-empty closed subsets as its underlying set and the map $\mathcal{C}_X: X \to \mathcal{C}(X)$ is defined by $x \mapsto \text{cl} x$.

(2) A non-empty closed subset $A$ belongs to the front-closure $\pi X$ of $X$ in $\mathcal{C}(X)$ if and only if for every finite family $\{U_1, \ldots, U_n\}$ of open subsets with $U_i \cap A \neq \emptyset$ ($i = 1, \ldots, n$), $\bigcap_i U_i \cap A \neq \emptyset$ if and only if $\bigcup_{x \in O(x)}$ is a (proper) open filter on $X$, where $O(x)$ denotes the open neighborhood filter of $x$ if and only if $A$ is irreducible. Hence $\pi X$ is the subspace of $\mathcal{C}(X)$ with the set of all non-empty irreducible closed subsets as its underlying set (see [6]). For any closed subset $F$ of $X$, let $\Sigma_F$ denote $\mathcal{P}_0(F) \cap \pi X$. Then it is obvious that $\{\Sigma_F|F$ closed subset of $X\}$ is precisely the family of closed subsets of $\pi X$. Moreover, the correspondence $F \mapsto \Sigma_F$ is a lattice isomorphism. Hence $X$ is
quasi-compact (Lindelöf, of second countability, connected respectively) if and only if $\pi X$ is (see [17]).

2.5 Definition. An open filter on a $T_0$-space $X$ is called a union filter if it is a union of open neighborhood filters on $X$.

2.6 Remark. For any $A \in \pi X$, its trace filter $T(A)$ on $X$ is precisely the union filter $\bigcup_{x \in A} O(x)$ and $\pi X$ is the strict extension of $X$ with all union filters as the filter trace (see [1]). Hence a $T_0$-space $X$ is sober if and only if every union filter on $X$ is already the open neighborhood filter of some point of $X$.

Likewise union filters, if $\bigvee_{x \in A} O(x)$ (see [2]) is a (proper) open filter, then so is $\bigvee_{x \in A} O(x)$ and

$$\bigvee_{x \in A} O(x) = \bigvee_{x \in A} O(x).$$

We note that a non-empty closed set $A$ belongs to the closure of $X$ in $\mathcal{C}(X)$ if and only if $\bigvee_{x \in A} O(x)$ is a proper open filter, and that the trace filter of $A \in \text{cl}_{\mathcal{C}(X)} X$ on $X$ is precisely the join filter $\bigvee_{x \in A} O(x)$. Contrary to the case of $\pi X$, the extension $\text{cl}_{\mathcal{C}(X)} X$ of $X$ is not relatively $\tau_0$ (see [1]).

3. Extensive subcategories of $\text{Top}_0$.

3.1 Definition. A subcategory $\mathcal{B}$ of $\text{Top}_0$ is said to be extensive in $\text{Top}_0$ if $\mathcal{B}$ is a reflective subcategory of $\text{Top}_0$ such that every $\mathcal{B}$-reflection map $r_X: X \to rX$ is an embedding for each $X \in \text{Top}_0$.

3.2 Remark. (1) Since every extensive subcategory $\mathcal{B}$ is epi-reflective and epimorphisms in $\text{Top}_0$ are exactly front-dense continuous maps (see [3]), every $\mathcal{B}$-reflection map $r_X: X \to rX$ of $X \in \text{Top}_0$ is an extension.

(2) Likewise $H$-closed spaces, every sobre space is front-closed in its $T_0$ superspace (see [14]). Thus every extensive subcategory of $\text{Top}_0$ contains all sobre spaces and obviously the category $\text{Sob}$ is the smallest extensive subcategory of $\text{Top}_0$. Moreover every reflective subcategory of $\text{Top}_0$ containing $\text{Sob}$ is also extensive in $\text{Top}_0$.

(3) Using Theorem 1.3 and Corollary 1.4, one can directly show that every extensive subcategory of $\text{Top}_0$ is front-closed-hereditary and productive.

3.3 Definition. An operator $l$ which associates every pair $(X, A)$, where $X$ is a sobre space and $A$ is a subset of $X$, a subset $l_X A$ of $X$ is said to be an idempotent semi-limit-operator if $l$ satisfies the following conditions:

(1) if $A$ is a subset of a sobre space $X$, then $A \subseteq l_X A \subseteq \text{cl}_X A$;

(2) if $f: X \to Y$ is a morphism in the category $\text{Sob}$ and $A$ is a subset of $X$ then $f(l_X A) \subseteq l_Y f(A)$;

(3) if $A$ is a subset of a sobre space $X$, then $l_X (l_X A) = l_X A$.

It is obvious that the restriction of an idempotent limit-operator to the category $\text{Sob}$ is an idempotent semi-limit-operator.
For an idempotent semi-limit-operator \( l \), a subset \( A \) of a sober space \( X \) with \( l_X A = A \) will be called \( l \)-closed.

Hereafter, by an extension space of a space \( X \) is meant a space of which \( X \) is a dense subspace.

Let \( l \) be an idempotent semi-limit-operator and let Sob\(_1\) be the subcategory of Top\(_0\) determined by \( T_0 \)-spaces which are \( l \)-closed in their Sob-reflection spaces.

3.4 Theorem. A subcategory \( \mathcal{B} \) of Top\(_0\) is extensive if and only if \( \mathcal{B} \) is of the form Sob\(_I\) for some idempotent semi-limit-operator \( I \).

Proof. For any \( x \in \text{Top}_0 \), let \( \pi x: X \rightarrow \pi X \) be the Sob-reflection of \( X \) such that \( X \) is a subspace of \( \pi X \) and \( \pi X \) is the natural embedding.

\( \leftarrow \) Let \( \pi \) be the subspace of \( \pi Y \) with \( \pi X \) as its underlying set. Obviously, the natural embedding \( j: \pi X \rightarrow \pi Y \) is a Sob-reflection of \( \pi Y \); hence \( \pi X \) belongs to Sob\(_1\). Now we wish to show that the natural embedding \( \pi Y \): \( X \rightarrow \pi X \) is the Sob\(_1\)-reflection of \( X \). For any \( Y \in \text{Sob}_1 \) and any morphism \( f: X \rightarrow Y \) in Top\(_0\), there is a unique \( f: \pi X \rightarrow \pi Y \) with \( \pi f \pi = \pi Y \). Since \( f \) is a morphism in Sob and \( l_y Y = Y \),

\[ f(\pi X) = f(l_x X) \subseteq l_y f(X) \subseteq l_y Y = Y. \]

Let \( f^i \) be the restriction and corestriction of \( f \) to \( \pi X \) and \( Y \) respectively. Then it is obvious that \( f^i \pi X = f \) and it is unique.

\( \Rightarrow \) For any subset \( A \) of a sober space \( X \), define

\[ l_X A = \cap \{ B \mid A \subseteq B \text{ and } B \text{ is an object of } \mathcal{B} \text{ as a subspace of } X \}. \]

Since every extensive subcategory of Top\(_0\) is front-closed-hereditary, \( \mathcal{B} \) is, in particular, closed-hereditary. Hence \( A \subseteq l_X A \subseteq cl_X A \). For any \( f: X \rightarrow Y \) in Sob and \( B \subseteq Y \) with \( B \subseteq Y \) as a subspace of \( Y \), \( f^{-1}(B) \) is also an object of \( \mathcal{B} \) as a subspace of \( X \). Take \( x \in l_X A \) and \( B \supseteq f(A) \) with \( B \subseteq \mathcal{B} \). Since \( f^{-1}(B) \) and \( f^{-1}(B) \) are objects of \( \mathcal{B} \), \( x \in f^{-1}(B) \), i.e., \( f(x) \in B \). Hence \( f,l_X A \subseteq l_Y f(A) \). Since \( \mathcal{B} \) is closed under the intersections, \( l_X A = A \) if and only if \( A \subseteq B \). Hence \( l_X l_Y X = A \) if and only if \( A \subseteq B \).

3.5 Remark. For any extensive subcategory \( \mathcal{B} \) of Top\(_0\) and for any \( X \in \text{Top}_0 \), its \( \mathcal{B} \)-reflection space is given by the intersection of all subspaces of \( \pi X \) which belong to \( \mathcal{B} \) and contain \( X \).

3.6 Corollary. For any coreflective subcategory \( \mathcal{C} \) of Top\(_0\), let Sob\(_\mathcal{C}\) be the subcategory of Top\(_0\) determined by \( T_0 \)-spaces which are closed in the \( \mathcal{C} \)-coreflection spaces of their Sob-reflection spaces. Then Sob\(_\mathcal{C}\) is also an extensive subcategory of Top\(_0\).

3.7 Examples. (1) Let \( k \) be an infinite cardinal number. For \( A \subseteq X \in \text{Top} \), we define \( l^k X = \{ x \in X \mid \text{for any family } (U_i)_{i \in I} \}\) of open neighborhoods of \( x \) with \( |I| < k, \cap_i U_i \cap A \neq \emptyset \). Then it is known [10] that \( l^k \) is an idempotent limit-operator. Moreover, for any extension \( Y \) of a space \( X \), \( l^k Y X = X \) if and
only if any point \( y \) of \( Y \) whose trace filter on \( X \) has the \( k \)-intersection property is already a point of \( X \) (see [10]). Hence a \( T_0 \)-space \( X \) belongs to \( \text{Sob}_k \) if and only if every union filter with the \( k \)-intersection property is itself the open neighborhood filter if and only if every non-empty irreducible closed subset \( A \) such that every open filter \( \mathcal{U} \) on \( X \) with \( U \cap A \neq \emptyset \) for every \( U \in \mathcal{U} \) has the \( k \)-intersection property, is a point closure. We note that \( \text{Sob}_{\kappa_0} = \text{Sob} \), \( \text{Sob}_{\kappa_0} \), the category of \( \varphi(k) \)-spaces which is simply generated by \( E(k) : E(\{0,1\}) = 2 \), \( E(k^+) = 2^* - \{1^*\} \), and

\[
E(k) = \bigtimes_{m<\kappa} E(m^+)
\]

for a limit cardinal number \( k \), where 2 is the space \( \{0,1\} \) with the topology \( \{\varnothing, \{1\}, \{0,1\}\} \) (see [15]).

(2) For \( A \subseteq X \in \text{Top} \), let

\[
I_xA = \{x \in X | \text{there is a point } b \text{ of } A \text{ with } x \in \cl b \}.
\]

Then it is known [8] that \( I \) is an idempotent limit-operator. For any extension \( Y \) of a space \( X \), \( l_YX = X \) if and only if any point \( y \) of \( Y \) whose trace filter on \( X \) is contained in the open neighborhood filter of some point of \( X \) is already a point of \( X \). Hence a \( T_0 \)-space \( X \) belongs to \( \text{Sob}_i \) if and only if every union filter which is contained in an open neighborhood filter on \( X \) is itself an open neighborhood filter if and only if every non-empty irreducible closed subset which is contained in a point closure is already a point closure.

Let \( X \) be an ordered set. Let \( X^+ \) denote the space with the right topology on \( X \), i.e. the topology whose base is \( \{[x, \to | x \in X] \} \). Then \( R^+ \in \text{Sob}_1 = \text{Sob} \) and \( R^+ - \{0\} \in \text{Top}_0 - \text{Sob} \), where \( \text{Sob} \) is the real line with the usual order.

(3) For \( A \subseteq X \in \text{Top} \), let

\[
I_xA = \{x \in X | \text{there is a sequence in } A \text{ which converges to } x \}.
\]

For the limit-operator \( I \) and for any extension \( Y \) of a space \( X \), \( l_YX = X \) if and only if any point \( y \) of \( Y \) whose trace filter on \( X \) is contained in a filter with a countable base is already a point of \( X \) (see [12]). Hence a \( T_0 \)-space \( X \) belongs to \( \text{Sob}_{\kappa_1} \), where \( I \) is the associated idempotent limit-operator with \( l \) (see [8]) if and only if every union filter which is contained in a filter with a countable base is already an open neighborhood filter. It is obvious that \( N^+ \in \text{Top}_0 - \text{Sob} \), where \( \text{Sob} \) is the set of natural numbers with the usual order. Let \( W(\omega_1) \) be the set of ordinals \( < \omega_1 \) with the usual order. Then \( W(\omega_1)_+ \in \text{Sob}_{\kappa_1} - \text{Sob} \).

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