TWISTED ALEXANDER POLYNOMIAL FOR THE LAWRENCE–KRAMMER REPRESENTATION

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In this paper, we prove that the twisted Alexander polynomial for the Lawrence–Krammer representation of the braid group $B_4$ is trivial. This gives an answer to the problem of whether the twisted Alexander polynomial for given faithful representations is always non-trivial.

1. INTRODUCTION

The twisted Alexander polynomial for finitely presentable groups was introduced by Wada in [5]. As a notable application, it was shown that the twisted Alexander polynomial can tell Kinoshita-Terasaka knot from Conway’s 11-crossing knot.

In [4], the twisted Alexander polynomial for Jones representations of the braid group $B_n (n \geq 3)$ is studied. One of the main results of [4] is that the twisted Alexander polynomial for the Burau representation is not trivial for $n = 3$ and trivial for $n \geq 4$. We know that the Burau representation is faithful for $n = 3$, not faithful for $n \geq 5$ and the faithfulness is still open for the case $n = 4$. Then it is mentioned in [4] that it would be interesting to study a relation between the faithfulness of the Burau representation and the twisted Alexander polynomial. In other words,

PROBLEM 1.1. If a given representation is faithful, is the twisted Alexander polynomial non-trivial?

In this paper, we present the answer to this question.

Krammer constructed in [2] a representation of the braid group, which is now called the Lawrence–Krammer representation, and showed that it is faithful for $n = 4$. Moreover, Bigelow [1] and Krammer [3] proved that the Lawrence–Krammer representation is faithful for all $n$. Then we may show a relation between the faithfulness of a representation and the twisted Alexander polynomial as a consequence of an explicit calculation of the twisted Alexander polynomial for the Lawrence–Krammer representation.

In this paper, we show the following. (See Section 3 for the precise statement.)
Theorem 1.2. The twisted Alexander polynomial for the Lawrence-Krammer representation of the braid group $B_4$ is trivial.

This gives the negative answer to Problem 1.1.

In Section 2, we briefly recall the definition of the Lawrence-Krammer representation of the braid group $B_4$. In Section 3, the twisted Alexander polynomial of $B_4$ is computed and we prove Theorem 1.2.

2. Lawrence-Krammer representation of $B_4$

Let $B_n$ be the braid group of $n$ strings, $B_n \to \mathbb{Z} \simeq \langle x \rangle$ the Abelianisation and $LK$ the Lawrence-Krammer representation

$$LK : B_n \to GL(n(n-1)/2; \mathbb{Z}[q^{\pm 1}, t^{\pm 1}]).$$

In this paper, we treat the case $n = 4$, and we discuss the definition of the braid group and the Lawrence-Krammer representation for only this case. The braid group $B_4$ admits the presentation:

$$B_4 = \langle \sigma_1, \sigma_2, \sigma_3 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2, \ \sigma_2\sigma_3\sigma_2 = \sigma_3\sigma_2\sigma_3, \ \sigma_1\sigma_3 = \sigma_3\sigma_1 \rangle.$$

The Lawrence-Krammer representation of $B_4$ is defined as follows (see [1, 2, 3] for general cases):

\[
LK(\sigma_1) = \begin{pmatrix}
tq^2 & 0 & 0 & 0 & 0 \\
tq(q-1) & 0 & 0 & q & 0 \\
tq(q-1) & 0 & 0 & 0 & q \\
0 & 1 & 0 & 1-q & 0 \\
0 & 0 & 1 & 0 & 1-q \\
0 & 0 & 0 & 0 & 1
\end{pmatrix},
\]

\[
LK(\sigma_2) = \begin{pmatrix}
1-q & q & 0 & 0 & 0 \\
1 & 0 & 0 & tq^2(q-1) & 0 \\
0 & 0 & 1 & tq(q-1)^2 & 0 \\
0 & 0 & 0 & tq^2 & 0 \\
0 & 0 & 0 & tq(q-1) & 0 \\
0 & 0 & 0 & 0 & 1 \\
1 & 1-q
\end{pmatrix},
\]

\[
LK(\sigma_3) = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1-q & q & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1-q & q \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & tq^2
\end{pmatrix}.
\]
3. Twisted Alexander Polynomial

In this section, we compute the twisted Alexander polynomial. All notations are the same as ones used in [4], unless we state otherwise.

First, we obtain a denominator in the twisted Alexander polynomial by an explicit calculation.

**Lemma 3.1.**

\[
\det(I_6 - xLK(\sigma_3)) = (1 - x)^3 (1 + qx)^2 (1 - q^2tx).
\]

Next, we calculate a numerator in the twisted Alexander polynomial. In our case, we have the $18 \times 12$-matrix $M_3$ which is obtained from the Alexander matrix removing the third column. The numerator which we need is the greatest common divisor of $\det M^I_3$ for all the choices of the indices $I$. Here $I = (i_1, i_2, \ldots, i_{12})$ and $M^I_3$ denotes the square matrix consisting of the $i_k$-th rows of the matrix $M_3$, where $1 \leq i_1 < \cdots < i_{12} \leq 18$.

**Lemma 3.2.** For any index $I$, $\det M^I_3$ has a common divisor $(1 - x)^3(1 + qx)^2 (1 - q^2tx)$.

**Proof:** For a given $18 \times 12$-matrix $A$, we denote by $A(i; a_1, \ldots, a_{12})$ the matrix obtained from $A$ adding $a_j$ times the $j$-th column to the $i$-th column. We note that

\[
\det A(i; a_1, \ldots, a_{12})^I = (1 + a_i) \det A^I.
\]

1. First, we consider

\[
M^{(1)} = M_3(4; -1 + q^2t, p, p, 0, 1, 0, 0, 0, 0, 0, 0, 0),
\]

where $p = -1 - qt + q^2t$. Then we can take a term $1 - x$ as a common divisor from the fourth column. Next, we observe

\[
M^{(2)} = M^{(1)}(12; 0, 0, 0, 0, 0, 0, 0, q^2, pq, (1 - q)^2qt, -1 + q^2t, p, 0)
\]

and

\[
M^{(3)} = M^{(2)}(8; -1 + q^2t, (-1 + q)qt, (-1 + q)qt, 0, 0, 0, -q, 0, 0, 0, 0, 0).
\]

Therefore the eighth and the twelfth columns have common divisors $1 - x$ and $\det M^I_3$ has a divisor $(1 - x)^3$ for any index $I$.

2. Similarly, it can be considered

\[
M^{(4)} = M_3(12; 0, 0, 0, 0, 0, 0, 0, q^2, pq^2, -1 + q^3t - q^4t + pq, -q^2(1 + qt), -pq, 0)
\]

and

\[
M^{(5)} = M^{(4)}(5; 0, -q^2, q, -q, 0, 0, -q^2, -q^2, 1 + q, 0, 0, 0).
\]

Then the fifth and the twelfth columns have common divisors $1 + qx$ and $\det M^I_3$ has a divisor $(1 + qx)^2$ for any index $I$. 
3. Finally, we set
\[
M^{(6)} = M_3(12; 0, q^3t(1 - q)(1 - q^2t), \\
q^2t(-1 + q)(1 - q^2t + q^4t^2 + pq), q^2t(1 - q)(1 - q^2t), \\
qt(-1 + q)(1 - q^2t + q^4t^2 + pq), (1 + qt)(1 - q^2t)^2, \\
(1 - q)q^4t, (-1 + q)q^4t^2, q^2t(-1 + q)(1 - q - qt + q^4t^2), \\
0, q(1 + qt - q^2t)(1 - q^2t^2)(1 - q - q^2t)(1 - q^2t^2)).
\]

The twelfth column of \( M^{(6)} \) has a common divisor \( 1 - q^2tx \). We need to note that the determinant of this matrix \( M^{(6)} \) is different from that of \( M^I_3 \). More precisely,
\[
\det M^{(6)}I = (1 + (1 - q - q^2t)(1 - q^3t^2)) \det M^I_3.
\]

However, the greatest common divisor of two polynomials
\[
1 + (1 - q - q^2t)(1 - q^3t^2) \quad \text{and} \quad 1 - q^2tx
\]
is a unit, that is, they are relatively prime. This deduces that \( \det M^I_3 \) has a divisor \( 1 - q^2tx \) for any index \( I \). Then it completes the proof.

**Lemma 3.3.** There exist indices \( I_1, I_2 \) such that
\[
\gcd(\det M^{I_1}_3, \det M^{I_2}_3) = (1 - x)^3(1 + qx)^2(1 - q^2tx).
\]

**Proof:** We select
\[
I_1 = (1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12), \\
I_2 = (2, 3, 4, 5, 6, 7, 9, 10, 11, 12, 15, 17)
\]
and calculate \( \det M^{I_1}_3, \det M^{I_2}_3 \) explicitly, then we get the conclusion.

The above two lemmas deduce that \( \det M^I_3 \) has a common divisor \( (1 - x)^3(1 + qx)^2(1 - q^2tx) \) and does not have any other common divisor, then the numerator is settled. It follows by the definition that

**Theorem 3.4.** The twisted Alexander polynomial \( \Delta_{B_4, LK}(x) \) for the Lawrence-Krammer representation with the Abelianisation \( B_4 \to \mathbb{Z} \cong \langle x \rangle \) is given by
\[
\Delta_{B_4, LK}(x) = 1.
\]

**Remark 3.5.** The twisted Alexander polynomial for the Lawrence-Krammer representation is not always trivial for \( n \). In fact, we get \( \Delta_{B_4, LK}(x) = 1 + q^3tx^3 \) by an easy calculation.
REFERENCES


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