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FINITE GROUP ACTIONS ON 4-MANIFOLDS

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Abstract

Let X be a closed, oriented, smooth 4-manifold with a finite fundamental group and with a non-vanishing Seiberg-Witten invariant. Let G be a finite group. If G acts smoothly and freely on X, then the quotient X/G cannot be decomposed as $X_1 \sharp X_2$ with $b_2^+(X_i) > 0$, $i = 1, 2$. In addition let X be symplectic and $c_1(X)^2 > 0$ and $b_2^+(X) > 3$. If σ is a free anti-symplectic involution on X then the Seiberg-Witten invariants on X/σ vanish for all spin$_C$ structures on X/σ, and if η is a free symplectic involution on X then the quotients X/σ and X/η are not diffeomorphic to each other.

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1. Introduction

Let X be a closed, oriented, smooth 4-dimensional manifold. In [W], Witten introduced Seiberg-Witten invariants on X. If X is decomposed as a connected sum $X_1 \sharp X_2$ with $b_2^+(X_i) > 0$, $i = 1, 2$, then the Seiberg-Witten invariants vanish for all spin$_C$ structures on X.

In [W1] and [W2], Wang studied free (anti-) holomorphic involutions on the simply connected Kähler surfaces X with $c_1(K_X)^2 > 0$, where K_X is the canonical complex line bundle of X. He showed that Seiberg-Witten invariants vanish for another class of 4-manifolds which are not diffeomorphic to a connected sum of two manifolds with both $b_2^+ > 0$. In this paper we study finite group actions on closed (symplectic) 4-manifolds with finite fundamental groups, rather than involutions on simply connected Kähler surfaces.

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In Section 2 we introduce the Seiberg-Witten invariants on 4-manifolds and their basic properties.

In Section 3 the following result is shown : Let $\overline{X}$ be a closed, oriented smooth 4-manifold with a finite fundamental group and with a non-vanishing Seiberg-Witten invariant. If a finite group G acts smoothly and freely on the 4-manifold $\overline{X}$, then the quotient $X = \overline{X}/G$ is not diffeomorphic to any smooth connected sum $X_1 \sharp X_2$ with both $b_2^+(X_i) > 0$.

In Section 4 we show that if $\overline{X}$ is a closed symplectic 4-manifold and $c_1(K_{\overline{X}})^2 > 0$ and $b_2^+(\overline{X}) > 3$ and σ a free anti-symplectic involution on $\overline{X}$, then the quotient manifold $X = \overline{X}/\sigma$ has vanishing Seiberg-Witten invariants.

In Section 5 we show the following : If $\overline{X}$ is a closed symplectic 4-manifold with a finite fundamental group and η, σ are two free involutions on $\overline{X}$, which are symplectic and anti-symplectic, respectively. If $c_1(\overline{X})^2 > 0$ and $b_2^+(\overline{X}) > 3$, then the quotient manifolds $X = \overline{X}/\eta$, $X' = \overline{X}/\sigma$ are not diffeomorphic to each other. If $\overline{X}$ is simply connected and X is not spin, then X and X' are homeomorphic to each other.

2. Review of Seiberg-Witten invariant

Let X be a closed, oriented, 4-dimensional manifold with $b_2^+(X) > 1$. For each $c \in H^2(X, \mathbb{Z})$ with $c = w_2(X) \mod 2$, there is a complex line bundle L called spin_C structure on X with $c = c_1(L)$. There are a pair of twisted spinor bundles $W^\pm$ associated with L on X. The Clifford multiplication $\sigma : W^+ \otimes T^*X \to W^-$ induces $c_+ : \Lambda^{2,+} \otimes \mathbb{C} \to \text{End}(W^+)_0$, there is a correspondence $\tau : W^+ \times W^+ \to \text{End}(W^+)_0$ given by $\tau(\phi, \phi) = (\phi \otimes \overline{\phi}^t)_0$, where $\text{End}(W^+)_0$ is the set of traceless endomorphisms of W^+. Let $\mathscr{A}(L)$ be the set of connections on L. Then the Levi-Civita connection on X together with a connection $A \in \mathscr{A}(L)$ on L induce a covariant derivative ∇_A on W^+.

The composition of ∇_A and the Clifford multiplication σ defines a Dirac operator

$$D_A : \Gamma(W^+) \longrightarrow \Gamma(W^-).$$

The Weitzenböck formula for D_A is

$$D_A^* D_A \phi = \nabla_A^* \nabla_A \phi + \frac{1}{4} R\phi + \frac{1}{2} F_A^+ \phi,$$

where R is the scalar curvature on X.

For each pair $(A, \phi) \in \mathscr{A}(L) \times \Gamma(W^+)$, as in [D2] we consider the functional

$$\begin{aligned} E(A, \phi) &= \int_X \left(|\nabla_A \phi|^2 + \frac{1}{2}|F_A|^2 + \frac{1}{8}(|\phi|^2 + R)^2 \right) d\mu \\ &= \int_X \left(|D_A \phi|^2 + |F_A^+ + \tau(\phi, \phi)|^2 \right) d\mu + \int_X \frac{R^2}{8}\, d\mu + 2\pi^2 c_1(L)^2. \end{aligned}$$

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The absolute minima of E yield the solutions of the Seiberg-Witten monopole equations

$$\begin{cases} D_A\phi = 0 \\ F_A^+ = -\tau(\phi, \phi). \end{cases}$$

The group $C^\infty(X, U(1))$ of gauge transformations to L acts on the space of solutions of the monopole equations. The quotient $\mathfrak{M}(L)$ of the space of solutions of the equations modulo the gauge transformation group is called the moduli space associated to the spin$_C$ structure L on X.

If we perturb the equations or find a generic metric on X, the moduli space $\mathfrak{M}(L)$ is a compact orientable d-dimensional manifold where $d = \frac{1}{4}(c_1(L)^2 - (2\chi + 3\sigma))$. Let x_0 be a fixed base-point in X, then the evaluation at x_0 gives a representation $\rho : C^\infty(X, U(1)) \to U(1)$, which induces a $U(1)$-vector bundle $E \to \mathfrak{M}(L)$ on $\mathfrak{M}(L)$. If the dimension d ($\equiv 2s$) of $\mathfrak{M}(L)$ is even, then we define an invariant

$$SW(L) = \langle c_1(E)^s, \mathfrak{M}(L)\rangle$$

which is called the Seiberg-Witten invariant of L on X.

We sum up the basic properties of the Seiberg-Witten invariants, which have been developed by many people: see for example [KM, T1, T2] and [W].

THEOREM 2.1. *Let X be a closed oriented 4-manifold with $b_2^+(X) > 1$.*

(1) *There is only a finite number of line bundles L for which $SW(L) \neq 0$.*

(2) *If $X = X_1 \sharp X_2$ with $b_2^+(X_i) > 0$, $i = 1, 2$, then $SW(L) = 0$ for all* spin$_C$ *structures L on X.*

(3) *For a* spin$_C$ *structure L on X, $SW(L)$ is independent on the metrics of X, and depends only on the class $c_1(L)$.*

(4) *If f is a self-diffeomorphism of X, then the invariant of L, $SW(L) = \pm SW(f^*L)$, is the same as the invariant of $f^*(L)$, up to sign.*

(5) *If X admits a metric of positive scalar curvature, then the Seiberg-Witten invariants on X vanish.*

(6) *If X is a closed symplectic 4-manifold with the canonical complex line bundle K_X, then $SW(K_X) = \pm 1$.*

3. Indecomposable 4-manifolds

Let X be a closed symplectic 4-manifold. The tangent bundle TX of X admits an almost complex structure which is an endomorphism $J : TX \to TX$ with $J^2 = -I$. The almost complex structure J defines a splitting

$$T^*X \otimes \mathbb{C} = T^{1.0} \oplus T^{0.1}$$

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where J acts on $T^{1,0}$ and $T^{0,1}$ as multiplication by $-i$ and i, respectively. The canonical bundle K_X on X associated to the almost complex structure J is defined by $K_X = \Lambda^2 T^{1,0}$.

A symplectic structure ω on X is defined as a closed two-form with $\omega \wedge \omega \neq 0$ everywhere. An almost complex structure J on X is said to be compatible with the symplectic structure ω if $\omega(Jv_1, Jv_2) = \omega(v_1, v_2)$ and $\omega(v, Jv) > 0$ for a non-zero tangent vector v.

The space of compatible almost complex structures of a given symplectic structure on X is non-empty and contractible. If an almost complex structure J is compatible with ω, then for any $v, w \in TX$, $g(v, w) = \omega(v, Jw)$ defines a Riemannian metric on X. For such a metric on X, the symplectic structure ω is self-dual and $\omega \wedge \omega$ gives the orientation on X. On the other hand, any metric on X for which ω is self-dual can define an almost complex structure J which is compatible with the symplectic structure ω.

Let (X, ω) be a closed, symplectic, 4-manifold. A diffeomorphism σ on X is symplectic, anti-symplectic if σ satisfies $\sigma^*\omega = \omega, -\omega$, respectively.

Let (X, ω) be a closed, symplectic, 4-manifold. Then an involution σ on X is symplectic, anti-symplectic if and only if it satisfies $\sigma_* J = J\sigma_*, -J\sigma_*$, respectively, for some compatible almost complex structure J on X with the symplectic structure ω. If (X, ω) is a Kähler surface with Kähler form ω, then an involution σ on X is symplectic, anti-symplectic if and only if it is holomorphic, anti-holomorphic, respectively.

In [W1] Wang showed the following results using Witten's vanishing theorem for Seiberg-Witten invariant.

THEOREM 3.1. *If X is a simply connected Kähler surface with $b_2^+(X) > 1$ and σ is a free involution, then the quotient manifold X/σ cannot be decomposed as $X_1 \sharp X_2$ with $b_2^+(X_i) > 0$, $i = 1, 2$.*

We study this theorem for closed, oriented 4-manifolds $\overline{X}$ with finite fundamental groups. Let G be a finite group. In this section we assume that G acts smoothly and freely on $\overline{X}$.

THEOREM 3.2. *Let $\overline{X}$ be a closed, oriented smooth 4-manifold with a finite fundamental group $\pi_1\overline{X}$ and with a non-vanishing Seiberg-Witten invariant. If a finite group G acts smoothly and freely on the 4-manifold $\overline{X}$, then the quotient manifold $X \equiv \overline{X}/G$ cannot be decomposed as a smooth connected sum $X_1 \sharp X_2$ with $b_2^+(X_i) > 0$, $i = 1, 2$.*

PROOF. If G is trivial, then there is nothing to prove. So we may assume that G is a non-trivial finite group. From the covering $G \to \overline{X} \to X$, we have an exact sequence $0 \to \pi_1\overline{X} \to \pi_1 X \to G \to 0$. Since the fundamental group $\pi_1\overline{X}$ is finite,

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the order $|\pi_1 X| \equiv n$ of $\pi_1 X$ is finite and ≥ 2. Assume that X can be decomposed as a connected sum $X_1 \sharp X_2$ with $b_2^+(X_i) > 0$, $i = 1, 2$. Since $\pi_1 X = \pi_1 X_1 * \pi_1 X_2$, which is the free product of $\pi_1 X_1$ and $\pi_1 X_2$, is finite, we may assume that $\pi_1 X \cong \pi_1 X_1$ and $\pi_1 X_2 \cong \{1\}$.

Let $\overline{\overline{X_1}}$ be the universal covering space of X_1. Then the universal covering space $\overline{\overline{X}}$ of X is decomposed as $\overline{\overline{X}} \cong \overline{\overline{X_1}} \sharp n X_2$. Since the order $|\pi_1 X| = n \geq 2$, the Seiberg-Witten invariants on $\overline{\overline{X}}$ vanish for all spin$_C$ structure on $\overline{\overline{X}}$.

Suppose that $\overline{L} \to \overline{X}$ is a spin$_C$ structure on $\overline{X}$ such that the Seiberg-Witten invariant $SW(\overline{L})$ is non-trivial.

Let $\pi : \overline{\overline{X}} \to \overline{X}$ be the covering projection and let $(\overline{A}, \overline{\phi})$ be an irreducible solution of the monopole equations for the spin$_C$ structure $\overline{L}$ on $\overline{X}$. The characteristic number $b_2^+(\overline{\overline{X}}) = b_2^+(\overline{\overline{X_1}}) + n b_2^+(X_2) \geq n \geq 2$. Let $\overline{\overline{L}} = \pi^* \overline{L}$, $\overline{\overline{g}} = \pi^* \overline{g}$, $\overline{\overline{A}} = \pi^* \overline{A}$, $\overline{\overline{\phi}} = \pi^* \overline{\phi}$ and $\overline{\overline{W}}^{\pm} = \pi^* \overline{W}^{\pm}$ be the pull-backs the spin$_C$ structure $\overline{L}$, a metric $\overline{g}$, a solution $(\overline{A}, \overline{\phi})$ of the monopole equations and the spinor bundles $\overline{W}^{\pm}$ associated with $\overline{L}$ on $\overline{X}$, respectively.

Then by Lemma 3.3, $\overline{\overline{L}}$ is a spin$_C$ structure on $\overline{\overline{X}}$ and $\overline{\overline{g}}$ is a G-invariant metric on $\overline{\overline{X}}$. The bundles $\overline{\overline{W}}^{\pm}$ are the spinor bundles corresponding to $\overline{\overline{L}}$ on $\overline{\overline{X}}$ and $(\overline{\overline{A}}, \overline{\overline{\phi}})$ is an irreducible solution of the monopole equations for the spin$_C$ structure $\overline{\overline{L}}$ on $\overline{\overline{X}}$. Thus we have a contradiction. □

LEMMA 3.3. *If $\overline{L} \to \overline{X}$ is a* spin$_C$ *structure on $\overline{X}$, then the pull-back $\overline{\overline{L}} \to \overline{\overline{X}}$ is a* spin$_C$ *structure on $\overline{\overline{X}}$.*

PROOF. Since $\overline{L}$ is a spin$_C$ structure on $\overline{X}$, the first Chern class $c_1(\overline{L}) = w_2(T\overline{X}) \mod 2$. By the naturalness of cohomology $\pi^* c_1(\overline{L}) = \pi^* w_2(T\overline{X}) \mod 2$, and $c_1(\overline{\overline{L}}) = w_2(T\overline{\overline{X}}) \mod 2$. □

COROLLARY 3.4. *Let $\overline{X}$ be a closed symplectic 4-manifold with a finite fundamental group $\pi_1 \overline{X}$. If a finite group G acts smoothly and freely on the manifold $\overline{X}$, then the quotient $X = \overline{X}/G$ cannot be decomposed as $X_1 \sharp X_2$ with $b_2^+(X_i) > 0$, $i = 1, 2$.*

In [C4] and [KMT] there are many examples which are 4-dimensional manifolds without symplectic structures but with non-trivial Seiberg-Witten invariants. These examples are the connected sums of symplectic manifolds and negative definite manifolds with certain fundamental groups.

4. A vanishing theorem

A map σ between two almost complex manifolds is called anti-holomorphic if $\sigma_* J_1 = -J_2 \sigma_*$ on the tangent bundles, where the J_i are the almost complex structures of the manifolds. Let K denote the canonical bundle of an almost complex manifold.

In [W1] Wang proved a vanishing theorem for Seiberg-Witten invariants on the quotients of Kähler surfaces under free anti-holomorphic involutions.

THEOREM 4.1 ([Wang]). *Let $\overline{X}$ be a simply-connected Kähler surface with $c_1(K_{\overline{X}})^2 > 0$ and $b_2^+(\overline{X}) > 3$. Suppose that $\sigma : \overline{X} \to \overline{X}$ is an anti-holomorphic involution without fixed points. Then the quotient manifold $\overline{X}/\sigma = X$ has vanishing Seiberg-Witten invariants.*

We extend Theorem 4.1 to closed, symplectic, 4-manifolds with finite fundamental groups.

THEOREM 4.2. *Let $\overline{X}$ be a closed, symplectic, 4-manifold with a finite fundamental group $\pi_1(\overline{X})$, $c_1(\overline{X})^2 > 0$ and $b_2^+(\overline{X}) > 3$. If $\sigma : \overline{X} \to \overline{X}$ is a free anti-symplectic involution on $\overline{X}$, then the Seiberg-Witten invariants vanish on the quotient $X = \overline{X}/\sigma$.*

PROOF. Let $p : \overline{X} \to X$ be the projection. The tangent bundle $T\overline{X} = p^*(TX)$ is the pull-back of the tangent bundle TX by p. We get the relations of the Euler characteristics $\chi(\overline{X}) = 2\chi(X)$ and the signatures sign $(\overline{X}) = 2$sign (X), since $\pi_1(\overline{X})$ is finite the first Betti numbers $b_1(\overline{X}) = b_1(X) = 0$ are zero, and $b_2^+(X) = \frac{1}{2}(b_2^+(\overline{X}) - 1) > 1$.

Assume that $L \to X$ is a spin$_C$ structure on X which has a non-vanishing Seiberg-Witten invariant. Let $W^{\pm}$ be the spinor bundles on X associated with L. By Lemma 3.3 the pull-back $\overline{L} = p^*L \to \overline{X}$ is a spin$_C$ structure on $\overline{X}$ and the pull-back $p^*(W^{\pm}) = \overline{W}^{\pm} \to \overline{X}$ is the spinor bundle on $\overline{X}$ associated with $\overline{L}$. For a generic metric g on X the pull-back $\overline{g} = p^*(g)$ is a σ-invariant metric on $\overline{X}$. Let $\overline{\omega}$ be a self-dual symplectic form on $\overline{X}$ and let J be an almost complex structure compatible with $\overline{\omega}$ on $\overline{X}$.

Suppose that (A, ϕ) is an irreducible solution to the Seiberg-Witten equations for the spin$_C$ structure L on X. As in the proof of Theorem 3.2 the pull-backs $\overline{A} = p^*A$ and $\overline{\phi} = p^*\phi$ also give a solution of the Seiberg-Witten equations to the spin$_C$ structure $\overline{L}$ on $\overline{X}$, since the connection $\overline{A}$ of $\overline{L}$ and the section $\overline{\phi}$ of $\overline{W}^+$ exist globally and locally the Seiberg-Witten equations to L and $\overline{L}$ are the same. Clearly the solution $(\overline{A}, \overline{\phi})$ is also irreducible.

Let us investigate the solution $(\overline{A}, \overline{\phi})$ of the Seiberg-Witten equations of the spin$_C$ structure $\overline{L}$ on the symplectic 4-manifold $(\overline{X}, \overline{\omega})$.

Let $\overline{W}^+ = \overline{E} \oplus (K_{\overline{X}}^{-1} \otimes \overline{E})$ and $\overline{L} = \det \overline{W}^+$ for some line bundle $\overline{E} \to \overline{X}$ on $\overline{X}$. For the section $\overline{\phi} = (\alpha, \beta) \in \Gamma(\overline{W}^+)$, consider the perturbed equations

$$(*) \qquad \begin{cases} \overline{\partial}_{\overline{A}}\alpha = -\overline{\partial}_{\overline{A}}^*\beta \\ F_{\overline{A}}^{0,2} = \overline{\alpha}\beta \\ iF_{\overline{A}}^{1,1}\overline{\omega} = (|\beta|^2 - |\alpha|^2 + \gamma^2), \end{cases}$$

where we use the notation of [D2] and γ is a real parameter.

Using the results of computations in [D2], we have

$$\int_{\overline{X}} |\nabla_{\overline{A}}\alpha|^2 \, d\,\mathrm{vol} = -2\int_{\overline{X}} (\beta, N \circ \partial_{\overline{A}}\alpha) \, d\,\mathrm{vol} - \int_{\overline{X}} |\alpha|^2|\beta|^2 \, d\,\mathrm{vol} - \int_{\overline{X}} (|\alpha|^2 - \gamma^2)^2 \, d\,\mathrm{vol} - \gamma^2 \int_{\overline{X}} (|\alpha|^2 - \gamma^2) \, d\,\mathrm{vol},$$

where N is the Nijenhuis tensor on the symplectic manifold $\overline{X}$. Since $\sigma : \overline{X} \to \overline{X}$ is an anti-symplectic involution on $\overline{X}$, we have

$$c_1(\overline{L})\overline{\omega} = \sigma^* c_1(\overline{L})\sigma^*\overline{\omega} = -c_1(\overline{L})\overline{\omega},$$

and so

$$\int_{\overline{X}} |\beta|^2 \, d\,\mathrm{vol} = \int_{\overline{X}} (|\alpha|^2 - \gamma^2) \, d\,\mathrm{vol}.$$

We have

$$\|\nabla_{\overline{A}}\alpha\|^2 + \gamma^2\|\beta\|^2 + \big\||\alpha|^2 - \gamma^2\big\|^2 \le \frac{1}{2}\|\nabla_{\overline{A}}\alpha\|^2 + c\|\beta\|^2,$$

and by rearranging

$$\frac{1}{2}\|\nabla_{\overline{A}}\alpha\|^2 + \big\||\alpha|^2 - \gamma^2\big\|^2 \le (c - \gamma^2)\|\beta\|^2$$

for some constant c, which depends only on the manifold $\overline{X}$. For details, see [D2].

If we choose the real parameter γ such that $\gamma^2 > c$, then $\nabla_{\overline{A}}\alpha = 0$, $|\alpha|^2 = \gamma^2$ and the solution will be $\overline{\phi} = (\alpha, \beta) = (\gamma, 0)$ up to gauge equivalence. Thus we have $F_{\overline{A}}^+ = 0$ and $F_{\overline{A}}$ is anti-self-dual and $c_1(\overline{L})^2 \le 0$.

The solution $\overline{\phi} = (\alpha, \beta) = (\gamma, 0)$ is a non-zero, covariantly constant, section of $\overline{W}^+ = \overline{E} \oplus (K_{\overline{X}}^{-1} \otimes \overline{E})$. The line bundle $\overline{E}$ is trivial and so the determinant line bundle $\overline{L} = \det W^+ = K_{\overline{X}}^{-1}$ is the inverse of the canonical line bundle $K_{\overline{X}}$. Since $c_1(\overline{L})^2 = c_1(K_{\overline{X}})^2 = c_1(\overline{X})^2 > 0$ by our hypothesis, thus we have a contradiction.

For such a real number $\gamma^2 > c$, the perturbed Seiberg-Witten equations $(*)$ have neither reducible nor irreducible solutions. The perturbation γ is generic and the Seiberg-Witten invariant is zero. Since L is an arbitrary spin$_C$ structure on X, the Seiberg-Witten invariants of X all vanish. We complete the proof of Theorem 4.2. □

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In Theorem 4.2 the quotient space $X = \overline{X}/\sigma$ is a smooth 4-manifold which does not have any symplectic structure in view of Taubes result [T1] and the vanishing of the Seiberg-Witten invariants of X.

By Theorem 3.2 and Theorem 4.2 the quotient $X = \overline{X}/\sigma$ cannot be decomposed as $X = X_1 \sharp X_2$ with both $b_2^+(X_i) > 0$ and the Seiberg-Witten invariants vanish for all spin$_C$ structures on X.

5. Smooth structures on some quotient manifolds

In [D1], Donaldson proved that the Dolgachev surface $D_{2,3}$ and $\mathbb{CP}^2 \sharp 9\overline{\mathbb{CP}^2}$ are homeomorphic but not diffeomorphic. This implies that the h-cobordism conjecture in 4-manifolds does not hold. After that many people have obtained good results on smooth structures of 4-manifolds using the Donaldson invariant. Recently in [W2], Wang showed that the quotients of a complex surface under free holomorphic, anti-holomorphic involutions are homeomorphic but not diffeomorphic to each other using the Seiberg-Witten invariants. A smooth map $\sigma : (X_1, J_1) \to (X_2, J_2)$ between complex manifolds is called anti-holomorphic if $\sigma_* J_1 = -J_2 \sigma_*$ on the tangent bundles, where J_1 and J_2 are the complex structures on X_1 and X_2, respectively. We denote by K_X the canonical bundle of an almost complex manifold X.

THEOREM 5.1 ([W2]). *Let $\overline{X}$ be a simply connected Kähler surface, and suppose that η, σ are two free involutions on $\overline{X}$, which are respectively holomorphic, anti-holomorphic.*

(1) *If $c_1(K_{\overline{X}})^2 > 0$ and $b_2^+(\overline{X}) > 3$, then the quotient manifolds $X = \overline{X}/\eta$, $X' = \overline{X}/\sigma$ are not diffeomorphic to each other.*

(2) *If X is not spin, then X and X' are homeomorphic to each other.*

In this section we study Theorem 5.1 for symplectic 4-manifolds with finite fundamental groups.

COROLLARY 5.2. *Let $\overline{X}$ be a closed symplectic 4-manifold with a finite fundamental group $\pi_1 \overline{X}$. Suppose that η, σ are two free involutions on $\overline{X}$, which are respectively symplectic, anti-symplectic.*

(a) *If $c_1(\overline{X})^2 > 0$ and $b_2^+(\overline{X}) > 3$, then the quotients $X = \overline{X}/\eta$, $X' = \overline{X}/\sigma$ are not diffeomorphic to each other.*

(b) *If $\overline{X}$ is simply connected and X is not spin, then X and X' are homeomorphic to each other.*

PROOF. (a) Let $\pi : \overline{X} \to X$ be the double covering projection. As in the proof of Theorem 4.2, $b_2^+(X) > 1$.

By averaging let $\overline{g}$ be a η-invariant metric on $\overline{X}$ and let $\overline{\omega}$ a self-dual symplectic form on $\overline{X}$ and J an almost complex structure on $\overline{X}$ which is compatible with $\overline{\omega}$. Since the free involution η preserves the symplectic structure $\overline{\omega}$, the push-downs $\omega = \pi_* \overline{\omega}$ of $\overline{\omega}$ and $g = \pi_* \overline{g}$ of $\overline{g}$ through the projection π are a symplectic structure (because the closedness and nondegeneracy are local) and a metric on X, respectively. Thus we have a symplectic manifold (X, ω).

By Taubes [T1] the Seiberg-Witten invariant for the canonical bundle K_X on X is non-trivial. While by Theorem 4.2 the quotient $X' = \overline{X}/\sigma$ has vanishing Seiberg-Witten invariant. Since the Seiberg-Witten invariant is a diffeomorphism invariant, the quotients X and X' are not diffeomorphic to each other.

(b) The proof is the same as the proof of (2) of Theorem 5.1. For details, see [W2]. □

REMARK. 1. In (b) of Corollary 5.2, we assume that $\overline{X}$ is simply connected, then the fundamental group of the quotient is $\mathbb{Z}_2$, and spin structures determine the topological structures of the quotients. If we know the classification of 4-manifolds with finite fundamental groups, then we may extend the Corollary 5.2 (b).

2. There are many (simply connected) closed non-Kähler, symplectic 4-manifolds. In fact Gompf [G] constructed infinite families of simply connected symplectic 4-manifolds which are non-Kähler. For instance, let X_1, X_2 be simply connected Dolgachev surfaces. By Dehn twisting fiber sum for the fibers of the relative prime multiplicities we have the symplectic fiber-sum $X = X_1 \sharp X_2$ which is not Kähler, but symplectic.

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