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A GENERALIZATION OF ROLLE'S THEOREM AND AN APPLICATION TO A NONLINEAR EQUATION

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Abstract

Given two C^1-functions $g\colon \mathbf{R} \to \mathbf{R}$, $u\colon [0,1] \to \mathbf{R}$ such that $u(0) = u(1) = 0$, $g(0) = 0$, we prove that there exists c, with $0 < c < 1$, such that $u'(c) = g(u(c))$. This result implies the classical Rolle's Theorem when $g \equiv 0$. Next we apply our result to prove the existence of solutions of the Dirichlet problem for the equation $x'' = f(t, x, x')$.

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0. Introduction

Let $f\colon [0,1] \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ be a continuous function and suppose that there exist a continuous function $\phi\colon [0,\infty) \to (0,\infty)$ and a constant $R > 0$ such that

$$f(t,x,0)x \geq 0 \quad \text{if } |x| = R,$$
$$|f(t,x,y)| \leq \phi(|y|) \quad \text{if } |x| \leq R.$$

It is well known that

0.1. THEOREM. *The Dirichlet problem*

(0.1) $$x'' = f(t,x,x'), \qquad x(0) = x(1) = 0$$

has at least one solution if

$$\int_0^\infty s\phi(s)^{-1}\,ds > R.$$

For instance see [1] or [2].

In this paper we prove a generalized Rolle's Theorem and we apply this result to obtain the following generalization of Theorem 0.1.

0.2. THEOREM. *Suppose that there are* $(r_0, s_0), (r_1, s_1) \in \mathbf{R}\times\mathbf{R}, r_1 < 0 < r_0$, *such that*

(i) $f(t, x, s_0) \geq 0$ *if* $r_0 \leq x \leq r_0\exp(K)$,

(ii) $f(t, x, s_1) < 0$ *if* $r_1\exp(K) \leq x \leq r_1$,

where $K = \max\{|s_0/r_0|, |s_1/r_1|\}$. *Assume further that*

(iii) $|f(t, x, y)| \leq \phi(|y|)$ *if* $r_1\exp(K) \leq x \leq r_0\exp(K)$,

(iv) $\int_0^\infty s\phi(s)^{-1}\,ds > \max\{-r_1, r_0\}\exp(K)$.

Then the problem (0.1) *has at least one solution* v *such that* $r_1\exp(K) \leq v \leq r_0 \exp(K)$.

1. A general existence principle

In the following, C_0^2 denotes the space of functions $u\colon [0, 1] \to \mathbf{R}$ of class C^2 such that $u(0) = u(1) = 0$, with the usual norm $\|u\|_2 = \max\{\|u^{(i)}\|_0, i = 0, 1, 2\}$, where $\|u^{(i)}\|_0 = \sup\{|u^{(i)}(t)|\colon 0 \leq t \leq 1\}$. For reference purposes, we state the following general, and now classical, result (see [2] for details).

1.1. THEOREM. *Let* U *be an open and bounded neighborhood of* $0 \in C_0^2$ *such that the problem*

$$x'' = \lambda f(t, x, x'), \qquad x(0) = x(1) = 0$$

has no solutions in the boundary ∂U *of* U *for* $0 < \lambda < 1$. *Then the problem* (0.1) *has at least one solution in the closure* $\mathrm{cl}(U)$ *of* U.

2. A Nagumo inequality

In this section we obtain a priori bounds for derivatives:

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2.1. PROPOSITION. *Let $v \in C_0^2$. If $v'(t_0) \neq 0$ then there is an interval $[a, b] \subset [0, 1]$ such that v and v' have constant sign in (a, b); $t_0 \in \{a, b\}$ and v' has a zero at one of the endpoints of $[a, b]$.*

PROOF. We consider two cases.

Case 1; $v(t_0) \neq 0$. Since $v(0) = v(1) = 0$ there is an interval $[c, d] \subset [0, 1]$ such that $v(t) \neq 0$ if $t \in (c, d)$, $v(c) = v(d) = 0$ and $c < t_0 < d$. In particular $v'(t_1) = 0$ for some $t_1 \in [c, d]$ and hence there is an interval $[a, b] \subset [c, d]$ such that $t_0 \in \{a, b\}$, $v'(a) \cdot v'(b) = 0$ and $v'(t) \neq 0$ for $t \in (a, b)$, as required.

Case 2; $v(t_0) = 0$. Since $v'(t_0) \neq 0$ there is an interval $[c, d] \subset [0, 1]$ such that $t_0 \in \{c, d\}$, $v(c) = v(d) = 0$ and $v(t) \neq 0$ if $t \in (c, d)$. The proof follows as in the first case.

2.2. COROLLARY. *Let $\phi\colon [0, \infty) \to (0, \infty)$ be a continuous function and let $v \in C_0^2$ be such that $|v''(t)| \leq \phi(|v'(t)|)$ $(0 \leq t \leq 1)$. Then*

$$\int_0^{|v'(t)|} s\phi(s)^{-1}\, ds \leq \|v\|_0 \qquad (0 \leq t \leq 1).$$

PROOF. Let $t_0 \in [0, 1]$ be such that $v'(t_0) \neq 0$ and take $[a, b] \subset [0, 1]$ as given by Proposition 2.1. If we follow the proof of Theorem 3.1 of [2] then we get

$$\int_0^{|v'(t_0)|} s\phi(s)^{-1}\, ds \leq |v(a) - v(b)|,$$

so the proof is complete, since v has constant sign in (a, b).

3. A generalized Rolle's Theorem

From now on $h\colon \mathbf{R} \to \mathbf{R}$ denotes a function of class C^1. Given $u \in C_0^2$ and $a \in [0, 1]$ we define

$$\begin{aligned} u_a(t) &= u(t) \exp\left(-\int_a^t h(u(s))\, ds\right), \\ M(u) &= \{a \in [0, 1]\colon \max u_a = u(a) > 0\}, \\ m(u) &= \{a \in [0, 1]\colon \min u_a = u(a) < 0\}. \end{aligned} \tag{3.1}$$

3.1. LEMMA. *If* $\max u > 0$ (*respectively* $\min a < 0$) *then* $M(u)$ (*respectively* $m(u)$) *is a nonempty set.*

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PROOF. If $\max u > 0$ we get $\max u_0 = u_0(a) > 0$ for some $a \in [0, 1]$. On the other hand $u_a = ku_0$ for some $k > 0$ and hence $0 < \max u_a = k \max u_0 = ku_0(a) = u_a(a) = u(a)$; or $a \in M(u)$. Similarly $m(u) \neq \varnothing$ if $\min u < 0$.

3.2. REMARKS. (a) If $a \in M(u)$ one has $u_a'(a) = 0$ and $u_a''(a) \leq 0$, which is equivalent to

$$u'(a) = u(a)h(u(a)) \tag{3.2}$$

and

$$u''(a) \leq u'(a) \cdot [h(u(a)) + u(a)h'(u(a))]. \tag{3.3}$$

(b) If $a \in m(u)$ we obtain (3.2) and the reverse of inequality (3.3).

Notice that $\max u_a = u_a(a)$ (respectively $\min u_a = u_a(a)$) if $a \in M(u)$ (respectively $a \in m(u)$).

REMARK. Let $u: [0, 1] \to \mathbf{R}$ a differentiable function and define u_a by (3.1) for $a \in [0, 1]$. If $u(0) = u(1) = 0$ we get $u_a(0) = u_a(1) = 0$ and hence $u_a'(c) = 0$ for some $c \in (0, 1)$. Therefore $u'(c) = u(c)h(u(c))$. This result implies Rolle's Theorem when $h \equiv 0$.

For each $r > 0$ let

$$U(r) = \{u \in C_0^2 \colon M(u) \neq \varnothing, u_a(t) < r \text{ if } (a, t) \in M(u) \times [0, 1]\},$$
$$V(-r) = \{u \in C_0^2 \colon m(u) \neq \varnothing, u_a(t) > r \text{ if } (a, t) \in m(u) \times [0, 1]\},$$
$$U(r, 0) = U(r) \cup U(0), V(-r, 0) = V(-r) \cup V(0),$$

where $U(0) = \{u \in C_0^2 \colon M(u) = \varnothing\}$ and $V(0) = \{u \in C_0^2 \colon m(u) = \varnothing\}$.

We give now some properties of the sets $U(r, 0), v(-r, 0)$, that we shall use in the next section.

3.3. PROPOSITION. (a) *If* $u \notin U(r, 0)$ *and* $u \in C_0^2$ (*respectively* $u \notin V(-r, 0)$) *then there is* $a \in M(u)$ (*respectively* $a \in m(u)$) *such that* $u(a) \geq r$ (*respectively* $u(a) \leq -r$).

(b) $U(r, 0), V(-r, 0)$ *are open sets* $(r > 0)$.

(c) $\partial(U(r_0, 0) \cap V(r_1, 0)) \subseteq (\partial U(r_0, 0)) \cup (\partial V(r_1, 0)), r_1 < 0 < r_0$.

(d) *If* $|h(x)| \leq K$ *for all* $x \in \mathbf{R}$ (*some* $K \geq 0$) *and* $u \in \text{cl}(U(r_0, 0) \cap V(r_1, 0))$ *for some* $r_1 < 0 < r_0$, *then* $r_1 \exp(K) \leq u(t) \leq r_0 \exp(K)$ $(0 \leq t \leq 1)$.

PROOF. (a) This is trivial.

(b) Let $\{u_n\}$ be a sequence in C_0^2 which tends to $u \in C_0^2$ in the $\| \ \|_2$-norm; then $\{u_{n,a_n}\}$ converges uniformly to u_a if $a_n \to a$. Since $[0, 1]$ is a compact set it is not difficult to prove that the complement of $U(r, 0)$ (respectively $V(r, 0)$) is a closed set.

(c) This is a consequence of (b).

Finally, to prove (d), notice first that $U(r_0, 0) \cap V(r_1, 0)$ is the union of the sets $U(r_0) \cap V(r_1)$, $U(r_0) \cap V(0)$, $V(r_1) \cap U(0)$ and $U(0) \cap V(0)$. Secondly, by Lemma 3.1, $U(0) = \{u \in C_0^2 \colon u \leq 0\}$ and $V(0) = \{u \in C_0^2 \colon u \geq 0\}$. If $u \neq 0$ it is easy to prove that one has the following cases: (i) there are $a, b \in [0, 1]$ such that $\max u_a \leq r_0$ and $\min u_b \geq r_1$; (ii) $u \geq 0$ and $\max u_a \leq r_0$ for some $a \in [0, 1]$;

(iii) $u \leq 0$ and $\min u_b \geq r_1$ for some $b \in [0, 1]$.

The proof follows from the fact that

$$u(t) = u_a(t)\exp\left(\int_a^t h(u(s))\,ds\right) \quad \text{for } a, t \in [0, 1].$$

4. The proof of Theorem 0.2

Let $\rho, \varepsilon_0 > 0$ be such that

$$\int_0^\rho \frac{sds}{\phi(s) + \varepsilon_0} > \max\{-r_1, r_0\}\exp(K).$$

For some $\varepsilon_1 > 0$ one has

$$\int_0^\rho \frac{sds}{\phi(s) + \varepsilon_0} = \max\{-r_1, r_0\}\exp(K + \varepsilon_1). \tag{4.1}$$

CLAIM. If there is $\varepsilon \in (0, \varepsilon_1)$ such that

$$|f(t, x, y)| \leq \phi(|y|) \text{ for } r_1 \exp(K + \varepsilon) \leq x \leq r_0 \exp(K + \varepsilon) \tag{4.2}$$

then the problem (0.1) has at least one solution V such that $r_1 \exp(K) \leq v(t) \leq r_0 \exp(K)$.

Proof of the claim. By the Tietze-Uryshon Lemma there is a continuous functin $\Delta \colon \mathbf{R} \times \mathbf{R} \to [-1, 1]$ such that $\Delta(x, s_0) = 1$ if $r_0 \leq x \leq r_0 \exp(k)$, and $\Delta(x, s_1) = -1$ if $r_1 \exp(K) \leq x \leq r_1$.

For each integer n such that $n\varepsilon_0 \geq 1$, we let $f_n(t, x, y) = f(t, x, y) + n^{-1}\Delta(x, y)$. Now fix n with $n\varepsilon_0 \geq 1$, and notice that there is $\delta = \delta_n > 0$ with $\delta \leq \min\{\varepsilon, 1/n\}$ such that

$$f_n(t, x, s_0) > 0 \quad \text{if } r_0 \leq x \leq r_0 \exp(K + \delta), \tag{4.3}$$

$$f_n(t, x, s_1) < 0 \quad \text{if } r_1 \exp(K + \delta) \leq x \leq r_1. \tag{4.4}$$

Choose a C^1-function $h = h_n \colon \mathbf{R} \to \mathbf{R}$ such that $h(r_i) = s_i/r_i, h'(r_i) = -s_i/r_i^2, h(x) = s_0/s$ if $x \geq r_0, h(x) = s_1/x$ if $x \leq r_1$, and $|h(x)| \leq K + \delta$ for $x \in \mathbf{R}$.

Given $u \in C_0^2$ and $a \in [0,1]$ define u_a by (3.1) and let U be the open and bounded neighborhood of $0 \in C_0^2$ defined by $u \in U$ if and only if

$$u \in U(r_0, 0) \cap V(r_1, 0), \qquad \|u'\|_0 < \rho, \qquad \|u''\|_0 < R,$$

where $R = R_n > 0$ is chosen such that

$$|f_n(t,x,y)| < R \text{ if } |x| \leq M := \max\{-r_1, r_0\}\exp(K+\delta), \tag{4.5}$$

and

$$|y| \leq \rho \ (0 \leq t, \lambda \leq 1).$$

We shall prove that the problem

$$x'' = \lambda f_n(t,x,x'), \qquad x(0) = x(1) = 0 \tag{4.6$_\lambda$}$$

has no solutions on ∂U for $0 < \lambda < 1$.

Suppose that $u \in \mathrm{cl}(U)$ is a solution of $(4.6)_\lambda$ for some $\lambda \in (0,1)$; by Proposition 3.3(d) we obtain

$$r_1 \exp(K+\delta) \leq u(t) \leq r_0 \exp(K+\delta) \tag{4.7}$$

and by $(4.6)_\lambda$, (4.5) and (4.2), $|u''(t)| \leq 1/n + \phi(|u'(t)|)$ since $\delta \leq \varepsilon$. On the other hand, $n\varepsilon_0 \geq 1$ and $\delta \leq \varepsilon < \varepsilon_1$, and therefore

$$\int_0^\rho s[1/n + \phi(s)|^{-1} ds > \max\{-r_1, r_0\}\exp(K+\delta) \geq \|u\|_0,$$

and by Corollary 2.2 we get $\|u'\|_0 < \rho$. Thus, by (4.5) and $(4.6)_\lambda$, $\|u''\|_0 < R$.

If $u \in \partial U$ then $u \in (\partial U(r_0, 0) \cup (\partial V(r_1, 0))$ and we suppose first that $u \in \partial U(r_0, 0)$. In this case, by Proposition 3.3(a), there is $a \in M(u)$ such that $\max u_a = u_a(a) = u(a) \geq r_0$ and by remarks 3.2 and the definition of h we have

$$u'(a) = u(a)h(u(a)) = s_0$$

and

$$u''(a) \leq s_0[h(u(a)) + u(a)h'(u(a))] = 0$$

as $h(u(a)) = s_0/u(a)$ and $h'(u(a)) = -s_0/u(a)^2$.

But this is a contradiction since, by (4.7) and (4.3), $u''(a) = \lambda f_n(a, u(a), s_0) > 0$. This contradiction proves that $u \notin \partial U(r_0, 0)$. Analagously $u \notin \partial V(r_1, 0)$ and then $u \notin \partial U$. So, by Theorem 1.1, the problem $(4.6)_1$ has at least one solution v_n such that $\|v_n'\|_0 \leq \rho, \|v_n''\| \leq R$ and $r_1 \exp(K + 1/n) \leq v_n(t) \leq r_0 \exp(K + 1/n)$. Remember that $\delta \leq 1/n$. Now it is easy to prove that $\{v_n\}$ has a subsequence which converges in C_0^2 to a solution of (0.1). So the proof of the claim is finished.

Now take an arbitrary $\varepsilon \in (0, \varepsilon_1)$ and a continuous function $\alpha\colon \mathbf{R} \to \mathbf{R}$ such that

$$\alpha(x) = x \quad \text{if } r_1 \exp(K) \leq x \leq r_0 \exp(K),$$

$$\alpha([r_1 \exp(K+\varepsilon), r_0 \exp(K+\varepsilon)]) \subset [r_1 \exp(K), r_0 \exp(K)],$$

and define $g(t,x,y) = f(t,\alpha(x),y)$. We have

$$\begin{aligned} g(t,x,s_0) &\geq 0 && \text{if } r_0 \leq x \leq r_0 \exp(K), \\ g(t,x,s_1) &\leq 0 && \text{if } r_1 \exp(K) \leq x \leq r_1, \\ |g(t,x,y)| &\leq \phi(|y|) && \text{if } r_1 \exp(K+\varepsilon) \leq x \leq r_0 \exp(K+\varepsilon). \end{aligned}$$

Then, by the claim, there exists at least one solution v of the problem

$$x'' = g(t,x,x'), \qquad x(0) = x(1) = 0$$

such that $r_1 \exp(K) \leq v(t) \leq r_0 \exp(K)$. In particular $\alpha(v(t)) = v(t)$ $(0 \leq t \leq 1)$ and hence v is a solution of (0.1). So the Proof of Theorem 0.2 is complete.

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