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On the Tempered Spectrum of Quasi-Split Classical Groups II

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Abstract. We determine the poles of the standard intertwining operators for a maximal parabolic subgroup of the quasi-split unitary group defined by a quadratic extension E/F of p-adic fields of characteristic zero. We study the case where the Levi component $M \simeq \mathrm{GL}_n(E) \times U_m(F)$, with $n \equiv m \pmod 2$. This, along with earlier work, determines the poles of the local Rankin-Selberg product L-function $L(s, \tau' \times \tau)$, with τ' an irreducible unitary supercuspidal representation of $\mathrm{GL}_n(E)$ and τ a generic irreducible unitary supercuspidal representation of $U_m(F)$. The results are interpreted using the theory of twisted endoscopy.

Introduction

One significance of the approach of Langlands-Shahidi to the definition of local L-functions is its connection with harmonic analysis and representation theory. More precisely, when the data is supercuspidal, the L-functions are defined precisely by means of the poles of standard intertwining operators. As investigated by the authors in several manuscripts, the residues of the operators at these poles can be studied by means of the theory of twisted endoscopy as developed by Kottwitz, Langlands, and Shelstad. When the groups are quasisplit special orthogonal or symplectic, the connection has already been studied by the authors. The purpose of this study is to complete the cases of quasisplit classical groups by completing the problem for quasisplit unitary groups.

It was first in [15], that the second named author studied the situation in which $\mathbf{G}$ is any of the split classical groups Sp_{2n} or SO_n, and $\mathbf{P}$ is the Siegel parabolic subgroup. In this case the L-functions in question are the symmetric square $L(s, \pi, \mathrm{Sym}^2 \rho_n)$ and the exterior square $L(s, \pi, \wedge^2 \rho_n)$, where π is a supercuspidal representation of $\mathrm{GL}_n(F)$ and ρ_n is the standard representation of $\mathrm{GL}_n(\mathbb{C})$ [15]. The first named author carried out a similar program for the Siegel parabolic subgroup of a quasi-split unitary group, determining the Asai L-functions $L(s, \pi, \Psi_n)$ and $L(s, \pi \otimes \mu, \Psi_n)$ [6, 7]. In fact, in each case the authors (separately) determined the L-functions $L(s, \pi, \mathrm{Sym}^2 \rho_n)$, $L(s, \pi, \wedge^2 \rho_n)$, and $L(s, \pi, \Psi_n)$ for any irreducible admissible representation π of $\mathrm{GL}_n(F)$ or $\mathrm{GL}_n(E)$ (as appropriate). The case of an arbitrary maximal parabolic subgroup of SO_{2n}, Sp_{2n}, or the non-split quasi-split special orthogonal group SO^*_{2n} was carried out in two further steps. The second named author studied SO_{2n} in [16] while we examined the remaining cases in [8]. In these cases the

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Levi subgroup is of the form $\mathbf{M} \simeq \mathrm{GL}_n \times \tilde{\mathbf{G}}$, where $\tilde{\mathbf{G}}$ is a classical group of the same type and lower rank. The standard intertwining operator for a supercuspidal $\tau' \otimes \tau$ of $M = \mathbf{M}(F)$ is normalized by a product of two L-functions, $L(s, \tau' \times \tau)L(2s, \tau', \wedge^2 \rho_n)$. Since the second factor, and in particular its poles, are known [15] and the product has at most a simple pole [14], the poles of the Rankin product L-function are determined by those of the intertwining operator which do not come from the exterior square L-function.

As explained before, in all the above cases the theory of twisted endoscopy [12] was used to describe the poles in question. This pattern continues in the case we study here, and is expected in general. Furthermore, in a general setting, the second named author has shown [13] that all poles of the intertwining operators come from twisted endoscopy in the case where the L-group ${}^L N$ of $\mathbf{N}$ is abelian. (A situation which does not apply in the case we discuss here.)

Here we let $\mathbf{G} = \mathbf{G}(r)$ be the quasi-split unitary group in r variables defined with respect to a quadratic extension E/F. Then a maximum parabolic subgroup is of the form $\mathbf{P} = \mathbf{MN}$ with $\mathbf{M} \simeq \mathrm{Res}_{E/F} \mathrm{GL}_n \times \mathbf{G}(m)$, with $r = n + m$. The case $m = 0$ was discussed in [7], so we assume that $m \geq 1$. We further restrict ourselves to the case where $n \equiv r \pmod 2$. We then consider a unitary supercuspidal representation $\sigma \simeq \tau' \otimes \tau$ of $M = \mathbf{M}(F)$, and form the unitarily induced representation $I(\tau' \otimes \tau)$ of G. The reducibility of $I(\tau' \otimes \tau)$ is determined by the analytic behavior of the standard intertwining operator $A(s, \tau' \otimes \tau, w_0)$ at $s = 0$ (see Section 1 for the definition). Furthermore, the product of L-functions, $L(s, \tau' \times \tau)L(2s, \tau', \Psi_n)$ normalizes $A(s, \tau' \otimes \tau, w_0)$, and hence has the same poles. Here Ψ_n is the Asai representation described in [7].

Let u_n be the alternating anti-diagonal matrix which defines our quasi-split unitary group U_n in n variables, *i.e.* $U(n/2, n/2)$ if n is even and $U([n/2], [n/2]+1)$ if n is odd. Let $\varepsilon \colon \mathrm{GL}_n(E) \to \mathrm{GL}_n(E)$ be the automorphism defined by $\varepsilon(g) = u_n {}^t\bar{g}^{-1} u_n^{-1}$, with $g \mapsto \bar{g}$ the coordinate action of the non-trivial Galois automorphism of E/F on $\mathrm{GL}_n(E)$. In the language of [16] we have $\varepsilon = \bar{\theta}^*$. As in [8, 16] computing the residue of $A(s, \tau' \otimes \tau, w_0)$ leads us to consider a correspondence between certain ε-conjugacy classes in $\mathrm{GL}_n(E)$ and ordinary conjugacy classes in $G(m)$. We again refer to this as the ε-norm correspondence (*cf.* Definition 2.8). This correspondence is surjective when $n \geq m$ (*cf.* Lemmas 2.13 and 2.15), and always has finite fibers (*cf.* Corollary 2.14, Lemma 2.15, and Lemma 3.9). When $n < m$, the image of the norm correspondence contains no regular elliptic conjugacy classes. We let $\mathcal{A}$ be the inverse image of the norm map (which is one to finite) extended by the scalars $F^\times/NE^\times$. This then agrees with the image map $\mathcal{A}_{\mathbf{G}(m)/\mathbf{G}'}$ of Kottwitz and Shelstad [12] (*cf.* the discussion prior to Lemma 3.5).

We break the residue of $A(s, \tau' \otimes \tau, w_0)$ into two terms. The first of these we call the *regular* term, since it is the contribution which comes from regular elliptic conjugacy classes in $G(m)$. The remaining term is called the *singular* term. Let $\psi_{\tau'}$ be a matrix coefficient for τ' and f_τ one for τ. Then the regular term can be given as

$$R_{\mathbf{G}}(f_\tau, \psi_{\tau'}) = \sum_{\mathbf{T}_i} \mu(T_i) |W(T_i)|^{-1} \int_{T_i} \tilde{\Phi}_\varepsilon\big(\mathcal{A}(\{\gamma\}), \psi_{\tau'}\big)\, \Phi(\{\gamma\}, f_\tau) |D(\gamma)|\, d\gamma,$$

with the sum over representatives for the G-conjugacy classes of elliptic tori in $G(\ell)$ defined over F, (see pg. 287 of [8]). Here $\ell = \min(n, m)$, $\tilde{\Phi}_\varepsilon$ is a certain twisted orbital integral, Φ is the ordinary orbital integral, $\mu(T_i)$ is the measure of $T_i = \mathbf{T}_i(F)$, and $|D(\gamma)|$ is the usual Harish-Chandra discriminant.

The singular term can be written as follows. Let $\{\mathbf{T}_i\}$ be a collection of representatives for the G-conjugacy classes of Cartan subgroups of $\mathbf{G}(\ell)$ defined over F. Let T_i' be the set of regular elements of $T_i = \mathbf{T}_i(F)$. For each i choose *any* compact subgroup ω_i of T_i'. Then,

$$R_{\text{sing}}(f_\tau, \psi_{\tau'}) = \operatorname*{Res}_{s=0} \sum_i |W(T_i)|^{-1} \int_{T_i' \backslash \omega_i} \varphi_{\mathcal{A}}(s, \gamma) |D(\gamma)| \, d\gamma,$$

where $\varphi_{\mathcal{A}}$ is a function defined in Section 3. Note that the form of the singular term given here differs from that of [8], where we took the limit of such residues as $\omega_i \to T_i'$. However, as we argue here, the residue of the integral is in fact independent of ω_i, and therefore the limit may be omitted (*cf.* the discussion prior to Lemma 3.9). This comment applies to the results of Section 4 of [8], and in particular to Theorem 4.8 and Corollary 4.9 of that paper.

We now state our main results.

Theorem *Let $c = (4n \log q_E)^{-1}$, with q_E the order of the residue field of E. The intertwining operator $A(s, \tau' \otimes \tau, w_0)$ has a pole at $s = 0$ if and only if, for some choice of matrix coefficients f_τ and $\psi_{\tau'}$ for τ and τ', respectively,*

$$cR_{\mathbf{G}}(f_\tau, \psi_{\tau'}) + R_{\text{sing}}(f_\tau, \psi_{\tau'}) \neq 0.$$

Suppose now that $\tau' \simeq (\tau')^\varepsilon$.

- *(a) The induced representation $I(\tau' \otimes \tau)$ is irreducible if and only if $cR_{\mathbf{G}} + R_{\text{sing}} \not\equiv 0$ (as an operator on the space, $\mathcal{A}(\tau) \times \mathcal{A}(\tau')$ of ordered pairs of matrix coefficients).*
- *(b) If τ is generic and $I(\tau' \otimes \tau)$ is irreducible, then, for $s \in \mathbb{R}$, $I(s, \tau' \otimes \tau)$ is reducible at $s = \pm 1/2$ or at $s = \pm 1$, and at exactly one of these pairs.* ■

As in [8], and [16], the non-vanishing of $R_{\mathbf{G}}(f_\tau, \psi_{\tau'})$ is expressible as a pairing between the distribution character χ_τ of τ and the ε-twisted character $\chi^\varepsilon_{\tau'}$ of τ'. Thus, we say that τ' is the twisted endoscopic transfer of τ if $R_{\mathbf{G}}(f_\tau, \psi_{\tau'}) \neq 0$ for some choice of f_τ and $\psi_{\tau'}$ (*cf.* Definition 4.1). Let μ be a character of $E^\times$ which extends the class field character $\omega_{E/F}$. From [7] we know that if $\tau' \simeq (\tau')^\varepsilon$, then exactly one of $L(s, \tau', \Psi_n)$ or $L(s, \tau' \otimes \mu, \Psi_n)$ has a pole at $s = 0$. Since such a pole depends only on τ' and not on τ, the non-vanishing of $R_{\mathbf{G}}(f_\tau, \psi_{\tau'})$ must point to a pole of $L(s, \tau' \times \tau)$.

On the other hand, $L(s, \tau', \Psi_n)$ has a pole at $s = 0$ if and only if τ' is an ε-twisted endoscopic transfer from U_n via one of the two maps described in [7] (and which one depends on the parity of n). Thus, in this case we expect $R_{\mathbf{G}} \equiv 0$, and therefore $R_{\text{sing}} \not\equiv 0$, for any choice of τ. We summarize below.

Proposition *(a) If n is odd and τ' comes from U_n via standard ε-twisted endoscopic transfer, then $L(s, \tau' \times \tau)$ is holomorphic at $s = 0$ for any τ. If n is even and τ' comes from U_n via κ-ε-twisted endoscopic transfer, then $L(s, \tau' \times \tau)$ is holomorphic at $s = 0$ for any τ.*

(b) *Otherwise, $L(s,\tau',\Psi_n)$ has no pole at $s=0$, so $L(s,\tau'\times\tau)$ has a pole at $s=0$ if and only if $cR_{\mathbf{G}}+R_{\mathrm{sing}}\not\equiv 0$.* ■

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1 Preliminaries

Let E/F be a quadratic extension of p-adic fields of characteristic zero with $x\mapsto\bar{x}=\tau(x)$ the Galois automorphism. Let $\sigma\in\mathrm{Gal}(\bar{F}/F)$ satisfy $\sigma\equiv\tau\big(\mathrm{mod}\,\mathrm{Gal}(\bar{F}/E)\big)$. Choose β such that $E=F(\beta)$, with $\bar{\beta}=-\beta$. We take $u_n=\begin{pmatrix} & & & \iddots\\ & & -1 & \\ & 1 & & \\ -1 & & & \end{pmatrix}$, where n is the size of the matrix, and set $w_n=\begin{pmatrix} & & 1\\ & \iddots & \\ 1 & & \end{pmatrix}$. Note that, if n is even, then $u_n=Jw_n$, where $J=\mathrm{diag}\{1,-1,1,\ldots,-1\}$, and $Jw_n=-w_nJ$. If n is odd, then $u_n=Jw_n=w_nJ$, where $J=\mathrm{diag}\{-1,1,\ldots,1,-1\}$. We further note that $u_n^{-1}=(-1)^{n+1}u_n$.

Now consider the group GL_r, defined over E and the cocycle given by $a_\sigma(g)=u_r{}^t\big(\sigma(g)\big)^{-1}u_r^{-1}$. Then set $\mathbf{G}=\mathbf{G}(r)=\mathbf{U}_r$, defined with respect to the form u_r. The F-points of $\mathbf{G}$ are given by

$$G(r)=\{g\in\mathrm{GL}_r(E)\mid g(u_r)^t\bar{g}=u_r\}.$$

The maximal torus of diagonal elements is

$$\mathbf{T}=\left\{\begin{pmatrix} x_1 & & & & & \\ & \ddots & & & & \\ & & x_{r/2} & & & \\ & & & \bar{x}_{r/2}^{-1} & & \\ & & & & \ddots & \\ & & & & & \bar{x}_1^{-1}\end{pmatrix}\;\middle|\; x_i\in\mathrm{Res}_{E/F}\,\mathbf{G}_m\right\},$$

if r is even, and

$$\mathbf{T}=\left\{\begin{pmatrix} x_1 & & & & & & \\ & \ddots & & & & & \\ & & x_{[r/2]} & & & & \\ & & & 1 & & & \\ & & & & \bar{x}_{[r/2]}^{-1} & & \\ & & & & & \ddots & \\ & & & & & & \bar{x}_1^{-1}\end{pmatrix}\;\middle|\; x_i\in\mathbf{G}_m\right\},$$

if r is odd. Let $\mathbf{B} = \mathbf{TU}$ be the Borel subgroup of upper triangular matrices in $\mathbf{G}(r)$. Denote by Δ the simple roots with respect to this choice of Borel subgroup. Suppose that $n \le [r/2]$, and let $m = [r/2] - n$. We let $\Theta = \Delta \setminus \{e_n - e_{n+1}\}$. We let $\mathbf{A} = \mathbf{A}_\Theta$, and $\mathbf{M} = \mathbf{M}_\Theta$. Then

$$\mathbf{M} = \left\{ \begin{pmatrix} g & & \\ & h & \\ & & \varepsilon(g) \end{pmatrix} \middle| \; g \in \operatorname{Res}_{E/F} \mathrm{GL}_n, h \in \mathbf{G}(m) \right\} \simeq \operatorname{Res}_{E/F} \mathrm{GL}_n \times \mathbf{G}(m).$$

Here $\varepsilon(g) = u_n({}^t\bar{g}^{-1})u_n^{-1}$. For $X \in M_n(E)$, we let $\tilde{\varepsilon}(X) = u_n{}^t\bar{X}u_n^{-1}$. Let $\mathbf{P} = \mathbf{MN}$ be the standard parabolic subgroup with Levi component $\mathbf{M}$.

Lemma 1.1 *If $v \in \mathbf{N}$, then $v = \begin{pmatrix} I_n & X & Y \\ 0 & I_m & X' \\ 0 & 0 & I_n \end{pmatrix}$, with $X' = u_m{}^t\bar{X}u_n$, $Y \in M_n(E)$, and $Y + (-1)^{n+m}\tilde{\varepsilon}(Y) = XX'$.*

Proof Straightforward computation. ∎

We denote such a v by $n(X, Y)$.

The equation

$$Y + (-1)^{n+m}\tilde{\varepsilon}(Y) = XX' \tag{1.1}$$

is crucial to understanding the poles of the intertwining operator. Let $\mathcal{N}$ be the set of ε-conjugacy classes, $\{Y\}$, in $\mathrm{GL}_n(E)$, for which there is an E-rational solution to (1.1).

Remark If $n = m$, we set $\tilde{\theta}(X) = w_n{}^tXw_n^{-1}$ and $\tilde{\bar{\theta}}(X) = \tilde{\theta}(\bar{X})$. Then we can rewrite (1.1) as

$$Y + Jw_n{}^t\bar{Y}w_nJ = XJw_n{}^t\bar{X}Jw_n$$

$$YJ + (-1)^{n+1}w_nJ{}^t\bar{Y}w_n = XJw_n{}^t\bar{X}Jw_nJ = -1^{(n+1)}XJw_n{}^t\bar{X}w_n$$

$$YJ + (-1)^{n+1}\tilde{\bar{\theta}}(YJ) = (-1)^{n+1}XJ\tilde{\bar{\theta}}(X).$$

Thus, when X is invertible, (1.1) has an interpretation in terms of equivalence of hermitian or skew hermitian forms.

We compute the functional $\tilde{\alpha}$, as defined in [14]. (Also see [7].) The restricted root system for $\mathbf{G}(r)$ is of type $C_{r/2}$, if r is even, and of type $BC_{(r-1)/2}$ if r is odd. The unrestricted roots are of type A_{r-1}. We let Δ be the simple restricted roots, and $\tilde{\Delta}$ be the simple unrestricted roots. Then $\Delta = \{\alpha_i\}_{i=1}^{[r/2]}$, with $\alpha_i = e_i - e_{i+1}$, for $i \neq [r/2]$, and $\alpha_{[r/2]} = 2e_{[r/2]}$. Similarly, we write $\tilde{\Delta} = \{\beta_i\}_{i=1}^{r-1}$, with $\beta_i = e_i - e_{i+1}$. The action of $\mathrm{Gal}(E/F)$ on $\tilde{\Delta}$ identifies β_i and β_{r-i}, and thus α_i is the restriction of both β_i and

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β_{r-i}. Since the subset Θ of Δ corresponding to $\mathbf{P}$ is $\Delta \setminus \{\alpha_n\}$, the corresponding subset $\tilde{\Theta}$ of $\tilde{\Delta}$ is $\tilde{\Delta} \setminus \{\beta_n, \beta_{r-n}\}$. Now, we compute $\rho_{\tilde{\Theta}}$.

$$\begin{aligned}
2\rho_{\tilde{\Theta}} &= \sum_{\beta>0, \beta\notin\Sigma(\tilde{\Theta})} \beta \\
&= \sum_{i=1}^{n}\sum_{j=n}^{r-1}\sum_{k=i}^{j}\beta_k + \sum_{i=n+1}^{r-n}\sum_{j=r-n}^{r-1}\sum_{k=i}^{j}\beta_k \\
&= (r-n)\sum_{i=1}^{n} i\beta_i + n\sum_{i=n+1}^{r-1}(r-i)\beta_i + n\sum_{i=n+1}^{r-n-1}(i-n)\beta_i + (r-2n)\sum_{i=r-n}^{r-1}(r-i)\beta_i \\
&= (r-n)\Big(\sum_{i=1}^{n} i(\beta_i+\beta_{r-i}) + n\sum_{i=n+1}^{r-n-1}\beta_i\Big).
\end{aligned}$$

To compute $\tilde{\alpha}$, we first compute, $\langle \rho_{\tilde{\Theta}}, \beta_n\rangle$, since β_n represents the unique simple root in $\mathbf{N}$. Note that

$$\begin{aligned}
\langle \rho_{\tilde{\Theta}}, \beta_n\rangle = \frac{2(\rho_{\tilde{\Theta}}, \beta_n)}{(\beta_n, \beta_n)} &= (\rho_{\tilde{\Theta}}, \beta_n) \\
&= (r-n)/2\big((n-1)\beta_{n-1} + n\beta_n + n\beta_{n+1}, \beta_n\big) = (r-n)/2.
\end{aligned}$$

Therefore, $\tilde{\alpha}$ is given by

$$\langle \rho_{\tilde{\Theta}}, \beta_n\rangle^{-1}\rho_{\tilde{\Theta}} = \sum_{i=1}^{n} i(\beta_i+\beta_{r-i}) + n\sum_{i=n+1}^{r-n-1}\beta_i.$$

Thus, if $m = (g,h) \in M$, then

$$q_F^{\langle\tilde{\alpha}, H_P(m)\rangle} = |\det g|_E.$$

Note that the adjoint representation r of ${}^LM = \big(\mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_m(\mathbb{C})\big) \ltimes W_F$ on ${}^L\mathfrak{n}$ decomposes into two irreducible pieces. Since $\langle\tilde{\alpha}, \alpha_n\rangle = 1$, we see that the subspace of ${}^L\mathfrak{n}$ containing $X_{\beta_n^\vee}$ should be labeled V_1 in the ordering given by [14]. Also note that if $\beta = \sum_{i=1}^{r-n+1}\beta_i$, then $X_{\beta^\vee} \notin V_1$. Furthermore, a straightforward (but tedious) calculation shows that, if β restricts to α', then $\langle\tilde{\alpha}, \alpha'\rangle = 2$, and thus $X_{\beta^\vee}$ indeed lies in V_2. Let r_i denote the restriction of r to V_i. Then $r_1|{}^LM^\circ \simeq \rho_n \otimes \tilde{\rho}_m \oplus \rho_m \otimes \tilde{\rho}_n$, where ρ_k is the standard representation of $\mathrm{GL}_k(\mathbb{C})$. Also, r_2 is the Asai representation of $\mathrm{GL}_n(\mathbb{C})$ described in [7].

Let $w_0 = \begin{pmatrix} & & (-1)^{n+m}I_n \\ & I_m & \\ I_n & & \end{pmatrix}$ represent the non-trivial element of $W\big(\mathbf{G}(r), \mathbf{A}\big)$.

Lemma 1.2 *Suppose $n(X,Y) \in N$. Then $w_0^{-1}n(X,Y) \in P\bar{N}$ if and only if $Y \in \mathrm{GL}_n(E)$, and then*

$$w_0^{-1}n(X,Y) = \begin{pmatrix} (-1)^{n+m}\varepsilon(Y) & -Y^{-1}X & I_n \\ 0 & I_m - X'Y^{-1}X & X' \\ 0 & 0 & (-1)^{n+m}Y \end{pmatrix}\begin{pmatrix} I_n & 0 & 0 \\ (Y^{-1}X)' & I_m & 0 \\ Y^{-1} & Y^{-1}X & I_n \end{pmatrix}$$

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Proof Straightforward. ∎

We note for future reference that $(Y^{-1}X)' = X'\varepsilon(Y)$.

For $n \in \mathbf{N}$ we let $\mathbf{M}_n = \{m \in \mathbf{M} \mid \mathrm{Ad}(m)n = n\}$. Similarly, for $\bar{n} \in \bar{N}$, we let $\mathbf{M}_{\bar{n}} = \{m \in \mathbf{M} \mid \mathrm{Ad}(m)\bar{n} = \bar{n}\}$. Finally, for $m_1 \in \mathbf{M}$, we let $\mathbf{M}_{w_0,m_1} = \{m \in \mathbf{M} \mid \mathrm{Ad}(w_0)(m)m_1m^{-1} = m_1\}$. Note that $\mathbf{M}_{w_0,m_1}$ is the twisted centralizer of m_1, and if $m_1 = (g,h)$, then $\mathbf{M}_{w_0,m_1} = \mathbf{G}'_{\varepsilon,g} \times \mathbf{G}_h$, where $\mathbf{G}'_{\varepsilon,g}$ is the ε-twisted centralizer of $g \in \mathbf{G}' = \mathrm{Res}_{E/F}\,\mathrm{GL}_n$ and $\mathbf{G}_h$ is the centralizer of $h \in \mathbf{G}(m)$.

Lemma 1.3 *Suppose that $n(X,Y)$ satisfies $w_0^{-1}n(X,Y) = p_1\bar{n}_1 = m_1n_1\bar{n}_1$, as in Lemma 1.2. Then the following hold:*

(a) $\mathbf{M}_{n(X,Y)} = \mathbf{M}_{n_1} = \mathbf{M}_{\bar{n}_1}$ and so the F-points of all of these groups are equal as well. Furthermore, $\mathbf{M}_{n_1} \subset \mathbf{M}_{w_0,m_1}$.

(b) $\mathbf{M}_{w_0,m_1} = \mathbf{G}'_{\varepsilon,Y} \times \mathbf{G}_Z$, where $Z = I - X'Y^{-1}X$;

Proof Statement (a) is taken directly from [13, Lemma 2.1]. Part (b) follows from Lemma 1.2 and the above discussion. ∎

For a connected reductive group $\mathbf{H}$, defined over F, with $H = \mathbf{H}(F)$, we let $\mathcal{E}_c(H)$ be the collection of (equivalence classes of) irreducible admissible representations of H. We denote by $\mathcal{E}(H)$ the unitary classes in $\mathcal{E}_c(H)$. Let $^\circ\mathcal{E}_c(H)$ be the collection of irreducible admissible supercuspidal representations of H, and we further denote by $^\circ\mathcal{E}(H)$ the collection $\mathcal{E}(H) \cap {}^\circ\mathcal{E}_c(H)$.

Let $\mathbf{G}' = \mathrm{Res}_{E/F}\,\mathrm{GL}_n$. Choose $(\tau', V') \in {}^\circ\mathcal{E}(G')$, and $(\tau, V) \in {}^\circ\mathcal{E}\big(G(m)\big)$. Let $\mathrm{Ind}_P^G\Big(\big(\tau' \otimes |\det(\ )|_E^s\big) \otimes \tau \otimes \mathbf{1}_N\Big)$ be the representation of G unitarily induced from $\big(\tau' \otimes |\det(\ )|_E^s\big) \otimes \tau$. We usually denote this representation by $I(s, \tau' \otimes \tau)$, and set $I(\tau' \otimes \tau) = I(0, \tau' \otimes \tau)$. We let $V(s, \tau' \otimes \tau)$ be the space of $I(s, \tau' \otimes \tau)$ and let $V(s, \tau' \otimes \tau)_0$ be the subspace of functions supported in $P\bar{N}$.

We wish to study the reducibility of $I(\tau' \otimes \tau)$, we need to study the poles of the intertwining operator $A(s, \tau' \otimes \tau, w_0)$ defined by

$$A(s, \tau' \otimes \tau, w_0)f(g) = \int_N f(w_0^{-1}ng)\,dn.$$

By Lemma 4.1 of [15] it is enough to assume that $g = e$ and $f \in V(s, \tau' \otimes \tau)_0$. Let L and L' be compact subsets in $M_n(E)$ and $M_{n\times m}(E)$, respectively. Let ξ_L and $\xi_{L'}$ be their characteristic functions. Set

$$h\left(\begin{pmatrix} I & 0 & 0 \\ X & I & 0 \\ Y & X' & I \end{pmatrix}\right) = \xi_L(Y)\xi_{L'}(X')(v' \otimes v),$$

for some $v \in V$, and $v' \in V'$. Choose $\tilde{v}$ and $\tilde{v}'$ in $\tilde{V}$ and $\tilde{V}'$ respectively. We let

$\psi_{\tau'}(g') = \langle \tilde{v}', \tau'(g')v' \rangle$, and $f_\tau(g) = \langle \tilde{v}, \tau(g)v \rangle$. Then

$$(1.2) \qquad \langle \tilde{v}' \otimes \tilde{v}, A(s, \tau' \otimes \tau, w_0)h(e) \rangle$$
$$= \int_{(X,Y)} \psi_{\tau'}\big((-1)^{n+m}\varepsilon(Y)\big)\, f_\tau(I - X'Y^{-1}X)|$$
$$\cdot \det Y|_E^{-s-\langle \rho_P, H_P(Y) \rangle} \xi_L(Y^{-1})\xi_{L'}(Y^{-1}X)\, d(X,Y).$$

2 Norm Correspondence

Lemma 2.1 *Suppose that (X, Y) is an E-rational solution to (1.1). Further suppose that $(Xg)(Xg)' = XX'$, for some $g \in \mathrm{GL}_m(E)$. Then $Xg = Xh$ for some $h \in G(m)$.*

Proof Consider E^n and E^m as row vectors, and let $U = E^n X \subset E^m$. Let $\langle\, , \rangle$ be the hermitian form given by $\beta_m u_m$. Here $\beta_m = \begin{cases} \beta & \text{if } m \text{ is even} \\ 1 & \text{if } m \text{ is odd.} \end{cases}$ Let $g^* : U \to E^m$ be given by $g^*(v) = vg$. If $v_1, v_2 \in U$, and $v_i = y_i X$, with $y_i \in E^n$, then

$$\begin{aligned} \langle v_1 g, v_2 g \rangle &= v_1 g \beta_m u_m{}^t \bar{g}^t \bar{v}_2 \\ &= y_1 X g \beta_m u_m{}^t \bar{g}^t \bar{X}^t \bar{y}_2 \\ &= y_1 X \beta_m u_m{}^t \bar{X}^t \bar{y}_2 = \langle v_1, v_2 \rangle. \end{aligned}$$

Thus, g^* is an isometry onto a subspace of E^m. By the hermitian version of Witt's Theorem, we can choose $h \in G(m)$ with $vg = vh$, for all $v \in U$. If $y \in E^n$, we have $yXh = yXg$, and thus, $Xh = Xg$. ■

Suppose that $Y + (-1)^{n+m}\tilde{\varepsilon}(Y) = XX'$, and $g \in \mathrm{GL}_n(E)$. Then

$$\begin{aligned} gY\varepsilon(g)^{-1} + (-1)^{n+m}\tilde{\varepsilon}\big(gY\varepsilon(g)^{-1}\big) &= gY\varepsilon(g)^{-1} + (-1)^{n+m} u_n{}^t\overline{(gYu_n{}^t\bar{g}u_n^{-1})}u_n^{-1} \\ &= gY\varepsilon(g)^{-1} + (-1)^{n+m} g u_n{}^t\bar{Y}u_n^{-1}\varepsilon(g)^{-1} \\ &= g\big(Y + (-1)^{n+m}\tilde{\varepsilon}(Y)\big)\varepsilon(g)^{-1} = gXX'\varepsilon(g)^{-1} \\ &= gX(gX)'. \end{aligned}$$

Thus, $\{X\} \in \mathrm{GL}_n(E) \setminus M_{n\times m}(E)$ parameterizes the ε-conjugacy classes for which (1.1) has a rational solution. Now suppose $\{X\}$ parameterizes $\{Y^{-1}\}$, *i.e.* (X, Y) satisfies (1.1). If $g \in \mathrm{GL}_n(E)$, then when we replace X by gX, we see that $I - X'Y^{-1}X$ is replaced by $I - X'\big(\varepsilon(g)^{-1}Y^{-1}g\big)X$. So, $I - X'Y^{-1}X$ is unchanged if we replace Y^{-1} by $\varepsilon(g)Y^{-1}g^{-1}$, which is also in $\{Y^{-1}\}$.

If X is changed to $X_1 = gX$ then we say that $\{X_1\}$ parameterizes the ε-conjugacy class $\{Y_1^{-1}\}$ with $Y_1 = gY\varepsilon(g)^{-1}$, since it is the pair (X_1, Y_1) which satisfies (1.1), even though the classes are the same. If $X_1 = Xh$, with $h \in \mathrm{GL}_m(E)$, then by Lemma 2.1 we may assume $h \in G(m)$. Therefore,

$$\begin{aligned} I - X_1'Y^{-1}X_1 &= I - (Xh)'Y^{-1}Xh = I - u_m{}^t\bar{h}\,{}^t\bar{X}u_nY^{-1}Xh \\ &= I - h^{-1}X'Y^{-1}Xh = h^{-1}(I - XY^{-1}X)h. \end{aligned}$$

Thus, the conjugacy class of $I - X'Y^{-1}X$ is well defined for a class $\{X\}$. Moreover, only finitely many conjugacy classes $\{I - X'Y^{-1}X\}$ are so attached to an ε-conjugacy class $\{Y^{-1}\}$.

Lemma 2.2 *Suppose that $\{X_1\}$ parameterizes $\{Y_1^{-1}\}$ with $X_1 = gXh$, for some $g \in \mathrm{GL}_n(E)$ and $h \in \mathrm{GL}_m(E)$, and $Y_1 = gY\varepsilon(g)^{-1}$. Then in fact $X_1 \in \mathrm{GL}_n(E)XG(m)$, and $\{X_1\}$ parameterizes precisely the same collection of ε-conjugacy classes as X.* ■

Lemma 2.3 *Suppose that $n(X, Y) \in N$, with $Y \in \mathrm{GL}_n(E)$. Then*

(a) $(I - X'Y^{-1}X)X' = -X'Y^{-1}\varepsilon(Y^{-1})$;
(b) $X(I - X'Y^{-1}X) = -\varepsilon(Y^{-1})Y^{-1}X$.

Proof Since $n(X, Y) \in N$, and $Y \in \mathrm{GL}_n(E)$, we know that $w_0^{-1}n(X, Y) \in P\bar{N}$, and furthermore

$$\begin{pmatrix} 0 & 0 & (-1)^{n+m}I \\ 0 & I & X' \\ I & X & Y \end{pmatrix}$$
$$= \begin{pmatrix} (-1)^{n+m}\varepsilon(Y) & -Y^{-1}X & I \\ 0 & I - X'Y^{-1}X & X' \\ 0 & 0 & (-1)^{n+m}Y \end{pmatrix} \begin{pmatrix} I & 0 & 0 \\ X'\varepsilon(Y) & I & 0 \\ Y^{-1} & Y^{-1}X & I \end{pmatrix}.$$

Comparing the $(2, 1)$-entries, we see that $(I - X'Y^{-1}X)X'\varepsilon(Y) + X'Y^{-1} = 0$, which implies (a). We then rewrite this as

$$(I - X'Y^{-1}X)u_m{}^t\bar{X}u_n = -u_m{}^t\bar{X}u_nY^{-1}u_n^{-1t}\bar{Y}u_n.$$

Which implies

$$u_m^{-1}(I - X'Y^{-1}X)u_m{}^t\bar{X} = -{}^t\bar{X}u_nY^{-1}u_n^{-1t}\bar{Y}.$$

This in turn implies

$${}^t\overline{(I - X'Y^{-1}X)}^{-1t}\bar{X} = -{}^t\bar{X}u_nY^{-1}u_n^{-1t}\bar{Y},$$

i.e.,

$$X(I - X'Y^{-1}X)^{-1} = -Yu_n{}^t\bar{Y}^{-1}u_n^{-1}X = -Y\varepsilon(Y)X.$$

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This immediately implies (b). ■

Definition 2.4 Suppose that $X \in M_{n\times m}$. Let $\tilde{X}$ be the square matrix obtained by adding rows, if $n \le m$, (respectively columns if $n \ge m$) of zeros to the bottom (respectively. right) of X. We call X a projection if $\tilde{X}$ is, *i.e.*, if $\tilde{X}^2 = \tilde{X}$.

Lemma 2.5 *Suppose that $X \in M_{n\times m}$ is a projection. Let $U = E^n X \subset E^m$. Set*

$$H_X = \{h \in G(m) \mid Xh = g_h X, \text{ for some } g_h \in \mathrm{GL}_n(E)\}.$$

If $\{0\} \subsetneq U \subsetneq E^m$, then $H_X \subsetneq G(m)$.

Proof By Corollary 4.3 of [16], we know that $h \in H_X$ if and only if $Uh = U$. So U is an H_X-invariant subspace. Since the standard representation of $G(m)$ is irreducible, we have the lemma. ■

Lemma 2.6 *Suppose that $\{X\} \in \mathrm{GL}_n(E) \setminus M_{n\times m}(E)/G(m)$. If there is some $Y \in \mathrm{GL}_n(E)$ so that (X, Y) satisfies (1.1), then*

$$X(I - X'Y^{-1}X) = -\varepsilon(Y^{-1})Y^{-1}X,$$

and if X is a projection, then $I - X'Y^{-1}X \in H_X$.

Proof This follows from Lemma 2.3(a) and the definition of H_X. ■

Lemma 2.7 *Suppose that $X \in M_{n\times m}(E)$, and set $U = E^n X$. We consider E^m as a hermitian space with the non-degenerate form $\beta_m u_m$. Set $H'_X = \{h \in G(m) \mid Uh = U\}$. If U is a non-degenerate subspace, then H'_X is the centralizer of an involution in $G(m)$. If U is degenerate, then H'_X is contained in a proper parabolic subgroup of $G(m)$. If X is a projection, then $H'_X = H_X$.*

Proof Suppose U is non-degenerate. Then $E^m = U \oplus U^\perp$. If $h \in H'_X$, then $Uh = U$, which implies that $U^\perp h = U^\perp$. Therefore, h is in the centralizer of $(-1_U \oplus 1_{U^\perp})$. Conversely, suppose h is in the centralizer of $(-1_U \oplus 1_{U^\perp})$. Let $v \in U$, and suppose that $vh = w \oplus w^\perp$. Then $v(-1_U \oplus 1_{U^\perp})h = -vh = -w \oplus -w^\perp$, while $vh(-1_U \oplus 1_{U^\perp}) = -w \oplus w^\perp$. Therefore, $w^\perp = 0$, and we see that $Uh \subset U$.

Now suppose that U is degenerate. Then $\mathrm{Rad}\, U \ne \{0\}$, and H'_X stabilizes $\mathrm{Rad}\, U$. Therefore, $H'_X \subset P_X = M_X N_X$, where P_X is the parabolic subgroup stabilizing precisely the isotropic subspace $\mathrm{Rad}\, U$. ■

Definition 2.8 Suppose that $Y \in \mathrm{GL}_n(E)$, and there is some X_0 for which (X_0, Y) is an E-rational solution to (1.1). Let $\{Y^{-1}\}$ be the ε-conjugacy class of Y^{-1}, and set $N_\varepsilon(\{Y^{-1}\})$ be the collection of $G(m)$-conjugacy classes $\big\{\{I - X_1'Y_1^{-1}X_1\}\big\}$ for which $Y_1^{-1} \in \{Y^{-1}\}$ and (X_1, Y_1) satisfies (1.1). We call $N_\varepsilon(\{Y^{-1}\})$ the ε-norm of $\{Y^{-1}\}$,

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and it gives a one to finite correspondence between the ε-conjugacy classes of $\mathrm{GL}_n(E)$ for which (1.1) has an E-rational solution, and conjugacy classes in $G(m)$.

Proposition 2.9 *Suppose that $n < m$, and take $X \in M_{n\times m}(E)$. Fix a Y (if such exists) in $\mathrm{GL}_n(E)$ satisfying (1.1) with X. Then $I - X'Y^{-1}X$ belongs to a proper parabolic subgroup of $G(m)$ or a proper centralizer of a singular elliptic element. Moreover, if $m > n+1$, then $N_\varepsilon(\{Y^{-1}\})$ never contains regular elliptic classes.*

Proof That $I - X'Y^{-1}X$ is in either a proper parabolic subgroup, or the proper centralizer of a singular elliptic element, follows from Lemmas 2.6 and 2.7. Further note that the rank of $X'Y^{-1}X$ is at most n, and therefore, at least $m - n$ eigenvalues of $I - X'Y^{-1}X$ are equal to 1. Thus, if $m > n + 1$ we see that $I - X'Y^{-1}X$ cannot be regular elliptic. ■

Lemma 2.10 *Suppose that $n = m$, that $S \in M_n(E)$, and that $I + S \in G(m)$. Then there is a projection $X \in M_n(E)$, and a choice of $Y \in \mathrm{GL}_n(E)$ for which $S = (-1)^{n+1}X'Y^{-1}X = (-1)^{n+1}X'Y^{-1} = Y^{-1}X$.*

Proof Note that $X \mapsto \tilde\varepsilon(X) = u_n{}^t\bar{X}u_n^{-1}$ is an anti-involution of $M_n(E)$. Moreover, since $u_n^{-1} = (-1)^{n+1}u_n$, we have $X' = (-1)^{n+1}\tilde\varepsilon(X)$.

Since $I + S \in G(m)$, we have $(I + S)(u_n)(I + {}^t\bar{S}) = u_n$, which implies $u_n{}^t\bar{S} + Su_n + Su_n{}^t\bar{S} = 0$. Therefore, $\tilde\varepsilon(S) = -(I + S)^{-1}S$. Now applying Lemma 5.6 of [16], we see that we can choose a projection X in $M_n(E)$ and a $Y \in \mathrm{GL}_n(E)$ for which $S = \tilde\varepsilon(X)Y^{-1}X = \tilde\varepsilon(X)Y^{-1} = Y^{-1}X$. Now X and Y have the desired property. ■

Definition 2.11 If $n = m$, and $\gamma \in G(m)$, and we choose X, Y as in Lemma 2.10 for which $\gamma = I - X'Y^{-1}X$, then we say that (X, Y) is a *canonical section* over γ if $\det(Y|_{\ker X}) \in NE^\times$. If $n > m$, and $n \equiv m \pmod 2$, then, letting $k = (n - m)/2$, we say that $X = \begin{pmatrix} 0_{k\times m} \\ X_0 \\ 0_{k\times m}\end{pmatrix} \in M_{n\times m}$ and $Y = \begin{pmatrix} I_k & 0 & 0 \\ 0 & Y_0 & 0 \\ 0 & 0 & -I_k\end{pmatrix} \in \mathrm{GL}_n(E)$ form a canonical section over $\{\gamma\}$ if (X_0, Y_0) is a canonical section over $\{\gamma\}$.

Lemma 2.12 *Suppose that $n = m$. Then N_ε is surjective. That is, if $\{\gamma\}$ is a conjugacy class in $G(m)$, then $\gamma \in N_\varepsilon(\{Y^{-1}\})$ for some Y.*

Proof Let $S \in M_n(E)$ with $I + (-1)^nS = \gamma \in G(m)$. By Lemma 2.10, we can choose a projection X and an element Y of $\mathrm{GL}_n(E)$ for which

$$S = -X'Y^{-1}X = -X'Y^{-1} = (-1)^nY^{-1}X,$$

Now,

$$\big(I + (-1)^nS\big)\, u_n{}^t\overline{(I + (-1)^nS)} = u_n,$$

which implies

$$(I + Y^{-1}X)u_n{}^t\overline{\big(I + (-1)^{n+1}X'Y^{-1}X\big)} = u_n.$$

Therefore,

$$Y^{-1}Xu_n + (-1)^{n+1}u_n{}^t\bar{X}\,{}^t\bar{Y}^{-1}\,{}^t\bar{X}' = (-1)^n Y^{-1}Xu_n{}^t\bar{X}\,{}^t\bar{Y}^{-1}\,{}^t\bar{X}',$$

or,

$$Y^{-1}Xu_n + (-1)^{n+1}u_n{}^t\bar{Y}^{-1}u_nXu_n = (-1)^n Y^{-1}Xu_n{}^t\bar{X}\,{}^t\bar{Y}u_nXu_n.$$

Thus,

$$Y^{-1}X + \tilde{\varepsilon}(Y^{-1})X = (-1)^n Y^{-1}XX'\tilde{\varepsilon}(Y^{-1})X.$$

In the case that n is even, we have

$$Y^{-1}X + \tilde{\varepsilon}(Y^{-1})X = Y^{-1}XX'\tilde{\varepsilon}(Y^{-1})X.$$

In the case where n is odd,we let $X_1 = -X$ and $Y_1 = -Y$. Then we have

$$I - X_1'Y_1^{-1}X_1 = I + X'Y^{-1}X = I - S = \gamma.$$

Also, we have

$$Y^{-1}X + \tilde{\varepsilon}(Y^{-1})X = -Y^{-1}XX'\tilde{\varepsilon}(Y^{-1})X,$$

which implies

$$Y_1^{-1}X_1 + \tilde{\varepsilon}(Y_1^{-1})X_1 = Y_1^{-1}X_1X_1'\tilde{\varepsilon}(Y_1^{-1})X_1.$$

Thus, for any n, we can choose X and Y so that $X^2 = \pm X$, $I - X'Y^{-1}X = \gamma$, and

$$Y^{-1}X + \tilde{\varepsilon}(Y^{-1})X = Y^{-1}XX'\tilde{\varepsilon}(Y^{-1})X. \tag{2.1}$$

Now, if v is in the right image of X, then (2.1) and $Xv = \pm v$ gives us

$$\left(Y^{-1} + \tilde{\varepsilon}(Y^{-1})\right)v = Y^{-1}XX'\tilde{\varepsilon}(Y^{-1})v,$$

which shows that (1.1) holds on the image of X.

Thus, we need to show that such a choice of X and Y can be made for which (1.1) holds on the kernel of X. Notice that

$$\ker X \subset \ker Y^{-1}XX'\tilde{\varepsilon}(Y^{-1})X = \ker Y^{-1}XX'\tilde{\varepsilon}(Y^{-1}),$$

so we need to show we can choose X and Y with $Y + \tilde{\varepsilon}(Y) = 0$ on $\ker X$.

We know that $Y^{-1}X = (-1)^{n+1}X'Y^{-1}$ and $\tilde{\varepsilon}(Y^{-1})X = (-1)^{n+1}X'\tilde{\varepsilon}(Y^{-1})$. Note that $(-1)^{n+1}X' = \tilde{\varepsilon}(X)$ is also a projection up to sign. Thus, Y^{-1} and $\tilde{\varepsilon}(Y^{-1})$ both define isomorphisms from $\ker X$ onto $\ker\tilde{\varepsilon}(X)$. For $v \in E^n$, we define $\tilde{\varepsilon}(v) = {}^t(\overline{u_nv})$. A straightforward calculation shows that $\tilde{\varepsilon}(v)\tilde{\varepsilon}(Y^{-1}) = \tilde{\varepsilon}(Y^{-1}v)$. Choose bases $\mathcal{B}$ and $\mathcal{B}'$ for $\ker X$ and $\ker X' = \ker\varepsilon(X)$, respectively. Then the above equality shows that the matrix of $Y^{-1}|_{\ker X}$ with respect to the bases $\mathcal{B}$ and $\mathcal{B}'$ is the same as the matrix of $\tilde{\varepsilon}(Y^{-1})|_{\ker X}$ with respect to the bases $\tilde{\varepsilon}(\mathcal{B})$ and $\tilde{\varepsilon}(\mathcal{B}')$. Note that on $\ker X$, one can choose Y for which Y^{-1} is $\tilde{\varepsilon}$-skew hermitian. Thus, such a Y satisfies $Y + \tilde{\varepsilon}(Y) = XX'$. ■

Lemma 2.13 *Suppose that n is any positive integer. If (X, Y) satisfies (1.1), then the conjugacy class $\{I - X'Y^{-1}X\}$ in $G(m)$ determines the semisimple part of the conjugacy class $\{\varepsilon(Y)^{-1}Y^{-1}\}$ uniquely.*

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Proof We assume that X is of rank r. Then we have $E^n = V \oplus W$, where W is the right kernel of X, $\dim V = r$, and right multiplication by X gives an embedding of V into E^m. If $v \in W$ then $v\big(Y + (-1)^{n+m}\tilde{\varepsilon}(Y)\big) = vXX' = 0$, so $(-1)^{n+m+1}v = v\tilde{\varepsilon}(Y)Y^{-1} = v\varepsilon(Y)^{-1}Y^{-1}$. Consequently, $\varepsilon(Y)^{-1}Y^{-1}|_W = (-1)^{n+m+1}\operatorname{Id}_W$. For $v \in V$, we know by Lemma 2.3(b) that

$$v\varepsilon(Y)^{-1}Y^{-1}X = -vX(I - X'Y^{-1}X).$$

Thus, the matrix of $\varepsilon(Y)^{-1}Y^{-1}$ with respect to a basis which respects the decomposition $E^n = V \oplus W$ is of the form $\begin{pmatrix} A & * \\ 0 & (-1)^{n+m+1}I \end{pmatrix}$, and A is determined by $I - X'Y^{-1}X$. Thus, the lemma holds. ∎

Corollary 2.14 *If $n = m$, then N_ε has finite fibers.*

Corollary 2.15 *If $n \geq m$, and $n \equiv m \pmod 2$ then N_ε is surjective with finite fibers.*

Proof Let $k = (n-m)/2$. Let $h \in U_m(F)$. By Lemma 2.12, there is an element $Y_0 \in \mathrm{GL}_m(E)$, and an $X_0 \in M_m(E)$ so that $Y_0 + \tilde{\varepsilon}(Y_0) = X_0X_0'$ and $I_m - X_0Y_0^{-1}X_0 = h$. Let $S = X_0'Y_0^{-1}X_0$. Let $X = \begin{pmatrix} 0_{k\times m} \\ X_0 \\ 0_{k\times m} \end{pmatrix}$ and $Y = \begin{pmatrix} I_k & 0 & 0 \\ 0 & (-1)^kY_0 & 0 \\ 0 & 0 & -I_k \end{pmatrix}$. Then $X' = \begin{pmatrix} 0_{m\times k} & (-1)^kX_0' & 0_{m\times k} \end{pmatrix}$, and thus

$$XX' = \begin{pmatrix} 0_{k\times k} & 0_{k\times m} & 0_{k\times k} \\ 0_{m\times k} & (-1)^kX_0X_0' & 0_{m\times k} \\ 0_{k\times k} & 0_{k\times m} & 0_{k\times k} \end{pmatrix}.$$

Furthermore,

$$\tilde{\varepsilon}(Y) = \begin{pmatrix} -I_k & 0 & 0 \\ 0 & (-1)^k\tilde{\varepsilon}(Y_0) & 0 \\ 0 & 0 & I_k \end{pmatrix}.$$

Thus,

$$\begin{aligned} Y + \tilde{\varepsilon}(Y) &= \begin{pmatrix} 0_{k\times k} & 0_{k\times m} & 0_{k\times k} \\ 0_{m\times k} & (-1)^k\big(Y_0 + \tilde{\varepsilon}(Y_0)\big) & 0_{m\times k} \\ 0_{k\times k} & 0_{k\times m} & 0_{k\times k} \end{pmatrix} \\ &= \begin{pmatrix} 0_{k\times k} & 0_{k\times m} & 0_{k\times k} \\ 0_{m\times k} & (-1)^kX_0X_0' & 0_{m\times k} \\ 0_{k\times k} & 0_{k\times m} & 0_{k\times k} \end{pmatrix} = XX'. \end{aligned}$$

Note that

$$\begin{aligned} X'Y^{-1}X &= \begin{pmatrix} 0_{m\times k} & (-1)^kX_0' & 0_{m\times k} \end{pmatrix} \begin{pmatrix} I_k & & \\ & (-1)^kY_0^{-1} & \\ & & -I_k \end{pmatrix} \begin{pmatrix} 0_{k\times m} \\ X_0 \\ 0_{k\times m} \end{pmatrix} \\ &= \begin{pmatrix} 0 & X_0'Y_0^{-1} & 0 \end{pmatrix} \begin{pmatrix} 0 \\ X_0 \\ 0 \end{pmatrix} = X_0'Y_0^{-1}X_0 = S. \end{aligned}$$

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The finiteness of the fibers follows from Lemma 2.13. ■

Lemma 2.16 *The involution ε fixes a splitting of $\mathbf{G}' = \operatorname{Res}_{E/F} \mathrm{GL}_n$.*

Proof Recall that $\mathbf{G}' = \mathrm{GL}_n \times (\mathrm{GL}_n)^\sigma$, where σ represents the Galois automorphism of E/F. Moreover, the action of $\operatorname{Gal} E/F$ on $\mathbf{G}'$ is by $\sigma(g,h) = \big(\sigma(h), \sigma(g)\big)$, where $\sigma(g) = \bar{g}$ is the coordinate-wise Galois action. We take $\mathbf{B}_0'$ to be the Borel subgroup of upper triangular matrices in $\mathbf{G}'$ and $\mathbf{T}_0'$ the maximal torus of diagonal elements. Let $\alpha_i = e_i - e_{i+1}$ be a simple root, and let $X_{\alpha_i}(t) = tE_{i(i+1)}$ be the standard splitting of GL_n over E. Here $E_{i(i+1)}$ is the elementary matrix for the $i,(i+1)$ entry. Now set $X_i = (X_{\alpha_i}, \bar{X}_{\alpha_i})$. Then $(\mathbf{B}_0', \mathbf{T}_0', \{X_i\}_i)$ is a splitting for $\mathbf{G}'$. Note that, to be precise, we should write $u_n' = (u_n, u_n)$, and

$$\varepsilon(g,h) = u_n'^{\,t}\big(\sigma(g,h)\big)^{-1}(u_n')^{-1} = \big(\varepsilon(h), \varepsilon(g)\big),$$

where $\varepsilon(g)$ is as before. Note that $\varepsilon(X_{\alpha_i}) = \bar{X}_{\alpha_i}$, and therefore,

$$\varepsilon(X_i) = \varepsilon(X_{\alpha_i}, \bar{X}_{\alpha_i}) = \big(\varepsilon(\bar{X}_{\alpha_i}), \varepsilon(X_{\alpha_i})\big) = X_i.$$

Consequently, ε fixes the splitting $(\mathbf{B}_0', \mathbf{T}_0', \{X_i\})$. ■

Suppose that F is algebraically closed. Then $E = F$, and the Galois map $x \mapsto \bar{x}$ is the identity. Note that $\mathbf{T}_0'$ is an ε-stable Cartan subgroup of $\mathbf{G}'$. Let $\bar{N}_\varepsilon : \mathbf{T}_0' \to \mathbf{T}_0'$ be given by $\bar{N}_\varepsilon(Y) = Y\varepsilon(Y)$. If $Y = \operatorname{diag}\{a_1, \dots, a_n\}$, then

$$\varepsilon(Y) = \operatorname{diag}\{\bar{a}_n^{-1}, \dots, \bar{a}_1^{-1}\} = \operatorname{diag}\{a_n^{-1}, \dots, a_1^{-1}\}.$$

Thus, $\bar{N}_\varepsilon(Y) = \operatorname{diag}\{a_1 a_n^{-1}, a_2 a_{n-1}^{-1}, \dots, a_n a_1^{-1}\}$. Therefore,

$$\ker \bar{N}_\varepsilon = \big\{ \operatorname{diag}\{a_1, \dots, a_{n/2}, a_{n/2}, \dots, a_1\}\big\},$$

if n is even, and

$$\ker \bar{N}_\varepsilon = \big\{ \operatorname{diag}\{a_1, \dots, a_{[n/2]}, a_{[n/2]+1}, a_{[n/2]}, \dots, a_1\}\big\},$$

if n is odd. Assume n is even. If $Y_0 = \operatorname{diag}\{a_1, \dots, a_{n/2}, 1, \dots, 1\}$, then

$$(I - \varepsilon)Y_0 = Y_0 \varepsilon(Y_0)^{-1} = \operatorname{diag}\{a_1, \dots, a_{n/2}, a_{n/2}, \dots, a_1\}$$

is in $\ker \bar{N}_\varepsilon$. Therefore, $\ker \bar{N}_\varepsilon = (I - \varepsilon)\mathbf{T}_0'$. Similarly, if n is odd, and $Y_0 = \operatorname{diag}\{a_1, \dots, a_{[n/2]+1}, 1, \dots, 1\}$, then

$$(I - \varepsilon)Y_0 = Y_0 \varepsilon(Y_0)^{-1} = \operatorname{diag}\{a_1, \dots, a_{[n/2]}, a_{[n/2]+1}^2, a_{[n/2]}, \dots, a_1\}$$

is in $\ker \bar{N}_\varepsilon$. Since F is algebraically closed, $\ker \bar{N}_\varepsilon = (I - \varepsilon)\mathbf{T}_0'$.

Now suppose that F is not necessarily algebraically closed. Let $\mathbf{T_H}$ be a Cartan subgroup of $\mathbf{G}(n)$ defined over F and $(\mathbf{B}', \mathbf{T}')$ be a ε-stable pair in $\mathbf{G}'$, also defined over F, for which there is an isomorphism $\mathbf{T_H} \xrightarrow{\sim} \mathbf{T}_\varepsilon'$, defined over F [12].

Lemma 2.17 *The map $Y \mapsto Y\varepsilon(Y)$ from $\mathbf{T}'$ to $\mathbf{T}'$ has $(\mathbf{T}')^\varepsilon$ as its image, and can be identified with the projection onto $\mathbf{T}_\varepsilon' = \mathbf{T}'/(I - \varepsilon)\mathbf{T}'$.*

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Proof Since $(\mathbf{B}'_0, \mathbf{T}'_0)$ and $(\mathbf{B}', \mathbf{T}')$ are both ε-stable pairs, there is an element g in the ε-fixed points of $\mathbf{G}'$ for which $g^{-1}\mathbf{T}'_0 g = \mathbf{T}'$ [18]. Let $\mathbf{K}$ and $\mathbf{K}_0$ be the kernels of $Y \mapsto Y\varepsilon(Y)$ in $\mathbf{T}'$ and $\mathbf{T}'_0$, respectively. Note that $\mathbf{K} = g^{-1}\mathbf{K}_0 g$. Since $\mathbf{K}_0 = (I - \varepsilon)\mathbf{T}'_0$, and $\varepsilon(g) = g$, we have $\mathbf{K} = (I - \varepsilon)\mathbf{T}'$. Similarly, since the image of $\bar{N}_\varepsilon$ on $\mathbf{T}'_0$ is $(\mathbf{T}'_0)^\varepsilon$, we see that the image of the map $Y \mapsto Y\varepsilon(Y)$ on $\mathbf{T}'$ is $(\mathbf{T}')^\varepsilon$. ■

Lemma 2.18 *Suppose that F is algebraically closed. Let $\{Y^{-1}\} \in \mathcal{N} = \mathcal{C}$ be ε-semisimple, with Y in an ε-stable pair $(\mathbf{B}', \mathbf{T}')$ of $\mathrm{GL}_n(F)$. Then there is an X in $M_n(F)$ for which $I - X'Y^{-1}X$ is semisimple in $\mathbf{G}(n)$, and $I - X'Y^{-1}X$ is $\mathrm{GL}_n(F)$ conjugate to $-\varepsilon(Y^{-1})Y^{-1}$. Furthermore, every $\mathrm{GL}_n(F)$-conjugate of $-\varepsilon(Y^{-1})Y^{-1}$ belongs to the image of $\{Y^{-1}\}$ under the norm correspondence N_ε.*

Proof Since Y is ε-semisimple, and lies in an ε-stable pair, there is an element $h \in (\mathbf{G}')^\varepsilon(F) = U_n(F) = G(n)$ for which $Y_1 = hYh^{-1} = \mathrm{diag}\{a_1, \dots, a_n\}$. Note that $Y_1 + \tilde{\varepsilon}(Y_1) = \mathrm{diag}\{a_1 + \bar{a}_n, a_2 + \bar{a}_{n-1}, \dots, a_n + \bar{a}_1\}$. Suppose n is even. Let $i = \sqrt{-1}$. Set $X_1 = i\,\mathrm{diag}\{a_1 + \bar{a}_n, \dots, a_{n/2} + \bar{a}_{n/2+1}, 1, \dots, 1\}$. Note that $X'_1 = -\tilde{\varepsilon}(X_1) = -i\,\mathrm{diag}\{1, \dots, 1, \bar{a}_{n/2} + a_{n/2+1}, \dots, \bar{a}_1 + a_n\}$. Thus,

$$X_1X'_1 = \mathrm{diag}\{a_1 + \bar{a}_n, \dots, a_n + \bar{a}_1\} = Y_1 + \tilde{\varepsilon}(Y_1).$$

Note that $I - X'_1Y_1^{-1}X_1 = \mathrm{diag}\{a_1^{-1}\bar{a}_n, \dots, a_n^{-1}\bar{a}_1\}$, is semisimple. If n is odd, then we take $y \in F$ with $y\bar{y} = y^2 = 2a_{[n/2]+1}$. Then let $X_1 = \mathrm{diag}\{a_1, \dots, a_{[n/2]}, y, 1, \dots, 1\}$. Then $X'_1 = \tilde{\varepsilon}(X_1) = \mathrm{diag}\{1, \dots, 1, \bar{y}, \bar{a}_{[n/2]}, \dots, \bar{a}_1\}$. Thus,

$$X_1X'_1 = \mathrm{diag}\{a_1 + \bar{a}_n, \dots, a_{[n/2]} + \bar{a}_{[n/2]+2}, 2a_{[n/2]+1}, \dots, a_n + \bar{a}_1\} = Y_1 + \tilde{\varepsilon}(Y_1).$$

If we set $X = h^{-1}X_1$, then $Y + \tilde{\varepsilon}(Y) = XX'$, and $I - X'Y^{-1}X = I - X_1Y_1^{-1}X_1$ is semisimple. Note that since Y is in an ε-stable Cartan, we have $\varepsilon(Y^{-1})Y^{-1} = Y^{-1}\varepsilon(Y^{-1})$ is semisimple, and in the fixed points of ε, *i.e.*, in $G(n)$.

Now choose Y_2 which is ε-conjugate to Y, and a projection X_2 satisfying (1.1) for which $I - X'Y^{-1}X = I - X'_2Y_2^{-1}X_2$. Since Y_2 and Y are ε-conjugate, we see that $\varepsilon(Y_2^{-1})Y_2^{-1}$ and $\varepsilon(Y^{-1})Y^{-1}$ are conjugate. We have already seen that $-\varepsilon(Y_2^{-1})Y_2^{-1}$ has matrix

$$\begin{pmatrix} (I - X'_2Y_2^{-1}X)|_{\mathrm{Im}\,X_2} & * \\ 0 & I \end{pmatrix}$$

with respect to the decomposition $F^n = \mathrm{Im}\,X_2 \oplus \ker X_2$. Thus, the eigenvalues of $-\varepsilon(Y^{-1})Y^{-1}$ different from 1 are among those of $I - X'_2Y^{-1}X_2 = I - X'Y^{-1}X$. Since $Y^{-1}\varepsilon(Y^{-1}) = \varepsilon(Y^{-1})Y^{-1}$, we see that $Y_2^{-1}\varepsilon(Y_2^{-1})$ has the same eigenvalues. Thus, the eigenvalues of $I - X'Y^{-1}X$ which are different from one are among the eigenvalues of $-\varepsilon(Y_2^{-1})Y_2^{-1}$. Consequently, $-\varepsilon(Y^{-1})Y^{-1}$ and $I - X'Y^{-1}X$ are $\mathrm{GL}_n(F)$-conjugate. ■

Lemma 2.19 *Assume that $n = m$. Suppose that $Y + \tilde{\varepsilon}(Y) = XX'$, and $Z = I - X'Y^{-1}X$. Let $g \in \mathbf{G}'_{\varepsilon,Y}(F)$, and suppose that $gX = Xh$. Then h belongs to $\mathbf{G}_Z(F)$, and is uniquely determined modulo the right stabilizer of X. Conversely, suppose that $h \in \mathbf{G}_Z(F)$, and (X, Y) forms a canonical section over Z. If there is a $g \in \mathbf{G}'(F)$ for which $gX = Xh$, then g can be chosen to lie in $\mathbf{G}'_{\varepsilon,Y}(F)$.*

Proof Since (X,Y) is a canonical section, we know that $X'Y^{-1}X = X'Y^{-1} = (-1)^{n+1}Y^{-1}X$, and $(-1)^{n+1}X'$ is a projection. We set $U_1 = XE^n$, $U_1' = X'E^n$, $U = E^nX$, $U' = E^nX'$, $U_2 = \{v \mid Xv = 0\}$, and $U_2' = \{v \mid X'v = 0\}$. Then we have isomorphisms $Y^{-1}\colon U_1 \xrightarrow{\sim} U_1'$ and $Y^{-1}\colon U_2 \xrightarrow{\sim} U_2'$. If we let $\tilde\varepsilon(v) = {}^t\overline{(u_n v)} = {}^t\bar{v}u_n^{-1}$, then $\tilde\varepsilon$ gives isomorphisms from U_1 to U' and U_1' to U. Now $\varepsilon(Y)$ defines an isomorphism from U' to U.

Suppose that $h \in \mathbf{G}(F)$, and $h^{-1}X'Y^{-1}Xh = X'Y^{-1}X$. Then, applying ε to both sides of this equality we have $h^{-1}X'\varepsilon(Y^{-1})Xh = X'\varepsilon(Y^{-1})X$.

We know that $I - X$ and $I + (-1)^nX'$ are projections and

$$\big(I + (-1)^nX'\big)\,Y^{-1}(I-X) = \big(I + (-1)^nX'\big)\,Y^{-1} = Y^{-1}(I-X).$$

Suppose that $g_0 \in \mathbf{G}'_{\varepsilon,Y}$ with $g_0X = Xg_0$. Since $\mathbf{G}'_{\varepsilon,\varepsilon(Y)} = \mathbf{G}'_{\varepsilon,Y^{-1}} = \varepsilon(\mathbf{G}'_{\varepsilon,Y})$, we have

$$\big(I + (-1)^nX'\big)\,Y^{-1}(I-X) = \varepsilon(g_0)^{-1}\big(I + (-1)^nX'\big)\,Y^{-1}(I-X)g_0.$$

Let g be defined by $g|_{U_1} = Xh|_{U_1}$, and $g|_{U_2} = g_0|_{U_2} = (I-X)g_0|_{U_2}$. If $v \in U_1$, then $gXv = gv = Xhv$. If $v \in U_2$, then $Y^{-1}Xhv = hY^{-1}Xv = 0$, which says that $Xhv = 0 = gXv$. Thus, $gX = Xh$.

We now show that $g \in \mathbf{G}'_{\varepsilon,Y}$. Suppose that $v \in U_1$.

$$\begin{aligned}\big((-1)^{n+1}X'\big)\,\varepsilon(g)^{-1}Y^{-1}gv &= (-1)^{n+1}X'\varepsilon(g)^{-1}Y^{-1}gXv = (-1)^{n+1}h^{-1}X'Y^{-1}Xhv\\ &= (-1)^{n+1}X'Y^{-1}v.\end{aligned}$$

Since $(I + (-1)^nX')\varepsilon(g)^{-1} = \varepsilon(g_0)^{-1}(I + (-1)^nX')$, we also see that

$$\begin{aligned}\big(I + (-1)^nX'\big)\,\varepsilon(g)^{-1}Y^{-1}gv &= \varepsilon(g_0)^{-1}\big(I + (-1)^nX'\big)\,Y^{-1}Xhv\\ &= \varepsilon(g_0)^{-1}\big(I + (-1)^nX'\big)\,X'Y^{-1}v\\ &= 0 = \big(I + (-1)^nX'\big)\,Y^{-1}v.\end{aligned}$$

Now suppose that $v \in U_2$. Then $Y^{-1}v \in U_2'$, and

$$\begin{aligned}\big((-1)^{n+1}X'\big)\,\varepsilon(g)^{-1}Y^{-1}gv &= h^{-1}\big((-1)^{n+1}X'\big)\,Y^{-1}(I-X)g_0v\\ &= 0 = (-1)^{n+1}X'Y^{-1}v.\end{aligned}$$

Furthermore,

$$\begin{aligned}\big(I + (-1)^nX'\big)\,\varepsilon(g)^{-1}Y^{-1}gv &= \varepsilon(g_0)^{-1}(I+X)'Y^{-1}(I-X)g_0v\\ &= \big(I + (-1)^nX'\big)\,Y^{-1}(I-X)v = \big(I + (-1)^nX'\big)\,Y^{-1}v.\end{aligned}$$

Thus, $\varepsilon(g) \in \mathbf{G}'_{\varepsilon,Y^{-1}}$, which says that $g \in \mathbf{G}'_{\varepsilon,Y}$. Therefore, g has the desired properties. ■

Corollary 2.20 *Suppose that $n > m$, with $n \equiv n \pmod 2$. Then the statement of Lemma 2.19 is true.*

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Proof That the first statement of Lemma 2.19 holds is straightforward. Let $k = (n-m)/2$. Suppose that $X = \begin{pmatrix} 0 \\ X_0 \\ 0 \end{pmatrix}$ and $Y = \begin{pmatrix} I_k & & \\ & (-1)^k Y_0 & \\ & & -I_k \end{pmatrix}$, with (X_0, Y_0) a canonical section over Z for $M_m(E) \times \mathrm{GL}_m(E)$. If $g = \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix}$, then

$$gX = \begin{pmatrix} g_{12}X_0 \\ g_{22}X_0 \\ g_{32}X_0 \end{pmatrix} \text{ and } Xh = \begin{pmatrix} 0 \\ X_0 h \\ 0 \end{pmatrix}.$$

Therefore, $g_{22}X_0 = X_0 h$, while $g_{12}X_0 = g_{32}X_0 = 0$. By our choice of X_0 and Y_0, we can assume that $g_{22} \in \mathbf{G}'_{\varepsilon, Y_0}(F)$. We let $g_0 = \begin{pmatrix} I_k & & \\ & g_{22} & \\ & & I_k \end{pmatrix}$. Then $g_0 X = gX = Xh$. Moreover,

$$\varepsilon(g_0) = u_n {}^t\overline{g_0}^{-1} u_n = \begin{pmatrix} I_k & & \\ & \varepsilon(g_{22}) & \\ & & I_k \end{pmatrix}.$$

Thus,

$$g_0^{-1} Y \varepsilon(g_0) = \begin{pmatrix} I_k & & \\ & (-1)^k g_{22}^{-1} Y_0 \varepsilon(g_{22}) & \\ & & -I_k \end{pmatrix} = \begin{pmatrix} I_k & & \\ & (-1)^k Y_0 & \\ & & -I_k \end{pmatrix} = Y.$$

Thus, g_0 has the desired properties. ∎

Lemma 2.21 *Let $n = m$, and denote by N_ε^{-1} the canonical section of the norm map, as defined in Definition 2.11. Then N_ε^{-1} is continuous.*

Proof Suppose that $n = m$. Further suppose that $S \in M_n(E)$ with $I - S \in G(m)$. Let X be a projection and $Y \in \mathrm{GL}_n(E)$ for which $S = X'Y^{-1}X = X'Y^{-1} = (-1)^{n+1}Y^{-1}X$. Then

$$\begin{aligned} \tilde{\varepsilon}(S) = \tilde{\varepsilon}(X'Y^{-1}) &= u_n {}^t\overline{(X'Y^{-1})} u_n^{-1} \\ &= u_n({}^t\bar{Y}^{-1} u_n X u_n) u_n^{-1} = (-1)^{n+1} \tilde{\varepsilon}(Y^{-1}) X. \end{aligned}$$

Since $Y + \tilde{\varepsilon}(Y) = XX'$, we have

$$Y^{-1} + \tilde{\varepsilon}(Y^{-1}) = \tilde{\varepsilon}(Y^{-1}) XX' Y^{-1} = (-1)^{n+1} \tilde{\varepsilon}(S) S.$$

Now suppose that $\{I - S_k\}$ converges to $I - S$ in $G(m)$. Let $\{Y_k^{-1}\} = N_\varepsilon^{-1}(\{I - S_k\})$, and let X_k be the associated projection. Since $S_k(V)$ converges to $S(V)$ (pointwise) we see that X_k' converges to X', which says that X_k converges to X. Therefore, there is a k_0 so that $k > k_0$ implies that $X_k = X$. We have

$$S = X'Y^{-1} = \lim_k S_k = \lim_k X_k' Y_k^{-1} = X' \lim_k Y_k^{-1}.$$

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Applying $\tilde{\varepsilon}$, we see that

$$\tilde{\varepsilon}(X' \lim_k Y_k^{-1}) = (-1)^{n+1} \lim_k \tilde{\varepsilon}(Y_k^{-1})X,$$

and thus,

$$\lim_k \tilde{\varepsilon}(Y_k^{-1})X = \tilde{\varepsilon}(Y^{-1})X.$$

Suppose that $v \in W$. Then $Xv = v$, and so $\lim_k \tilde{\varepsilon}(Y_k^{-1})v = \tilde{\varepsilon}(Y^{-1})v$, and therefore, $\tilde{\varepsilon}(Y_k^{-1})|_W$ converges to $\tilde{\varepsilon}(Y^{-1})|_W$. Moreover,

$$Y_k^{-1} + \tilde{\varepsilon}(Y_k^{-1}) = \tilde{\varepsilon}(Y_k^{-1})X_k X_k' Y_k^{-1} = (-1)^{n+1}\tilde{\varepsilon}(S_k)S_k.$$

Consequently,

$$\begin{aligned}\lim_k \big(Y_k + \tilde{\varepsilon}(Y_k^{-1})\big) &= \lim_k Y_k^{-1} + \lim_k \tilde{\varepsilon}(Y_k^{-1}) \\ &= (-1)^{n+1} \lim_k \big(\tilde{\varepsilon}(S_k)S_k\big) = (-1)^{n+1}\tilde{\varepsilon}(S)S = Y^{-1} + \tilde{\varepsilon}(Y^{-1}).\end{aligned}$$

Thus, on $\ker X$, we have

$$\lim_k Y_k^{-1} + \tilde{\varepsilon}(\lim_k Y_k^{-1}) = 0 = Y^{-1} + \tilde{\varepsilon}(Y^{-1}).$$

However, for $k > k_0$, we have $\ker X_k = \ker X$, and since $\det(Y_k|_{\ker X}) \equiv \det(Y|_{\ker X}) \bmod NE^\times$, and both are ε-skew hermitian, we have $Y_k|\ker X$ is ε-conjugate to $Y|_{\ker X}$. Thus, we clearly have $\{Y_k\} \to \{Y\}$. ■

3 The Pole

Let $\mathbf{G}' = \mathrm{Res}_{E/F}\,\mathrm{GL}_n$. If $Y \in \mathbf{G}'$, then we define

$$\mathbf{G}'_{\varepsilon,Y} = \{g \in \mathbf{G}' \mid g^{-1}Y\varepsilon(g) = Y\}.$$

We also define

$$\tilde{\mathbf{G}}'_{\varepsilon,Y} = \{g \mid g^{-1}Y\varepsilon(g) = zY, \text{ for some } z \in NE^\times\}.$$

Note that $\chi_Y : \tilde{\mathbf{G}}'_{\varepsilon,Y} \to F^\times$, given by $\chi_Y(g) = z$ if $g^{-1}Y\varepsilon(g) = zY$, is a homomorphism. If $a \in E^\times$, then $(aI)^{-1}Y\varepsilon(aI) = N_{E/F}(a)Y$. Thus, χ_Y is surjective, and

$$\mathbf{G}'_{\varepsilon,Y}(F) \setminus \tilde{\mathbf{G}}'_{\varepsilon,Y}(F) \simeq NE^\times,$$

for any $\{Y\} \in \mathcal{N}$.

Suppose that ω is a character of $E^\times$. Let $\psi \in C^\infty(G', \omega)$. We define

$$\tilde{\Phi}_\varepsilon(Y, \psi) = \int_{\tilde{G}'_{\varepsilon,Y}\setminus G'} \psi\big(g^{-1}Y\varepsilon(g)\big)\, d\dot{g}.$$

Suppose that $\psi_{\tau'}$ is a matrix coefficient of τ'. For any $\{Y\} \in \mathcal{N}$, set

$$\begin{aligned}
&\Phi_\varepsilon(Y, \psi_{\tau'}|\ |^s)\\
&\quad= \int_{G'_{\varepsilon,Y}\backslash G'} \psi_{\tau'}\big(g^{-1}Y\varepsilon(g)\big)\,\big|\det\big(g^{-1}Y\varepsilon(g)\big)\big|_E^s\,d\dot g\\
&\quad= \int_{G'_{\varepsilon,Y}\backslash \tilde G'_{\varepsilon,Y}} \int_{\tilde G'_{\varepsilon,Y}\backslash G'} \psi_{\tau'}\big(g^{-1}h^{-1}Y\varepsilon(h)\varepsilon(g)\big)\,\big|\det\big(g^{-1}h^{-1}Y\varepsilon(h)\varepsilon(g)\big)\big|_E^s\,d\dot g\,d\dot h\\
&\quad= \int_{NE^\times} \int_{\tilde G'_{\varepsilon,Y}\backslash G'} \psi_{\tau'}\big(g^{-1}zY\varepsilon(g)\big)\,\big|\det\big(g^{-1}zY\varepsilon(g)\big)\big|_E^s\,d\dot g\,dz\\
&\quad= \int_{NE^\times} \omega'(z)^{-1}|z|_E^{ns}\tilde\Phi_\varepsilon(Y,\psi_{\tau'}|\ |^s)\,dz.
\end{aligned}$$

Proposition 3.1 *The twisted orbital integral $\Phi_\varepsilon(Y, \psi_{\tau'}|\ |^s)$ converges for* $\operatorname{Re} s > 0$.

Proof Since $\psi_{\tau'}$ is compactly supported modulo the center Z of G', and $Z \subset \tilde{\mathbf{G}}'_{\varepsilon,Y}$, we see that $\tilde\Phi_\varepsilon(Y, \psi_{\tau'}|\ |^s)$ converges for all s. Since the integral over $NE^\times$ converges for $\operatorname{Re} s > 0$, we have the proposition. ■

Lemma 3.2 *Let n be even and $\alpha \in F^\times/NE^\times$. Set $\alpha_0 = \operatorname{diag}\{\alpha, 1, \alpha, 1, \dots, \alpha, 1\} \in GU_n$. Then, for any $\{\gamma'\} \in \mathcal{N}$, we have $N_\varepsilon(\{\alpha\gamma'\}) = \alpha_0^{-1}N_\varepsilon(\{\gamma'\})\alpha_0$. If n is odd, and $\alpha \in NE^\times$, then we choose $\lambda \in E^\times$ so that $\lambda\bar\lambda = \alpha$. We then set $\alpha_0 = \operatorname{diag}\{\alpha I_{(n-1)/2}, \lambda, I_{(n-1)/2}\} \in GU_n$. Then, again we have $N_\varepsilon(\{\alpha\gamma'\}) = \alpha_0^{-1}N_\varepsilon(\{\gamma'\})\alpha_0$.*

Proof Let $\alpha^\vee = \alpha I_n$. Then, for all n, we have $\alpha^\vee = \alpha_0\tilde\varepsilon(\alpha_0)$. Suppose $Y^{-1} \in \{\gamma'\}$, and Y satisfies (1.1) with X. Then

$$X\alpha^\vee X' = \alpha\big(Y + \tilde\varepsilon(Y)\big) = (\alpha Y) + \tilde\varepsilon(\alpha Y).$$

On the other hand, we observe that

$$\begin{aligned}
X\alpha^\vee X' &= X\alpha_0\tilde\varepsilon(a_0)X' = X\alpha_0 u_n{}^t\bar\alpha_0 u_n^{-1}u_n{}^t\bar X u_n\\
&= X\alpha_0 u_n{}^t\overline{(X\alpha_0)}u_n = (X\alpha_0)(X_0\alpha_0)'.
\end{aligned}$$

Thus, $N_\varepsilon(\{\alpha\gamma'\}) = N_\varepsilon(\{\alpha Y\})$ is the collection of conjugacy classes given by

$$\{I - (X\alpha_0)'(\alpha Y)^{-1}(X\alpha_0)\} = \alpha_0^{-1}\{I - X'Y^{-1}X\}\alpha_0,$$

as X ranges over all its possible choices. This completes the lemma. ■

Proposition 3.3 *Assume $n = m$. Suppose the ε-conjugacy class $\{Y^{-1}\}$ is ε-regular. Then $N_\varepsilon(\{Y^{-1}\})$ consists of a single conjugacy class in G of a regular semisimple element in G. Assuming that Y and $\varepsilon(Y)$ commute, i.e., that $Y^{-1}\varepsilon(Y^{-1})$ is in G, then the converse is true, i.e., if $N_\varepsilon(\{Y^{-1}\})$ is regular, then $\{Y^{-1}\}$ is ε-regular (and hence ε-semisimple).*

Proof First we suppose the ε-conjugacy class $\{Y^{-1}\}$ is ε-regular. Then, up to $\mathrm{GL}_n(\bar{F})$-conjugation, $\varepsilon(Y^{-1})Y^{-1}$ is a regular semisimple element of $\mathbf{G}$. Choose Y_2^{-1}, ε-conjugate to Y^{-1}, and a projection X_2 satisfying (1.1) with Y_2, such that $I-X_2'Y_2^{-1}X_2 = I-X'Y^{-1}X$. By Lemma 2.13, the eigenvalues of $\varepsilon(Y^{-1})Y^{-1}$ different from 1 are among those of the semisimple part of $I-X_2'Y_2^{-1}X_2$. Since $Y^{-1}\varepsilon(Y^{-1})$ is $\mathrm{GL}_n(E)$-conjugate to $\varepsilon(Y^{-1})Y^{-1}$, one sees that the eigenvalues of $-\varepsilon(Y_2)^{-1}Y_2^{-1}$ and $-Y_2^{-1}\varepsilon(Y_2)^{-1}$ are the same. Therefore, one can apply the argument of Lemma 5.10 of [16] to the equation in Lemma 2.3(a) to show that the eigenvalues of the semisimple part of $I-X_2'Y_2^{-1}X_2$ which are not 1 are also among those of $-\varepsilon(Y_2)^{-1}Y_2^{-1}$. Then the semisimple parts of $I-X'Y^{-1}X$ and $-\varepsilon(Y)^{-1}Y^{-1}$ are $\mathrm{GL}_n(\bar{F})$-conjugate. But $-\varepsilon(Y^{-1})Y^{-1}$ is $\mathrm{GL}_n(\bar{F})$-conjugate to a regular element in $\mathbf{G}$, and therefore, $I-X'Y^{-1}X$ must be semisimple and regular.

Suppose now that $Y+\tilde{\varepsilon}(Y) = XX'$, with $Y^{-1}\varepsilon(Y)^{-1} \in \mathbf{G}(F)$, and assume $N_\varepsilon(\{Y^{-1}\})$ contains a regular semisimple element $\{I-X'Y^{-1}X\}$. Again by Lemma 2.3(a), and the argument of Lemma 2.13, the conjugacy class of $I-X'Y^{-1}X$ is completely determined by the semisimple part of the conjugacy class $\{-Y^{-1}\varepsilon(Y^{-1})\}$ in $\mathbf{G}(F)$. That is, the eigenvalues of the first are among those of the second. Moreover, by Lemma 2.13, the semisimple part of the conjugacy class of $-\varepsilon(Y^{-1})Y^{-1}$ is completely determined by $I-X'Y^{-1}X$. Since $\{I-X'Y^{-1}X\}$ is regular semisimple in $\mathbf{G}(F)$, and $Y^{-1}\varepsilon(Y^{-1}) \in \mathbf{G}(F)$, we conclude that $Y^{-1}\varepsilon(Y^{-1})$ is regular and semisimple. Therefore, $\{Y^{-1}\}$ is ε-regular. In fact, let $\tilde{Y} = (Y,\varepsilon)$ represent an element in the non-identity component of $\mathrm{GL}_n \rtimes \{1,\varepsilon\}$. Write $\tilde{Y} = su$, with s semisimple and u unipotent. Then $\tilde{Y}^2 = s^2u^2 = Y^{-1}\varepsilon(Y^{-1})$. If $Y^{-1}\varepsilon(Y^{-1})$ is semisimple, then $u^2 = u = 1$, and thus Y is ε-semisimple and ε-regular. ■

Corollary 3.4 *Suppose $n > m$ and $n \equiv m \pmod 2$. Then, for almost all regular elliptic conjugacy classes $\{h\} \in G(m)$, the collection of ε-conjugacy classes, $N_\varepsilon^{-1}(\{h\})$ is parameterized by a unique ε-regular ε-conjugacy class in $\mathrm{GL}_m(\bar{F})$.*

Proof For almost all regular semisimple classes in $G(m)$, there is a choice of $Y_2 \in \mathrm{GL}_m(F)$ which satisfies (1.1) with $X_2 = I_m$, so that $I-X_2'Y_2^{-1}X_2 \in \{h\}$. In particular $Y_2 + \tilde{\varepsilon}(Y_2) = I_m'$. By Proposition 3.3, the ε-conjugacy class of Y_2 is ε-regular and uniquely determined by h. Let $k = (n-m)/2$, as in Corollary 2.15. Let $X = \begin{pmatrix} 0_{k\times m} \\ I_m \\ 0_{k\times m}\end{pmatrix} \in M_{n\times m}(E)$. Then

$$XX' = \begin{pmatrix} 0_{k\times k} & 0_{k\times m} & 0_{k\times k} \\ 0_{m\times k} & (-1)^k I_m' & 0_{m\times k} \\ 0_{k\times k} & 0_{k\times m} & 0_{k\times k} \end{pmatrix}.$$

Let

$$Y = \begin{pmatrix} I_k & 0 & 0 \\ 0 & (-1)^k Y_2 & 0 \\ 0 & 0 & -I_k \end{pmatrix}.$$

Then as computed before, $Y + \tilde{\varepsilon}(Y) = XX'$ (Corollary 2.15). We have also seen that $I - X'Y^{-1}X = h$.

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It remains to check that for almost all Y_2, the class of Y satisfying (1.1) is, up to $\mathrm{GL}_n(\bar{F})-\varepsilon$-conjugacy, of the form given in the previous paragraph. First observe that for almost all Y satisfying (1.1) with $X = \begin{pmatrix} 0 \\ I \\ 0 \end{pmatrix}$, Y acts semisimply on the direct sum of the image and the kernel of XX', both of which are invariant under ε, since Y_2 is ε-regular. Moreover, Y must be ε-skew symmetric on $\ker(XX')$ and can therefore be given in the form $\mathrm{diag}(J_1, Y_2, J_2)$ with $\mathrm{diag}(J_1, J_2)$ ε-symmetric or ε-skew symmetric proving our assertion. ■

Let $\mathcal{C}$ be the set of conjugacy classes in $G = G(m)$ and denote by $\mathcal{C}'$ the set of ε-conjugacy classes in $\mathbf{G}'(F) = \mathrm{GL}_n(E)$. Suppose that $n = m$, and that $\mathbf{T}$ is a Cartan subgroup of $\mathbf{G}$, defined over F. Let $\mathbf{T}'$ be an ε-stable Cartan subgroup of $\mathbf{G}'$ with $\mathbf{T} \xrightarrow{\sim} \mathbf{T}'_\varepsilon$ defined over F. This isomorphism induces the image map $\mathcal{A}_{\mathbf{G}/\mathbf{G}'}$, as defined by Kottwitz and Shelstad, between semisimple classes in $\mathcal{C}$ and ε-semisimple classes in $\mathcal{C}'$ [12]. Now we have

$$N_\varepsilon \colon \mathbf{T}' \to \mathbf{T}'_\varepsilon \xrightarrow{\sim} \mathbf{T},$$

with all the maps defined over F.

Lemma 3.5 *Suppose that $\mathbf{T} \xrightarrow{\sim} \mathbf{T}'_\varepsilon$ is defined over F as above. If $\delta \in \mathbf{T}'$ is strongly ε-regular, then $\mathbf{G}'_{\varepsilon,\delta} = (\mathbf{T}')^\varepsilon$.*

Proof If δ is strongly ε-regular, then $\mathbf{G}'_{\varepsilon,\delta}$ is a torus which is stable under $\mathrm{Int}(\delta) \circ \varepsilon$, and which is maximal with respect to this condition. Since $\mathbf{T}'$ is a maximal torus, we know that $\mathbf{T}'_\delta = \mathbf{T}'$. Moreover, $\mathbf{T}'$ is ε-stable, so $(\mathbf{T}')^\varepsilon$ is the desired twisted centralizer. ■

As in [8] we wish to integrate over all the twisted conjugacy classes in $\mathcal{N}$. By Proposition 3.3 and the surjectivity of the norm correspondence, up to a set of measure zero these classes are parameterized by regular semisimple conjugacy classes in G. To account for the fact that more than one regular conjugacy class in G can parameterize a class in $\mathcal{N}$, we integrate over all the Cartan subgroups of G. Fix a representative $\mathbf{T}$ for each conjugacy class of Cartan subgroups of $\mathbf{G}$ which are defined over F. Let $d\gamma$ be a Haar measure for $T = \mathbf{T}(F)$. For $\{\gamma'\} \in \mathcal{C}'$, we define

$$D_\varepsilon(\gamma') = \det\bigl(\mathrm{Ad}(\gamma') \circ \varepsilon - 1\bigr)\,|_{\mathfrak{g}/\mathfrak{g}_{\varepsilon,\gamma'}}$$

as in [12]. Now by Lemma 2.17, Lemma 3.5, Proposition 3.3, and by computing the Jacobian of the open immersion in page 227 of [1] (or Theorem 3.2 of [17]), the measure $|W(\mathbf{T})|^{-1}|D_\varepsilon(\gamma')|\,d\gamma$ as $\mathbf{T}$ ranges, will provide us with a measure for $\mathcal{N}$. Here $\{\gamma\}$ is in $N_\varepsilon(\{\gamma'\})$ for each ε-regular $\{\gamma'\}$ in $\mathcal{N}$.

By Lemma 4.5.A of [12], the function

$$\kappa_1(\gamma, \gamma') = |D_\varepsilon(\gamma')|\,d\gamma'/|D(\gamma)|\,d\gamma$$

is continuous on

$$\bigl\{\,(\gamma, \gamma') \mid \{\gamma\} \in N_\varepsilon(\{\gamma'\})\bigr\}\,.$$

We define

$$\kappa(\gamma,\gamma') = \begin{cases} \kappa_1(\gamma,\gamma') & \text{if } \{\gamma\} \in N_\varepsilon(\{\gamma'\}) \text{ and } \gamma' \text{ is } \varepsilon-\text{regular} \\ 0 & \text{otherwise.} \end{cases}$$

For $\{\gamma\} \in \mathcal{C}$, we define

$$\mathcal{A}(\{\gamma\}) = \big\{ \{\alpha\gamma'\} \in \mathcal{N} \mid \{\gamma\} \in N_\varepsilon(\{\gamma'\}), \alpha \in NE^\times \setminus F^\times \big\}.$$

We then set $\Delta(\gamma, \alpha\gamma') = \omega'(\alpha)\kappa(\gamma,\gamma')$.

We set

$$\tilde{\Phi}_\varepsilon\big(\mathcal{A}(\{\gamma\}), \psi_{\tau'}\big) = \sum_{\{\gamma'\}\in\mathcal{A}(\{\gamma\})} \Delta(\gamma,\gamma')\tilde{\Phi}_\varepsilon(\gamma',\psi_{\tau'}).$$

As in [8] one part of the residue of $A(s, \tau'\otimes\tau, w_0)$ at $s = 0$ will come from the Weyl integration formula, applied to the class function $\tilde{\Phi}_\varepsilon\big(\mathcal{A}(\{\gamma\}), \psi_{\tau'}\big)$. Thus, we expect a contribution from these classes of the "regular term"

(3.1)

$$R_{\mathbf{G}}(f_\tau, \psi_{\tau'}) = \sum_{\{\mathbf{T}_i\}} \mu(T_i)|W(T_i)|^{-1} \int_{T_i} \Phi(\{\gamma\}, f_\tau)\tilde{\Phi}_\varepsilon\big(\mathcal{A}(\{\gamma\}),\psi_{\tau'}\big)\,|D(\gamma)|\,d\gamma,$$

with $\{\mathbf{T}_i\}$ running over all the conjugacy classes of elliptic Cartan subgroups of $\mathbf{G}$, $T_i = \mathbf{T}_i(F)$, and $\mu(T_i)$ the measure of T_i. Note that

$$R_{\mathbf{G}}(f_\tau, \psi_{\tau'}) = \sum_{\{\mathbf{T}_i\}} \mu(T_i)|W(T_i)|^{-1} \int_{T_i} \sum_{\{\gamma\}\in N_\varepsilon(\{\gamma'\})} \cdot \sum_{\alpha\in NE^\times\setminus F^\times} \omega'(\alpha)\Phi_\varepsilon(\alpha\gamma',\psi_{\tau'})\Phi(\gamma, f_\tau)|D_\varepsilon(\gamma')|\,d\gamma$$

$$= \int_{\mathcal{N}_{\mathrm{reg}}} \sum_\alpha \omega'(\alpha) f_\tau(I - X'Y^{-1}X)\psi_{\tau'}(Y^{-1})|\det Y|^{-\langle\rho_{\mathbf{P}},\tilde{\alpha}\rangle}\,d(X,Y). \tag{3.2}$$

We now address the question of the convergence of $R_{\mathbf{G}}(f_\tau, \psi_{\tau'})$. By Lemma 2.2 of [16], we need to show that

$$\int_{\mathbf{T}(F)} \tilde{\Phi}_\varepsilon\big(\mathcal{A}(\{\gamma\}), \psi_{\tau'}\big)\,\Phi(\gamma, f_\tau)|D(\gamma)|\,d\gamma$$

converges on any elliptic Cartan $\mathbf{T}$ of $\mathbf{G}$. By Theorem 14 of [9], the function $|D(\gamma)|^{1/2}\cdot \Phi(\gamma, f_\tau)$ is bounded on the intersection of $\mathbf{T}(F)$ with the regular set G_{reg}. Therefore, we need to prove the convergence of

$$\int_{\mathbf{T}(F)} \tilde{\Phi}_\varepsilon\big(\mathcal{A}(\{\gamma\}), \psi_{\tau'}\big)\,|D(\gamma)|^{1/2}\,d\gamma.$$

That is, we need to look at the absolute convergence of

$$\int_{\mathbf{T}(F)} \sum_{\{\gamma\}\in N_\varepsilon(\{\gamma'\})} \sum_{\alpha\in NE^\times\backslash F^\times} \omega'(\alpha)\tilde{\Phi}_\varepsilon(\alpha\gamma',\psi_{\tau'})\kappa(\gamma,\gamma')|D(\gamma)|^{1/2}\,d\gamma.$$

It is enough to look at each term in the sum, namely, enough to show the convergence of

$$\int_{\mathbf{T}'_\varepsilon(F)} |\tilde{\Phi}_\varepsilon(\alpha\gamma',\psi_{\tau'})|\,|D(\gamma)|^{-1/2}|\,|D_\varepsilon(\gamma')|\,d\gamma'. \tag{3.3}$$

Since $\tilde{\Phi}_\varepsilon(-,-)$ is a tempered distribution, the results of [9], and [4], imply the function $|D_\varepsilon(\gamma')|^{1/2}\tilde{\Phi}_\varepsilon(\gamma',\psi_{\tau'})$ is bounded on the intersection of $\mathbf{T}'_\varepsilon(F)$ with the ε-regular set. Moreover, $\kappa(\gamma,\gamma')^{1/2}$ is continuous, and thus (3.3) converges.

We let $M=\mathrm{GL}_n(E)\times G$ act on N by the adjoint action. The orbit of $n(X,Y)$ is the set of pairs $n\big(gXh^{-1},gY\varepsilon(g)^{-1}\big)$, with $g\in G'=\mathrm{GL}_n(E)$ and $h\in G$. The stabilizer $\Delta^\vee$ of this action at $n(X,Y)$ consists of all those pairs $(g,h)\in M$ with $g\in\mathbf{G}'_{\varepsilon,Y}(F)$ and for which $gX=Xh$. Then, by Lemma 2.19, we see that $h\in\mathbf{G}_Z(F)$, with $Z=I-X'Y^{-1}X$. We consider $\Delta^\vee$ as a subgroup of both $\mathbf{G}'_{\varepsilon,Y}(F)$ and $\mathbf{G}_Z(F)$ via its projections onto its components. We now reformulate (1.2) by first integrating over each M-orbit in N. Note that by Lemma 2.3 of [13] $d^*(X,Y)=|\det Y|^{-\langle\rho_{\mathbf{P}},\tilde{\alpha}\rangle}\,d(X,Y)$ is an invariant measure on these orbits.

Now, making the change of variables, and using our assumption that m and n have the same parity, we see that (1.2) can be rewritten as

$$\int_{(X,Y)} \psi_{\tau'}(Y)f_\tau(I-X'Y^{-1}X)|\det Y|_E^s\xi_L\big(\varepsilon(Y^{-1})\big)\,\xi_{L'}(X)\,d^*(X,Y). \tag{3.4}$$

We consider the map from the orbit of $n(X,Y)$ under M to $G'/\Delta^\vee\times\Delta^\vee\backslash G$ given by $n\big(gXh^{-1},gY\varepsilon(g)^{-1}\big)\mapsto(g\Delta^\vee,\Delta^\vee h)$. The fiber of this map is homeomorphic to $X\Delta^\vee$. Thus, integrating over the orbit of $n(X,Y)$ can be accomplished by integrating over the product of $G'/\Delta^\vee\times\Delta^\vee\backslash G$ and $X\Delta^\vee$. Consequently, the contribution to (3.4) from the orbit of $n(X,Y)$ may be written as

$$\int_{g\in G'/\Delta^\vee}\int_{h\in\Delta^\vee\backslash G}\int_{X\Delta^\vee} \psi_{\tau'}\big(gY\varepsilon(g)^{-1}\big)\,f_\tau(h^{-1}Zh)\big|\det\big(gY\varepsilon(g)^{-1}\big)\big|_E^s$$
$$\cdot\,\xi_{\tilde{L}}\big(gY\varepsilon(g)^{-1}\big)\,\xi_{L'}(gXh_0h)\,d(Xh_0)\,dh\,dg$$

(3.5)

$$=\int_{g\in G'/\tilde{G}'_{\varepsilon,Y}}\int_{g_1\in\tilde{G}'_{\varepsilon,Y}/G'_{\varepsilon,Y}}\int_{G_Z\backslash G}\int_{G'_{\varepsilon,Y}/\Delta^\vee}\int_{XG_Z}\psi_{\tau'}\big(gg_1Y\varepsilon(gg_1)^{-1}\big)\,f_\tau(h^{-1}Zh)$$
$$\cdot\,\big|\det\big(gg_1Y\varepsilon(gg_1)^{-1}\big)\big|_E^s\xi_{\tilde{L}}\big(gg_1Y\varepsilon(gg_1)^{-1}\big)\,\xi_{L'}(gg_1g_0Xh_0h)\,d(Xh_0)\,dg_0\,dh\,dg_1\,dg.$$

Here $\tilde{L}=\tilde{\varepsilon}(L)$. Now we use the fact that $\tilde{G}'_{\varepsilon,Y}/G'_{\varepsilon,Y}\simeq NE^\times$. For $z\in NE^\times$, let $g_1(z)$ be a choice of representative in $\tilde{G}'_{\varepsilon,Y}$ which satisfies $g_1(z)Y\varepsilon\big(g_1(z)\big)^{-1}=zY$. In

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fact, we can take $g_1 = a_z I_n$, where a_z satisfies $a_z \bar{a}_z = z$. Then (3.5) becomes

$$\int_{G'/\tilde{G}'_{\varepsilon,Y}} \int_{NE^\times} \int_{G_Z\backslash G} \int_{G'_{\varepsilon,Y}/\Delta^\vee} \int_{XG_Z} \psi_{\tau'}\big(g(zY)\varepsilon(g)^{-1}\big)\, f_\tau(h^{-1}Zh)\big|$$

$$\cdot \det\big(g(zY)\varepsilon(g)^{-1}\big)\big|_E^s \xi_{\tilde{L}}\Big(g\big((zY)\varepsilon(g)^{-1}\big)\Big)\, \xi_{L'}(ga_z g_0 Xh_0 h)\, d(Xh_0)\, dg_0\, dh\, d^\times z\, dg$$

$$= \int_{NE^\times} \omega'(z)|z|_E^{2ns} \int_{G'/\tilde{G}'_{\varepsilon,Y}} \int_{G_Z\backslash G} \int_{G'_{\varepsilon,Y}/\Delta^\vee} \int_{XG_Z} \psi_{\tau'}\big(gY\varepsilon(g)^{-1}\big)\, f_\tau(h^{-1}Zh)$$

$$\cdot \big|\det\big(g^{-1}Y\varepsilon(g)^{-1}\big)\big|_E^s \xi_{\tilde{L}}\big(gzY\varepsilon(g)^{-1}\big)\, \xi_{L'}(ga_z g_0 Xh_0 h)\, d(Xh_0)\, dg_0\, dh\, dg\, d^\times z.$$

However, since $(\tau')^\varepsilon \simeq \tau'$, we know that $\omega'(z) \equiv 1$ on $NE^\times$. Now, summing over $\alpha \in F^\times/NE^\times$, we get

$$\sum_{\alpha\in F^\times/NE^\times} \omega'(\alpha) \int_{NE^\times} |z|_E^{2ns} \int_{G'/\tilde{G}'_{\varepsilon,Y}} \int_{G_Z\backslash G} \int_{G'_{\varepsilon,Y}/\Delta^\vee} \int_{XG_Z} \psi_{\tau'}\big(gY\varepsilon(g)^{-1}\big)\, f_\tau(h^{-1}Zh)$$

$$\cdot \big|\det\big(g^{-1}\alpha Y\varepsilon(g)^{-1}\big)\big|_E^s \xi_{\tilde{L}}\big(gz\alpha Y\varepsilon(g)^{-1}\big)\, \xi_{L'}(ga_z g_0 Xh_0 h)\, d(Xh_0)\, dg_0\, dh\, dg\, d^\times z.$$

We denote this last expression by $\tilde{\psi}(s, Z)$.

Now suppose that $X = I$. Then $Z = I \pm Y^{-1}$. Moreover, if (Y, I) satisfies (1.1) and $g \in G'_{\varepsilon,Y}$, then

$$Y + \tilde{\varepsilon}(Y) = XX' = \pm I$$

which implies

$$\begin{aligned} Y + \tilde{\varepsilon}(Y) &= gY\varepsilon(g)^{-1} + \tilde{\varepsilon}\big(gY\varepsilon(g)^{-1}\big) = (gX)(gX)' \\ &= gXX'\varepsilon(g)^{-1} = g(\pm I)\varepsilon(g)^{-1} \end{aligned}$$

and therefore, $g\varepsilon(g)^{-1} = I$, which says that $g \in \mathbf{G}(F)$. Now we clearly have $G_{\varepsilon,Y} = G_Z$. Also note that $gX = Xh$ implies $g = h$, and so $\Delta^\vee = \{(g, g) \mid g \in G'_{\varepsilon,Y}\} \simeq G'_{\varepsilon,Y}$. We further assume that such a Y is ε-regular and therefore Z is regular and semisimple. Thus, in this case (3.6) becomes

$$\psi(s, Z) = \sum_{F^\times/NE^\times} \omega'(\alpha) \int_{NE^\times} |z|_E^{2ns} \int_{G'/\tilde{G}'_{\varepsilon,Y}} \int_{G_Z\backslash G} \int_{G_Z} \psi_{\tau'}\big(g^{-1}Y\varepsilon(g)^{-1}\big)\big|$$

$$\cdot \det\big(g\alpha Y\varepsilon(g_0^{-1})\big)\big|_E^s f_\tau(h^{-1}Zh)\xi_{\tilde{L}}\big(g\alpha zY\varepsilon(g)^{-1}\big)\, \xi_{L'}(ga_z h_0 h)\, dh_0\, dh\, dg\, d^\times z$$

We may assume that L is a basic open neighborhood about zero, that is, for some integer t,

$$L = M_n(\mathfrak{p}_E^t) = \big\{x = (x_{ij}) \bigm| |x_{ij}|_E \le q_E^{-t} \text{ for all } i, j\big\}.$$

Then $\tilde{L} = L$.

Lemma 3.6 *For any Y we have $\{g \mid gY\varepsilon(g)^{-1} \in \operatorname{supp}(\psi_{\tau'})\}$ is compact.*

Proof Let $L_0 = \{y \in M_n(E) \mid q_E^{-t-1} \leq \|y\|_\infty \leq q_E^t\}$, where $\| \; \|_\infty$ is the supremum norm. Therefore, if $z \in NE^\times$, with $|z|_E > 1$, then $zy \notin L$ for any $y \in L_0$. Let $g \in G'$. By changing g by an element of $\tilde{G}'_{\varepsilon,Y}$, we may assume that $gY\varepsilon(g)^{-1} \in L_0$. We fix our representatives for $G'/\tilde{G}'_{\varepsilon,Y}$ to have this property.

Now let C be a compact subset of G' so that $\operatorname{supp}(\psi_{\tau'}) \subset CZ(G')$. Then, since C is compact, we can choose integers k_1, k_2, j_1 and j_2 so that, for any $c \in C$, $q_E^{k_1} \leq \|c\|_\infty \leq q_E^{k_2}$ and $q_E^{j_1} \leq |\det c|_E \leq q_E^{j_2}$. Note that if $cz \in L_0$, then

$$q_E^{-t-1} \leq \|cz\|_\infty = \|c\|_\infty |z|_E \leq q_E^t,$$

which then implies

$$q_E^{-t-1-k_2} \leq |z|_E \leq q_E^{t-k_1}.$$

Let $\Omega = \{z \mid q_E^{-t-1-k_2} \leq |z|_E \leq q_E^{t-k_1}\}$. Then Ω is compact and $C\Omega \supset CZ(G') \cap L_0$. Thus,

$$\{gY\varepsilon(g)^{-1}\} \cap \operatorname{supp}(\psi_{\tau'}) \subset L_0 \cap CZ(G') \subset C\Omega$$

is a closed subset of a compact set, hence compact. ■

Let $\operatorname{supp}(g) = \{g \mid gY\varepsilon(g)^{-1} \in \operatorname{supp}(\psi_{\tau'})\}$. We see that

$$\psi_{\tau'}\big(gY\varepsilon(g)^{-1}\big)\,\xi_{\tilde{L}}\big(\alpha zgY\varepsilon(g)^{-1}\big) = 0,$$

unless $gY\varepsilon(g)^{-1} \in \operatorname{supp}\psi_{\tau'} \cap \alpha^{-1}z^{-1}\tilde{L}$, and since $\operatorname{supp}(g)$ is compact, we have $|\det z|_E \geq \eta$, for some η, which depends only on $\psi_{\tau'}$ and L. Also, since $\operatorname{supp} f_\tau$ is compact modulo $Z(G)$, and $Z(G) \simeq E^1$ is compact, we see that $\operatorname{supp} f_\tau$ is compact. Note that $\{h \mid hZh^{-1} \in \operatorname{supp} f_\tau\}$ is then also compact, and we call this $\operatorname{supp}(h)$. Let $h_0 \in \mathbf{G}_Z(F)$. We may assume that h_0 is diagonal, and we further assume that

$$h_0 = \operatorname{diag}\{a_1 I_{k_1}, a_2 I_{k_2} \dots, a_b I_{k_b}, \bar{a}_b^{-1} I_{k_b}, \dots, \bar{a}_2^{-1} I_{k_2}, \bar{a}_1^{-1} I_{k_1}\},$$

with the pairs $(a_i, \bar{a}_i^{-1})$ distinct. Note that $\xi_{L'}(ga_zh_0h) = 0$ unless $a_zh_0 \in \big(\operatorname{supp}(g)\big)^{-1} \cdot L'\big(\operatorname{supp}(h)\big)^{-1}$, which is compact. Further note that if T is the compact part of G_Z, then there is some κ so that if, for all i, $|a_za_i|_E \leq \kappa$ and $|a_za_i^{-1}|_E \leq \kappa$, then

$$\operatorname{supp}(g)a_zh_0T\operatorname{supp}(h) \subseteq L'.$$

Thus, for such a z and h_0, the element h ranges over all of $\operatorname{supp}(h)$, which then gives the term $\Phi(Z, f_\tau)$, which vanishes when the split component of $\mathbf{G}_Z$ is non-trivial. Thus, we may assume that $\mathbf{G}_Z$ is compact.

Let $\kappa_1 \geq \kappa$, and further suppose that $\kappa = q_E^{-m}$ and $\kappa_1 = q_E^{-m'}$. Suppose that $|a_z|_E = q_E^{-\ell}$. Then, since $|a_za_i|_E \leq q_E^{-m}$ and $|a_za_i^{-1}|_E \leq q_E^{-m}$, then we have $|a_i|_E \leq q_E^{-m+\ell}$ and $|a_i|_E \geq q_E^{m-\ell}$. Thus for each $\ell \geq m'$, we have a contribution to (3.6) of

$$\Big(\int_{\mathfrak{o}^\times} d\beta \prod_i \int_{q_E^{m-\ell} \leq |a_i|_E \leq q_E^{-m+\ell}} d^\times a_i \int_T dt q_E^{-4\ell ns}\Big)\,\Phi_{\varepsilon,s}(Y, \psi_{\tau'})\Phi(Z, f_\tau).$$

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Summing over all $\ell \geq m'$, we have

$$\Phi_{\varepsilon,s}(Y,\psi_{\tau'})\Phi(Z,f_\tau)\sum_{\ell\geq m'} q_E^{-4\ell ns}\mu(\ell),$$

where

$$\mu(\ell)=\prod_i \int_{q_E^{m-\ell}\leq|a_i|_E\leq q_E^{-m+\ell}} d^\times a_i \int_T dt.$$

The series converges for Re $s>0$. Integrating over other values of z can only lead to an entire function. Summing over $F^\times/NE^\times$, we have proved the following result.

Lemma 3.7 *There is an entire function $E_Z(s) = E(s,Z,Y,f_\tau,\psi_{\tau'},L,L')$ so that $\psi(s,Z)=E_Z(s)$ is entire if Z is regular and non-elliptic and*

$$\psi(s,Z)=E_Z(s)+\sum_{F^\times/NE^\times}\omega'(\alpha)\Phi_{\varepsilon,s}(Y,\psi_{\tau'})\Phi(Z,f_\tau)\mu\big(G_Z(F)\big)\,q_E^{b(Y,Z)s}L(\mathbf{1}_E,4ns),$$

for Re $s>0$ *if Z is regular and elliptic. The integer $b(Y,Z)$ depends on $Y,Z,\psi_{\tau'},f_\tau,L$ and L'. Thus,*

$$\operatorname*{Res}_{s=0}\psi(s,Z)=(4n\log q_E)^{-1}\sum_{F^\times/NE^\times}\omega'(\alpha)\Phi_e(Y,\psi_{\tau'})\Phi(Z,f_\tau)\mu\big(\mathbf{G}_Z(F)\big)$$

if Z is regular and elliptic and $\operatorname{Res}_{s=0}\psi(s,Z)=0$ *if Z is regular and non-elliptic.*

Corollary 3.8 *Let $\mathbf{T}_i$ be a Cartan subgroup of $\mathbf{G}$. Let T_i' be the regular set of $\mathbf{T}_i(F)$. Let ω_i be a compact subset of T_i'. Then given f_τ, $\psi_{\tau'}$, L and L', the integer $b(Y,Z)$ can be chosen independently for all Y and all $Z\in\omega_i$.*

Proof It is enough to show that one can choose the compact subsets supp(h) and supp(g) independently of Y and $Z\in\omega_i$. For supp(h) this follows from the corollary to Lemma 19 of [9]. For supp(g) this follows from the fact that supp$(\psi_{\tau'})$ is compact modulo $Z(G')$ and Lemma 2.1 of [1]. ∎

In order to calculate the residue of the intertwining operator, we integrate over all the orbits of M acting on N. We accomplish this by integrating over the ε-regular ε-conjugacy classes in $\mathcal{N}$. First suppose that $n=m$. Then we must integrate $\psi(s,Z)$ over the orbits of N under M. By Proposition 3.3 almost all such orbits are parameterized by single ε-regular ε-conjugacy classes in $\mathcal{N}$. Thus, by removing a set of measure zero from these classes, we can instead integrate $\psi(s,Z)$ over ε-semisimple ε-regular ε-conjugacy classes $\{Y\}$ in $\mathcal{N}$. Then $N_\varepsilon(\{Y^{-1}\})$ is regular and semisimple. Let $\{\mathbf{T}_i\}$ denote a complete set of conjugacy classes of Cartan subgroups of $\mathbf{G}$ defined over F. We can integrate over $\cup_i T_i$ using measures

$$|W(T_i)|^{-1}\kappa_1(\{\gamma_i\},\{\gamma_i'\})|D(\gamma_i)|\,d\gamma_i=|W(T_i)|^{-1}|D_\varepsilon(\gamma_i')|\,d\gamma_i.$$

Now suppose that $n > m$ and $n \equiv m \pmod 2$. Then, for almost all $\{Y\}$, we can choose a representative $\operatorname{diag}\{J_1, Y_2, J_2\}$ as in Corollary 3.4, *i.e.*, so that $Y_2 \in \mathrm{GL}_m(E)$ is ε-regular and that $X = \begin{pmatrix} 0 \\ I \\ 0 \end{pmatrix}$. Outside of a set of measure zero, the twisted conjugacy classes in $\mathcal{N}$ form a fiber bundle with finite fibers coming from the twisted conjugacy classes of $\operatorname{diag}\{J_1, J_2\}$. The base of the fiber bundle is parameterized by ε-conjugacy classes $\{Y_2^{-1}\}$ in $\mathrm{GL}_m(E)$ such that (I_m, Y_2) is a solution to (1.1) when $\mathrm{GL}_m(E) \times G(m)$ is considered as a Levi subgroup of $G(3m)$. Thus we may use ε-stable Cartan subgroups of GL_m and their F-isomorphisms with the T_i's as in the case $n = m$. We get measures $\kappa_1(\{\gamma_i\}, \{\gamma_i'\})|D(\gamma_i)|\, d\gamma_i$ on the T_i's, and by surjectivity, we can integrate over $\cup_i T_i$ and then use the image correspondence $\mathcal{A}$. Thus, for $n \ge m$, and $n \equiv m \pmod 2$, and (X, Y) as above, we have

$$\tilde{\psi}(s, Z) = \sum_{F^\times/NE^\times} \omega'(\alpha) \int_{NE^\times} |z|_E^{2ns} \int_{G'/G'_{\varepsilon,Y}} \int_{G_Z\backslash G} \int_{G_Z} \psi_{\tau'}\big(gY\varepsilon(g)^{-1}\big) f_\tau$$

$$\cdot (h^{-1}Zh)| \det gY\varepsilon(g)^{-1}|_E^s$$

$$\cdot\, \xi_L\big(gzY\varepsilon(g)^{-1}\big)\, \xi_{L'}(ga_zXh_0h)\, dh_0\, dh\, dg\, d^\times z. \tag{3.6}$$

Applying Lemma 3.6 and Corollary 3.8 we find

$$\psi(s, Z) = E_s(Z) + \sum_\alpha \omega'(\alpha) q_E^{b(Y,Z)s} L(\mathbf{1}_E, s) \Phi_{\varepsilon,s}(Y, \psi_{\tau'}) \Phi(Z, f_\tau),$$

and thus the residue at $s = 0$ is as above.

Let $\mathbf{T}_i$ be a representative of a conjugacy class of Cartan subgroups of $\mathbf{G}$. Let T_i' be the subset of regular elements of $T_i = \mathbf{T}_i(F)$. Now let

$$R(s, Z) = (4n \log q_E)^{-1} \sum_\alpha \omega'(\alpha) \Phi_\varepsilon(\alpha Y, \psi_{\tau'}) \Phi(Z, f_\tau) \mu\big(\mathbf{G}_Z(F)\big)$$

Then $\varphi(s, Z) = \psi(s, Z) - R(s, Z)$ is an entire function in s while it is locally constant in $Z \in T_i'$. For each $\gamma \in T_i'$ we let

$$\psi_{\mathcal{A}}(s, \gamma) = \sum_{\{Y\} \in \mathcal{A}(\{\gamma\})} \psi(s, \gamma).$$

Recall that $\mathcal{A}(\{\gamma\}) = \{\{Y\} | \{\gamma\} \in N_\varepsilon(\{Y^{-1}\})\}$. Then

$$\sum_i |W(T_i)|^{-1} \int_{T_i'} \psi_{\mathcal{A}}(s, \gamma) |D(\gamma)|\, d\gamma$$

$$- \sum_i |W(T_i)|^{-1} \mu(T_i) \int_{T_i'} \Phi_\varepsilon\big(\mathcal{A}(\{\alpha\}), \psi_{\tau'}\big) \Phi(\gamma, f_\tau) |D(\gamma)|\, d\gamma$$

$$= \sum_i |W(T_i)|^{-1} \int_{T_i'} \varphi_{\mathcal{A}}(s, \gamma) |D(\gamma)|\, d\gamma,$$

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where

$$\varphi_{\mathcal{A}}(s,\gamma) = \sum_{\mathcal{A}(\{\gamma\})} \varphi(s,\gamma).$$

Now we let ω_i denote a compact subset of T_i'. Then

$$\begin{aligned}
\lim_{\omega_i \to T_i'} \operatorname*{Res}_{s=0} \sum_i |W(T_i)|^{-1} \int_{\omega_i} \psi_{\mathcal{A}}(s,\gamma)|D(\gamma)|\, d\gamma \\
&= cR_G(f_\tau, \psi_{\tau'}) \\
&\quad + \lim_{\omega_i' \to T_i'} \operatorname*{Res}_{s=0} \sum_i |W(T_i')|^{-1} \int_{T_i' \setminus \omega_i} \psi_{\mathcal{A}}(s,\gamma)|D(\gamma)|\, d\gamma \\
&= \lim_{\omega_i \to T_i'} \sum_i |W(\mathbf{T}_i)|^{-1} \mu(T_i) \int_{T_i' \setminus \omega_i} \Phi_\varepsilon\big(\mathcal{A}(\{\gamma\}), \psi_{\tau'}\big)\, \Phi(\gamma, f_\tau)|D(\gamma)|\, d\gamma \\
&\quad + \lim_{\omega_i \to T_i'} \sum_i |W(\mathbf{T}_i)|^{-1} \int_{\omega_i} \varphi_{\mathcal{A}}(s,\gamma)|D(\gamma)|\, d\gamma. \qquad (3.7)
\end{aligned}$$

The first limit is zero since the normalized orbital integrals are locally bounded on T_i'. Since $\varphi_{\mathcal{A}}(s,\gamma)$ is entire in s,

$$\operatorname*{Res}_{s=0} \sum_i |W(\mathbf{T}_i)|^{-1} \int_{T_i' \setminus \omega_i} \varphi_{\mathcal{A}}(s,\gamma)|D(\gamma)|\, d\gamma$$

is independent of the choice of ω_i and we can drop the limit in front of it. Thus, the residue is

$$cR_G(f_\tau, \psi_{\tau'}) + \operatorname*{Res}_{s=0} \sum_i |W(\mathbf{T}_i)|^{-1} \int_{T_i' \setminus \omega_i} \varphi_{\mathcal{A}}(s,\gamma)|D(\gamma)|\, d\gamma,$$

and

$$\operatorname*{Res}_{s=0} \sum_i |W(\mathbf{T}_i)|^{-1} \int_{T_i' \setminus \omega_i} \varphi_{\mathcal{A}}(s,\gamma)|D(\gamma)|\, d\gamma$$

is independent of ω_i, and thus depends only on the singular part of T_i. Here $c = (4n \log q_E)^{-1}$.

Now suppose $n < m$ and $n \equiv m \pmod 2$. Let $k = (m-n)/2$. Consider the injection

$$h \mapsto \begin{pmatrix} I_k & & \\ & h & \\ & & I_k \end{pmatrix},$$

of $G(n)$ into $G(m)$. Let $N_\varepsilon \colon \mathcal{N} \to \mathcal{C}$ be the ε-norm correspondence from ε-conjugacy classes $\mathcal{N}$ in $\mathrm{GL}_n(E)$ to conjugacy classes of $G(m)$. Suppose $X \in M_{n\times m}(E)$ and $Y \in \mathrm{GL}_n(E)$ satisfy $Y + \tilde{\varepsilon}(Y) = XX'$. Note that rank $X'Y^{-1}X \le n$, so at most $n \le m-2$ eigenvalues of $I - X'Y^{-1}X$ are different from 1. Thus the conjugacy class $\{I - X'Y^{-1}X\}$ has a representative in the image of $G(n)$ under the above injection. Let $\mathcal{C}^\vee$ be the subset of $\mathcal{C}$ consisting of those conjugacy classes of $G(m)$ whose semisimple parts meet $G(n)$. Then we see that $N_\varepsilon : \mathcal{N} \to \mathcal{C}^\vee$.

Lemma 3.9 *If $n < m$ the norm correspondence N_ε has finite fibers.*

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Proof We need only show that if $Y \in \mathrm{GL}_n(E)$ and $X \in M_{n\times m}(E)$ satisfy (1.1), then $I - X'Y^{-1}X$ determines the semisimple part of the conjugacy class of $\varepsilon(Y^{-1})Y^{-1}$. By Lemma 2.3, we have $\varepsilon(Y^{-1})Y^{-1}X = -X(I - X'Y^{-1}X)$. We suppose that X is in row echelon form with the last $n-r$ rows identically zero. We consider the decomposition $E^n \oplus E^{n-r}$, with $X|_{E^{n-r}} \equiv 0$ and $X|_{E^r}$ an injection into E^m. Thus, the matrix of $\varepsilon(Y^{-1})Y^{-1}$ with respect to a basis which respects the above decomposition is $\left(\begin{smallmatrix} A & * \\ 0 & I \end{smallmatrix}\right)$, with A determined by $I - X'Y^{-1}X$. ■

Note that almost all of the ε-conjugacy classes in $\mathcal{N}$ are parameterized by regular semisimple conjugacy classes in $G(n)$, and thus by classes in $\mathcal{C}^\vee$. To see this, note that if k is even, then we can take $X_1 = \begin{pmatrix} 0_{n\times k} & I_n & 0_{n\times k} \end{pmatrix}$. Then

$$X_1' = \begin{pmatrix} & & (-1)^n u_k \\ & u_n & \\ u_k & & \end{pmatrix} \begin{pmatrix} 0_{k\times n} \\ I_n \\ 0_{k\times n} \end{pmatrix} u_n = \begin{pmatrix} 0 \\ I_n' \\ 0 \end{pmatrix}.$$

Thus, $X_1X_1' = I_n' = Y + \tilde{\varepsilon}(Y)$. Furthermore,

$$I_m - X_1'Y^{-1}X_1 = \begin{pmatrix} I_k & & \\ & I_n + (-1)^n Y^{-1} & \\ & & I_k \end{pmatrix},$$

which represents a conjugacy class in $\mathcal{C}^\vee$. We further note that $I_n + (-1)^n Y^{-1}$ is in the image of the ε-norm correspondence in the case $n = m$. If k is odd, and n is even, we let

$$w_n = \begin{pmatrix} & & & 1 \\ & & \iddots & \\ & 1 & & \\ 1 & & & \end{pmatrix}$$

and take $X_1 = \begin{pmatrix} 0_{n\times k} & w_n & 0_{n\times k} \end{pmatrix}$. Then

$$X_1' = \begin{pmatrix} & & -u_k \\ & -u_n & \\ u_k & & \end{pmatrix} \begin{pmatrix} 0_{k\times n} \\ w_n \\ 0_{k\times n} \end{pmatrix} u_n = \begin{pmatrix} 0 \\ -u_n w_n u_n \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -w_n \\ 0 \end{pmatrix}.$$

Thus, $X_1X_1' = -I_n = Y + \tilde{\varepsilon}(Y)$. Furthermore,

$$I_m - X'Y^{-1}Y = \begin{pmatrix} I_k & & \\ & I_n + (w_nY^{-1}w_n^{-1}) & \\ & & I_k \end{pmatrix}.$$

Since n is even, $\varepsilon(w_n) = -w_n = -w_n'$, and so we have

$$\left(w_nY\varepsilon(w_n)^{-1}\right) + \tilde{\varepsilon}\left(w_nY\varepsilon(w_n)^{-1}\right) = w_nw_n' = I_n.$$

Therefore, $w_nYw_n + \tilde{\varepsilon}(w_nYw_n) = -I_n = I_nI_n'$. Therefore,

$$I_n + w_nY^{-1}w_n = I_n - (I_n)'(w_nYw_n)^{-1}I_n,$$

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is in the image of the norm correspondence for $n = m$. For the case where k and n are both odd, we let $Y_1 = -Y$, and $X_1 = \begin{pmatrix} 0 & I_n & 0 \end{pmatrix}$. Then $X_1' = \left(\begin{smallmatrix} 0 \\ -I \\ 0 \end{smallmatrix}\right)$, and $X_1X_1' = -I = -\big(Y + \tilde{\varepsilon}(Y)\big) = Y_1 + \tilde{\varepsilon}(Y_1)$. Furthermore,

$$I - X_1'Y_1^{-1}X_1 = \begin{pmatrix} I_k & 0 & 0 \\ 0 & I_n - Y^{-1} & 0 \\ 0 & 0 & I_k \end{pmatrix},$$

and we know $I_n - Y^{-1}$ is in the image of the norm correspondence when $n = m$.

Considering the pairs (X_1, Y) as above, we see that $\Delta^\vee$ projects surjectively onto $\mathbf{G}'_{\varepsilon,Y}(F)$, and that for almost all Y we have $X_1\mathbf{G}_Z(F) \simeq \mathbf{G}'_{\varepsilon,Y}(F)$. Thus, summing over $F^\times/NE^\times$, (3.5) becomes

$$\begin{aligned}
\tilde{\psi}(s,Z) = &\sum_{F^\times/NE^\times} \omega'(\alpha) \int_{NE^\times} |z|_E^{2ns} \\
&\cdot \int_{G'/G'_{\varepsilon,Y}} \int_{G_Z\backslash G} \int_{X_1G_Z} \psi_{\tau'}\big(gY\varepsilon(g)^{-1}\big)\, f_\tau(h^{-1}Zh) \big| \det\big(gY\varepsilon(g)^{-1}\big) \big|_E^s \\
&\cdot \xi_L\big(gzY\varepsilon(g)^{-1}\big)\, \xi_{L'}(ga_zX_1h_0h)\, d(X_1h_0)\, dh\, dg\, d^\times z \\
= &\sum_{F^\times/NE^\times} \omega'(\alpha) \int_{NE^\times} |z|_e^{2ns} \\
&\cdot \int_{G'/G'_{\varepsilon,Y}} \int_{G_Z\backslash G} \int_{G'_{\varepsilon,Y}} \psi_{\tau'}\big(gY\varepsilon(g)^{-1}\big)\, f_\tau(h^{-1}Zh) |\det|gY\varepsilon(g)^{-1}|_E^s \\
&\cdot \xi_L\big(zgY\varepsilon(g)^{-1}\big)\, \xi_{L'}(zgg_0X_1h)\, dg_0\, dh\, dg\, d^\times z.
\end{aligned} \tag{3.8}$$

Note that expression (3.8) has the same form as the expression for $\tilde{\psi}(s,Z)$ when $n > m$, with the roles of $G'_{\varepsilon,Y}$ and G_Z as well as those of g_0 and h_0 exchanged. We again define $\psi_{\mathcal{A}}(s,\gamma) = \sum_{Y\in\mathcal{A}(\{\gamma\})} \psi(s,\alpha\gamma)$. We also define $\varphi_{\mathcal{A}}$ as before. The integration can again be transferred to integration over $\cup_i T_i$, where $\{\mathbf{T}_i\}$ is a collection of representatives for the conjugacy classes of Cartan subgroups of $\mathbf{G}(n)$. We then argue as in the case $n \geq m$. Note that, since $n < m$, if $\{\gamma\}$ is elliptic in $\mathcal{C}^\vee$, then $\Phi(\gamma, f_\tau) = 0$. Therefore, $R_{\mathbf{G}}(f_\tau, \psi_{\tau'}) = 0$. We have now proved the following result.

Theorem 3.10 *Let $\ell = \min(n,m)$, and denote by $\{\mathbf{T}_i\}$ a collection of representatives for the conjugacy classes of Cartan subgroups of $\mathbf{G}(\ell)$. For each i choose a compact subset ω_i of the set of regular elements T_i' of T_i. Then the intertwining operator $A(s,\tau'\otimes\tau,w_0)$ has a pole at $s = 0$ if and only if*

$$cR_{\mathbf{G}}(f_\tau,\psi_{\tau'}) + \operatorname*{Res}_{s=0} \sum_i |W(\mathbf{T}_i)|^{-1} \int_{T_i'\backslash\omega_i} \varphi_{\mathcal{A}}(s,\gamma)|D(\gamma)|\, d\gamma \neq 0,$$

for some choice of matrix coefficients $\psi_{\tau'}$ and f_τ of τ' and τ, respectively. Here $c = (4n\log q_E)^{-1}$. If $n < m$, then $R_{\mathbf{G}}(f_\tau,\psi_{\tau'}) \equiv 0$, and thus the residue is determined by the second term above. ■

Corollary 3.11 *Suppose that $\tau' \simeq (\tau')^{\varepsilon}$. Fix a choice of compact subsets ω_i of T_i'.*

(a) The induced representation $I(\tau' \otimes \tau)$ is irreducible if and only if, for some choice of data,

$$cR_G(f_\tau, \psi_{\tau'}) + \sum_i |W(\mathbf{T}_i)|^{-1} \operatorname*{Res}_{s=0} \int_{T_i' \setminus \omega_i} \psi_{\mathcal{A}}(s, \gamma) |D(\gamma)| \, d\gamma \neq 0.$$

(b) Assume τ is generic. If $I(\tau' \otimes \tau)$ is irreducible, then $I(s, \tau' \otimes \tau)$, for $s \in \mathbb{R}$, is reducible exactly at $s = \pm 1/2$ or at $s = \pm 1$, and only at one of these two pairs.

Remark We take this opportunity to correct some typographical errors in Section 4 of [8], which is the section corresponding to the current one. On pages 284–285, the three sums that appear are all missing factors $\omega'(\alpha)$.

4 Connection With Twisted Endoscopy

We now describe the connection between the results of Section 3 and the theory of twisted endoscopy and L-functions. Our work here is similar to Section 5 of [8].

Let χ_τ be the distribution character of τ, and also denote by χ_τ the locally integrable function, [9], supported on the regular set of G satisfying

$$\chi_\tau(f) = \int_{Z \setminus G} f(x) \chi_\tau(x) \, dx.$$

From [5, 11] we can choose a pseudocoefficient, f_τ, for τ, *i.e.*, a matrix coefficient satisfying

$$\Phi(\gamma, f_\tau) = \begin{cases} \chi_\tau(\gamma) & \text{for all elliptic regular semisimple } \gamma \in G, \\ 0 & \text{for all non-elliptic regular semisimple } \gamma \in G. \end{cases}$$

We are assuming that $\tau' \simeq (\tau')^{\varepsilon}$. Therefore, we choose an equivalence $\tau'(\varepsilon)$ between τ' and $(\tau')^{\varepsilon}$. Then for locally constant functions ψ which transform according to $(\omega')^{-1}$, the ε-twisted character of τ' is defined [4] by

$$\chi^{\varepsilon}_{\tau'}(\psi) = \operatorname{trace}\left(\tau'(f)\tau'(\varepsilon)\right).$$

Furthermore, [4] there is a locally integrable function, also denoted by $\chi^{\varepsilon}_{\tau'}$ so that

$$\chi^{\varepsilon}_{\tau'}(\psi) = \int_{Z' \setminus G'} \psi(x) \chi^{\varepsilon}_{\tau'}(x) \, dx.$$

An ε-twisted pseudocoefficient ψ for τ', is a matrix coefficient satisfying

$$\Phi_\varepsilon(\gamma', \psi) = \begin{cases} \chi^{\varepsilon}_{\tau'}(\gamma') & \text{for all elliptic } \varepsilon\text{-regular } \varepsilon\text{-semisimple elements } \gamma' \in G', \\ 0 & \text{for all non-elliptic } \varepsilon\text{-regular } \varepsilon\text{-semisimple } \gamma' \in G'. \end{cases}$$

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In general, the existence of twisted pseudocoefficients is yet to be determined. One expects such functions to exist and we shall assume their existence for all ε-invariant irreducible supercuspidal representations of G'. We choose ψ'_τ to be such a function. Then

$$\Phi_\varepsilon\big(\mathcal{A}(\{\gamma\}), \psi_{\tau'}\big) = \sum_{\gamma' \in \mathcal{A}(\{\gamma\})} \Delta(\gamma, \gamma')\chi^\varepsilon_{\tau'}(\gamma'),$$

which we define to be $\chi^\varepsilon_{\tau'}\big(\mathcal{A}(\{\gamma\})\big)$.

Suppose $n = m$. Then we suppose f_τ and $\psi_{\tau'}$ are as above. Then

$$R_{\mathbf{G}}(f_\tau, \psi_{\tau'}) = \sum_{\mathbf{T}_i} \mu(\mathbf{T}_i)|W(\mathbf{T}_i)|^{-1} \int_{T'_i} \chi_\tau(\gamma)\chi^\varepsilon_{\tau'}\big(\mathcal{A}(\{\gamma\})\big)\,|D(\gamma)|\,d\gamma,$$

where now the sum is over representatives for the conjugacy classes of elliptic Cartan subgroups of $\mathbf{G}(n)$. Therefore, the operator $R_{\mathbf{G}}$ defines an elliptic pairing between the character χ_τ and the ε-twisted character $\chi^\varepsilon_{\tau'}$. Thus, one expects the non-vanishing of $R_{\mathbf{G}}(f_\tau, \psi_{\tau'})$ must be related to τ' coming from τ via twisted endoscopy [2, 3, 12]. For our purposes, we make the following definition, expecting it to agree with those referred to above.

Definition 4.1 A supercuspidal representation, τ' of $\mathrm{GL}_n(E)$ which satisfies $\tau' \simeq (\tau')^\varepsilon$ is said to be the *ε-twisted endoscopic transfer* of a discrete series representation τ of $G(n)$ if $R_{\mathbf{G}}(f_\tau, \psi_{\tau'}) \neq 0$ for some matrix coefficients f_τ of τ and $\psi_{\tau'}$ of τ'.

Now assume $n \geq m$ and suppose τ and τ' are both supercuspidal. We expect the Rankin-Selberg product L-function $L(s, \tau' \times \tau)$, formally defined in [14], satisfies the following defining condition:

$L(s, \tau' \times \tau)$ has a pole at $s = 0$ if and only if $R_{\mathbf{G}}(f_\tau, \psi_{\tau'}) \neq 0$ for some choice of f_τ and $\psi_{\tau'}$, or equivalently, if and only if τ' comes from τ via twisted endoscopic transfer.

We will now discuss why this seems to agree with the definitions given in [7]. Since we now wish to discuss the contributions from singular orbits, we continue with the assumption that $n \geq m$. Let $\omega_i \subset T'_i$ be a compact subset. As we have seen in Section 3, the residues of the intertwining operator will be independent of this choice of ω_i, and we therefore fix our choice for the rest of this discussion. We set

$$R_{\mathrm{sing}}(f_\tau, \psi_{\tau'}) = \sum_i |W(\mathbf{T}_i)|^{-1} \operatorname*{Res}_{s=0} \int_{T'_i \setminus \omega_i} \psi_{\mathcal{A}}(s, \gamma)|D(\gamma)|\,d\gamma.$$

So,

$$\operatorname*{Res}_{s=0} A(s, \tau' \times \tau, w_0)(h) = cR_{\mathbf{G}}(f_\tau, \psi_{\tau'}) + R_{\mathrm{sing}}(f_\tau, \psi_{\tau'}), \tag{4.1}$$

with $c = (4n \log q_F)^{-1}$. Note that (4.1) suppresses the dependence of the function $h \in V(s, \tau' \times \tau)$ on L and L'.

We wish to distinguish the poles of $L(s, \tau' \times \tau)$ from those of the Asai L-function $L(s, \tau', \Psi_n)$. Recall that Ψ_n is the representation r_2 of ${}^LM$ on ${}^L\mathfrak{n}$ discussed in Section 1. By Lemma 3.7 we know that $R_{\mathrm{sing}}(f_\tau, \psi_{\tau'}) = 0$ if and only if the process of taking

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residues can be interchanged with the integration at hand. On the other hand, the theory of L-functions says that there will be poles of $A(s, \tau' \otimes \tau, w_0)$ which come from those of $L(s, \tau', \Psi_n)$. Such poles depend only on τ', and therefore cannot depend on the non-vanishing of $R_{\mathbf{G}}(f_\tau, \psi_{\tau'})$, and thus we expect $R_{\text{sing}} \not\equiv 0$. Therefore, the ability to interchange residues and integrals in this setting reflects some deep arithmetic content.

Let us suppose that τ' comes from $U(n)$ via standard ε-twisted endoscopic transfer if n is odd, and by κ-ε-twisted endoscopic transfer if n is even (see [7] for the precise definition). This is equivalent to assuming that $L(s, \tau', \Psi_n)$ has a pole at $s = 0$. By the simplicity of the pole of the standard intertwining operators, we know that $L(s, \tau' \times \tau)$ must be holomorphic at $s = 0$. Then we expect that $R_{\mathbf{G}}(f_\tau, \psi_{\tau'}) = 0$ for any choice of f_τ and $\psi_{\tau'}$. By the theory of L-functions [14] we have

$$L(s, \tau' \times \tau)^{-1} L(2s, \tau', \Psi_n)^{-1} A(s, \tau' \otimes \tau, w_0)$$

is non-zero and holomorphic. Therefore, $R_{\text{sing}}(f_\tau, \psi_{\tau'}) \neq 0$ for some choice of f_τ, $\psi_{\tau'}$, L, and L'.

Proposition 4.2 *Suppose $n \geq m$. Assume that τ' comes from $G(n)$ via standard ε-twisted endoscopic transfer if n is odd, and comes from $G(n)$ via κ-ε-twisted endoscopic transfer if n is even. Then $R_{\text{sing}} \not\equiv 0$ for any unitary supercuspidal representation τ of $G(m)$ from which τ' does not come via ε-twisted endoscopy.*

For $n < m$, we know that $R_{\mathbf{G}} \equiv 0$, so the residue is R_{sing}. Thus, the poles of both L-functions are determined, in some manner, by the non-vanishing of R_{sing}. To make sense of the poles of these L-functions in terms of twisted endoscopic transfer, one must further analyze the term R_{sing}. Note, however, that the poles of $L(s, \tau', \Psi_n)$ are completely known, and therefore, one must try to separate these out from other poles of R_{sing}.

We conclude by stating the result in general context. That is we allow m and n to be any positive integers with the same parity and assume that $\tau' \simeq (\tau')^\varepsilon$.

Proposition 4.3 *(a) Suppose that $L(s, \tau', \Psi_n)$ has a pole at $s = 0$, i.e., τ' comes from $U(n)$ via standard ε-twisted endoscopic transfer if n is odd and via κ-ε-twisted endoscopic transfer if n is even. Then $L(s, \tau' \times \tau)$ is holomorphic at $s = 0$.*

(b) If τ' is not as in (a), then $L(s, \tau' \times \tau)$ has a pole at $s = 0$ if and only if $cR_{\mathbf{G}}(f_\tau, \psi_{\tau'}) + R_{\text{sing}} \not\equiv 0$.

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