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TOWARDS NUMERICALLY ESTIMATING HAUSDORFF DIMENSIONS

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Abstract

This paper gives a numerical method for estimating the Hausdorff–Besicovitch dimension where this differs from the fractal (or capacity or box-counting) dimension. The method has been implemented, and numerical results obtained for the set $\{1/n \mid n \in \mathbb{N}\}$ and the Cantor set. Comments about the practical use of the estimation algorithms are made.

1. Introduction

The Hausdorff–Besicovitch dimension is one of a number of "dimensions" that can be ascribed to closed bounded sets in $\mathbb{R}^n$ [1]. The other main dimension is known as either the fractal, or box-counting dimension, or the capacity of the set [7]. While for certain self-similar sets these quantities are equal [5], they are not identical in general.

The Hausdorff–Besicovitch dimension (or HB dimension) is defined in terms of a d-dimensional Hausdorff measure defined by

$$\mathscr{H}_d(A) = \lim_{\delta \downarrow 0} \inf_{\{U_j\}} \sum_j (\operatorname{diam} U_j)^d$$

where the infimum is taken over all countable covers $\{U_j\}$ of A with diameter $U_j \leq \delta$. Note that the above limit exists since the infimum taken as a function of δ is non-decreasing as δ decreases. It is easily shown (see [1]) that $\mathscr{H}_d(A)$, considered as a function of d, is non-increasing. The Hausdorff–Besicovitch dimension is given by

$$\dim_H A = \inf\{d \mid \mathscr{H}_d(A) = 0\} = \sup\{d \mid \mathscr{H}_d(A) = +\infty\}.$$

Note that the infimum and the supremum have the same value.

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The lower-fractal dimension is given by similar formulae to those for the Hausdorff–Besicovitch dimension. There is a "fractal measure"

$$\mathscr{F}_d(A) = \limsup_{\delta\downarrow 0} \inf_{\{U_j\}} \sum_j (\operatorname{diam} U_j)^d$$

where the infimum is taken over covers where $\operatorname{diam} U_j = \delta$ for each U_j in the cover, instead of over covers where $\operatorname{diam} U_j \le \delta$ as in the Hausdorff–Besicovitch dimension. Then the fractal dimension is given by

$$\dim_F A = \inf\{d \mid \mathscr{F}_d(A) = 0\} = \sup\{d \mid \mathscr{F}_d(A) = +\infty\}.$$

This is equivalent to the definition

$$\dim_F A = \limsup_{\delta\downarrow 0} \frac{\log N(\delta)}{\log(1/\delta)}$$

where $N(\delta)$ is the minimum number of sets of diameter δ (or smaller) required to cover A.

The lower-fractal dimension is given by

$$\dim_{LF} A = \liminf_{\delta\downarrow 0} \frac{\log N(\delta)}{\log(1/\delta)}.$$

Numerical methods for estimating the upper- and lower-fractal dimensions are well known and are commonly based on "box-counting" methods where the covering(s) are taken to be hyper-cubes organised on a regular grid in $\mathbb{R}^n$ [9]. As long as $\dim_F A = \dim_{LF} A$ it can be shown that box-counting methods do indeed give estimates that converge to $\dim_F A$ as the fineness of the grid goes to zero.

For any closed bounded set $A \subset \mathbb{R}^n$ the following inequalities hold (see [10, p. 279]):

$$\dim_H A \le \dim_{LF} A \le \dim_F A.$$

However, these quantities are often different. For example, consider

$$B_\alpha = \{0\} \cup \{1/n^\alpha \mid n \in \mathbb{N}\} \subset [0, 1].$$

The set has lower-fractal dimension $1/(1+\alpha)$ [10], but has Hausdorff–Besicovitch dimension zero.

The problem addressed here is to give numerical methods for estimating the Hausdorff–Besicovitch dimension *where it is different to the fractal dimension*. Recent work has been directed at improving algorithms for estimating the fractal dimension, or related quantities such as the information dimension. See, for example, Hunt and Sullivan [4], Hunt [3], and Hall and Wood [2]. Direct estimation of the Hausdorff–Besicovitch dimension by numerical methods without knowledge of the nature of the generation process has not been described in the literature, either in the work of Hunt or elsewhere.

1.1. Form of input to algorithms The model of computation used here is that some source (typically a simulation) will generate output as a sequence of points $\mathbf{x}_0$, $\mathbf{x}_1, \ldots, \mathbf{x}_N, \ldots$ where $\mathbf{x}_k \in \mathbb{R}^n$. The set that is described by this sequence is

$$A = \overline{\{\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \ldots\}}.$$

The sets for which we will seek to estimate dimensions are thus always closed. It will also be assumed that they are bounded. Note that there is no point in considering the problem of estimating the Hausdorff–Besicovitch dimension of $\{\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \ldots\}$ itself as the Hausdorff–Besicovitch dimension of a countable set is always zero.

Note that at any stage of the computation, only a finite number of points could have been used. Both the Hausdorff–Besicovitch and fractal dimensions of such a finite set are zero. This means that we need to relate the amount of data that has been "seen" to the resolution used in the algorithms. The problem of estimating how much data is needed for obtaining good results has been studied by others, most notably by F. Hunt [3] who takes a statistical approach to this problem. To obtain the results in [3], however, one needs to make the assumption that the "samples" $\mathbf{x}_k$ are taken more or less independently according to a probability distribution on the set. While these results are most important, this assumption about the source of the samples is not made. This avoids spurious questions about what probability distribution to impose on B_α. On the other hand, no estimate of the number of samples needed for a precise estimate of the Hausdorff–Besicovitch dimension is attempted here, except to say that it is dependent on the set and the way it is generated, in as yet unspecified ways.

2. Box-counting for the Hausdorff–Besicovitch dimension

Box-counting approaches to dimension operate in terms of grids or nested sequences of grids in $\mathbb{R}^n$ and use rectangular covers based on these grids. First, assume that the set A lies in $[0, 1]^n$. Then consider a sequence of grids in each co-ordinate where the k'th grid consists of points $i/2^k$ for $i = 0, 1, \ldots, 2^k$. Associated with the k'th grid is a family of hypercubes

$$\begin{aligned} R_k(i_1, i_2, \ldots, i_n) = [2^{-k}i_1, 2^{-k}(i_1+1)] \times [2^{-k}i_2, 2^{-k}(i_2+1)] \times \cdots \\ \times [2^{-k}i_n, 2^{-k}(i_n+1)] \end{aligned}$$

where $0 \le i_p \le 2^k$ for $p = 1, \ldots, n$. The index vector $(i_1, \ldots, i_n)$ will be denoted by $\mathbf{i}$. To simplify later calculations, we use the l^∞ metric in $\mathbb{R}^n$ given by

$$d(\mathbf{x}, \mathbf{y}) = \max_i |x_i - y_i| = \|\mathbf{x} - \mathbf{y}\|_\infty.$$

This metric is equivalent to the Euclidean metric with

$$\left(1/\sqrt{n}\right)\|\mathbf{x}-\mathbf{y}\|_2 \le \|\mathbf{x}-\mathbf{y}\|_\infty \le \|\mathbf{x}-\mathbf{y}\|_2.$$

Equivalent metrics give equivalent Hausdorff–Besicovitch and fractal dimensions as if d_1 and d_2 are two metrics where $(1/c)d_1(\mathbf{x},\mathbf{y}) \le d_2(\mathbf{x},\mathbf{y}) \le cd_1(\mathbf{x},\mathbf{y})$, then the corresponding Hausdorff measures satisfy

$$(1/c^d)\mathscr{H}_{1,d}(A) \le \mathscr{H}_{2,d}(A) \le c^d\mathscr{H}_{1,d}(A).$$

Similar results hold for the lower- and upper-fractal dimensions.

In the l^∞ norm on $\mathbb{R}^n$, the rectangular sets all have diameter 2^{-k}. If we have an arbitrary set $E \subseteq [0,1]^n$ of diameter $\le 2^{-k}$ then consider the set of rectangles $R_k(\mathbf{i})$ which intersect E. There are no more than 2^n of such rectangles that E can intersect: if E intersects $R_k(\mathbf{i})$ and $R_k(\mathbf{j})$ then $|j_p - i_p| \le 1$ for all p. Of course E must intersect at least one such rectangle if $E \ne \emptyset$. From this it can be shown (after considerable work) that the restriction to covers consisting of gridded rectangular sets gives identical fractal dimensions. A proof of this can be developed following that of Theorem 1.

Given $A \subseteq [0,1]^n$ and k, let

$$A_k = \bigcup \{R_k(\mathbf{i}) \mid R_k(\mathbf{i}) \cap A \ne \emptyset\}.$$

Clearly $A \subset A_k$, and for A closed, $A = \bigcap_k A_k$. Note that as A_k is a finite union of rectangles, there is only a finite amount of data needed to reconstruct A_k. The number of data points needed for this depends not only on k and A, but also on the method of generating the $\mathbf{x}_i$'s. This upper bound on the amount of data needed is, however, extremely conservative; it is not necessary to reproduce A_k in its entirety to obtain good estimates of the dimension.

For estimating the Hausdorff–Besicovitch dimension, information about covers with rectangles from different levels is needed. Since the resolution used should be limited for finite data, we need to have a maximum limit on k, which is denoted $k_{\max}$. Since the Hausdorff–Besicovitch dimension is defined in terms of coverings by sets whose diameter is bounded above, the discrete coverings must be constructed from rectangles with $k \ge k_{\min}$; instead of taking $\delta \downarrow 0$, we take $k_{\min} \to \infty$. Clearly, we also need to take $k_{\max} \to \infty$, and increase the amount of data, correspondingly.

2.1. Numerically estimating $\mathscr{H}_d(A)$ The next task is to consider the problem of obtaining estimates of the Hausdorff measure of a set $A \subseteq [0,1]$ using these nested rectangular covers. Recall that

$$\mathscr{H}_d(A) = \limsup_{\delta\downarrow 0} \inf_{\{U_j\}} (\operatorname{diam} U_j)^d.$$

We now replace the infimum by an infimum over covers of gridded rectangular sets $R_k(\mathbf{i})$ where $k_{\min} \le k \le k_{\max}$. This infimum can be computed efficiently using a dynamic programming algorithm by virtue of the hierarchical nature of these gridded sets.

For efficient implementations, 2^n-ary trees should be used. (In one, two and three dimensions these are known as binary, quad-, and oct-trees respectively.) The root of these trees is on level zero and corresponds to the rectangle $[0, 1)^n$; at level k each node corresponds to $R_k(\mathbf{i})$ for some $\mathbf{i}$ and contains pointers to rectangles on level $k+1$. Each rectangle that is "hit" by a data point $\mathbf{x}_i$ is "marked". (In efficient implementations, "marked" means "created". This approach and a fast $O(N)$ algorithm based on multiway trees for estimating the fractal dimension is described by Hunt and Sullivan [4], who ascribe it to a private communication of F. Varosi. This was rediscovered more recently by Molteno [8].) Once the sequence $(\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_N)$ has been incorporated into the tree, the estimates of $\mathscr{H}_d(A)$ are computed for various d. This is done by the following algorithm. Note that the "level" of a node in the tree is the distance in arcs from the root of the tree.

function *H_measure*(d, *node*, $k_{\min}$, $k_{\max}$)
begin
  $k \leftarrow$ level of *node*
  if $k = k_{\max}$ **then**
    return 2^{-kd}
  measure $\leftarrow \sum_{s:s \text{ child of } node}$ *H_measure*(d, s, $k_{\min}$, $k_{\max}$)
  if $k \ge k_{\min}$ **then**
    if $2^{-kd} <$ *measure* **then**
      measure $\leftarrow 2^{-kd}$
  return *measure*
end

FIGURE 1. Algorithm for estimating $\mathscr{H}_d(A)$.

This algorithm is, in fact, a dynamic programming algorithm to find the optimal covering amongst covers consisting of the $R_k(\mathbf{i})$ sets.

Consider what happens as $k_{\min}$ increases. Since for increasing $k_{\min}$ the optimisation is done over a smaller family of covers, the Hausdorff measure estimate must increase. The compensatory action is to increase $k_{\max}$, which increases the family of covers over which the optimisation is performed and therefore reduces the Hausdorff measure estimate. If we set $k_{\min} = k_{\max}$ and take this to infinity we just get the fractal dimension. If we set $k_{\max} = k_{\min} + l$ and we take $k_{\max}$ to infinity the lower bound on the resulting measure is 2^{-ld} times the fractal "measure" of A. It is therefore necessary to consider schemes where $k_{\max} - k_{\min}$ becomes arbitrarily large.

2.2. Convergence results The main result of this section is that the Hausdorff measure can be estimated to within a constant factor by the above algorithm.

THEOREM 1. *Assume that A is compact, and $R_k(\mathbf{i})$ is "marked" whenever $A \cap R_k(\mathbf{i}) \neq \emptyset$. If $d > 0$ and $\mathscr{H}_d(A) < +\infty$, then for any $\epsilon > 0$ there is a function $k_{\max} = k_{\max}(k_{\min}, A, n, d)$ and positive constants c_1 and c_2 where for sufficiently large $k_{\min}$,*

$$c_1 \mathscr{H}_d(A) < H\_measure(root, d, k_{\min}, k_{\max}) < c_2(\mathscr{H}_d(A) + \epsilon).$$

Further, under the same conditions, except that $\mathscr{H}_d(A) = +\infty$, then

$$\lim_{k_{\min} \to \infty} H\_measure(root, d, k_{\min}, k_{\max}) = +\infty.$$

Some points should be made: The hypothesis that A is compact (that is, closed and bounded in $\mathbb{R}^n$) cannot be significantly weakened. Regarding boundedness, without this assumption there is no guarantee that any of the approximate calculations would converge to a finite value. Regarding the assumption that A is closed, if $A = [0, 1] \cap \mathbb{Q}$ then $A_k = [0, 1]$ for all k and the algorithm has no way of distinguishing between A (which is countable and therefore has Hausdorff–Besicovitch dimension zero) and $[0, 1]$ which has dimension one.

It should also be noted that the theorem requires that "$k_{\max}$ is sufficiently large for given $k_{\min}$". It may be possible to weaken this assumption, but it is not clear how to at this stage; or it may be possible to weaken it for certain classes of sets.

Note that no "rate" of convergence is proven for this algorithm.

PROOF. We begin by assuming that $\mathscr{H}_d(A) < +\infty$. The case where $\mathscr{H}_d(A) = +\infty$ will be dealt with later.

Step 0. Without loss of generality, assume that the $\|\cdot\|_\infty$ norm is used to define the metric, and that the Hausdorff measure is defined in terms of this metric.

Step 1. Note that the algorithm in Figure 1 also generates, implicitly, a cover of the set A which will be denoted $\{W_j\}$. If $\delta = 2^{-k_{\min}}$ then

$$\inf_{\{U_j\},\ \operatorname{diam} U_j \leq \delta} \sum_j (\operatorname{diam} U_j)^d \leq \sum_j (\operatorname{diam} W_j)^d$$

since the infimum is taken over *all* covers of A with maximum diameter $\leq \delta = 2^{-k_{\min}}$.

Step 2. To obtain a reverse inequality let $\epsilon > 0$ be given, and consider

$$\mathscr{H}_d(A) = \lim_{\delta \downarrow 0} \inf_{\{U_j\},\ \operatorname{diam} U_j \leq \delta} \sum_j (\operatorname{diam} U_j)^d.$$

Choose $\delta > 0$ such that

$$\left| \mathscr{H}_d(A) - \inf_{\{U_j\},\ \operatorname{diam} U_j \le \delta} \sum_j (\operatorname{diam} U_j)^d \right| < \epsilon/3.$$

Suppose that $\{U_j'\}$ is a covering of A where diam $U_j' \le \delta$, then

$$\sum_j (\operatorname{diam} U_j')^d < \inf_{\{U_j\},\ \operatorname{diam} U_j \le \delta} \sum_j (\operatorname{diam} U_j)^d + \epsilon/3$$

for given $\epsilon > 0$. With an arbitrarily small increase in the sum this covering can be made an open covering (replacing U_j' with the set of points of distance less than η_j from U_j' for a sequence η_j sufficiently small and decreasing sufficiently fast). Since A is compact, this open covering has a finite sub-covering $\{U_1', \dots, U_N'\}$.

Choose $k_{\min}$ for the grids such that $k_{\min} \ge \log_2(1/\delta)$. We now find a gridded covering $\{V_1', \dots, V_M'\}$ where the diam $V_j' \le 2^{-k_{\min}}$ which gives a sum within a constant of that for the covering $\{U_1', \dots, U_N'\}$. For each U_j' choose $k = \lceil \log_2(1/\operatorname{diam} U_j') \rceil$, and find gridded rectangles of diameter 2^{-k} which intersect U_j'. Note that we need

$$k_{\max} \ge \max_{j=1,\dots,N} \left\lceil \log_2 \left(1/\operatorname{diam} U_j'\right) \right\rceil .$$

There are at most 2^n of these gridded rectangles. The diameter of these gridded rectangles is also at most twice that of U_j'. Thus

$$\sum_j (\operatorname{diam} V_j')^d \le 2^n \sum_j (2 \operatorname{diam} U_j')^d = 2^{n+d} \sum_j (\operatorname{diam} U_j')^d < 2^{n+d}(\mathscr{H}_d(A) + \epsilon).$$

Since the gridded covering $\{W_j\}$ defined through the above algorithm minimises this sum over all gridded coverings over the prescribed range of levels $k_{\min}$ to $k_{\max}$, we get the inequalities (using the above values of $k_{\min}$ and $k_{\max}$)

$$\inf_{\{U_j\},\ \operatorname{diam} U_j \le \delta} \sum_j (\operatorname{diam} U_j)^d \le \sum_j (\operatorname{diam} W_j)^d \le 2^{n+d}(\mathscr{H}_d(A) + \epsilon).$$

Taking $\delta > 0$ sufficiently small we obtain the desired result for $\mathscr{H}_d(A) < +\infty$.

Step 3. To show that *H_measure*$(root, d, k_{\min}, k_{\max})$ approaches $+\infty$ as $k_{\min} \to \infty$, the above proof can be modified by choosing, for any $M > 0$ a finite value δ where the $\inf_{\{U_j\},\ \operatorname{diam} U_j \le \delta} \sum_j (\operatorname{diam} U_j)^d \ge M$. Thus for $2^{-k_{\min}} \le \delta$, the gridded cover $\{W_j\}$ computed by the algorithm, $\sum_j (\operatorname{diam} W_j)^d \ge M$. Hence the computed approximation to the Hausdorff measure must go to infinity as $k_{\min} \to \infty$.

To use this result to estimate Hausdorff–Besicovitch dimensions we need to relate the Hausdorff measure to the corresponding dimension. The Hausdorff–Besicovitch dimension of A is the value d^* where $\mathscr{H}_{d^*-\epsilon}(A) = +\infty$ and $\mathscr{H}_{d^*+\epsilon}(A) = 0$ for any $\epsilon > 0$. Since the estimates from *H_measure* can be made to approach zero and infinity respectively for $d = d^* - \epsilon$ and $d = d^* + \epsilon$, it follows that it can be used to estimate the Hausdorff–Besicovitch dimension.

Note that the case $d = 0$ is a fairly simple special case to consider; then $\mathscr{H}_0(A)$ and $\mathscr{F}_0(A)$ are both just the counting measure (that is, $\mathscr{H}_0(A) = \mathscr{F}_0(A) = \#A$).

3. Practical procedures

The algorithm in Figure 1 has been implemented in 'C' [6] based on a sparse 2^n-ary tree implementation of a conventional box-counting estimator for the fractal dimension similar to the procedure described by Molteno [8]. The main test example is the set $B_1 = \{1/n \mid n = 1, 2, \dots\} \cup \{0\}$. This set is a compact subset of $[0, 1]$ and has fractal dimension $1/2$ but Hausdorff–Besicovitch dimension zero (being countable). For comparison with a set where the fractal and Hausdorff dimensions are the same, some tests were performed for the standard Cantor set as generated by the iterated function system $\{x \mapsto x/3,\ x \mapsto 1 - x/3\}$; both functions having probability $1/2$, and with initial value zero.

As always with these methods care must be taken to match the amount of data with the resolution (that is, $k_{\min}$ and $k_{\max}$) used for the estimation procedures.

Coupled with the Hausdorff–Besicovitch dimension estimator is a fractal dimension estimator, which uses a number of heuristics to achieve an accuracy of about 0.01 to 0.02 for most medium- to low-dimensional fractals. For $\hat{B}_1 = \{1/n \mid 1 \le n \le 10^5\} \cup \{0\}$ it gave an estimate of the fractal dimension of 0.4968, which has a relative error of about 0.6%.

Probably the simplest approach to estimating d^* is to find (given $k_{\min}$ and $k_{\max}$) the value $\hat{d}$ where *H_measure*$(root, \hat{d}, k_{\min}, k_{\max}) = 1$. With sufficient data and sufficiently large $k_{\min}$ and $k_{\max}$ (with $k_{\max}$ sufficiently large given $k_{\min}$) we can make $\hat{d}$ approach d^*. However, this suffers from a lack of accuracy and/or convergence speed that makes it less useful than the following approach. On the test example $\hat{B}_1$, using 10^5 points to approximate B_1, this technique gave an estimate of $\dim_H B_1$ of 0.3, compared with the true value of zero, and fractal dimension of 0.5. This could be used as an indication that the Hausdorff–Besicovitch dimension may be lower than the fractal dimension, but it is not a sharp indication of this.

A better way to use the Hausdorff measure estimation procedure in practice appears to be in a diagnostic fashion. Figure 2 was generated by computing the $\mathscr{H}_d(\hat{B}_1)$ estimates for different values of d and for different values of $k_{\max} \le 15$ given $k_{\min} = 5$. Note that since the dynamic programming algorithm gives the minimising gridded cover using levels $k_{\min} \le k \le k_{\max}$, keeping $k_{\min}$ constant and increasing $k_{\max}$ can never

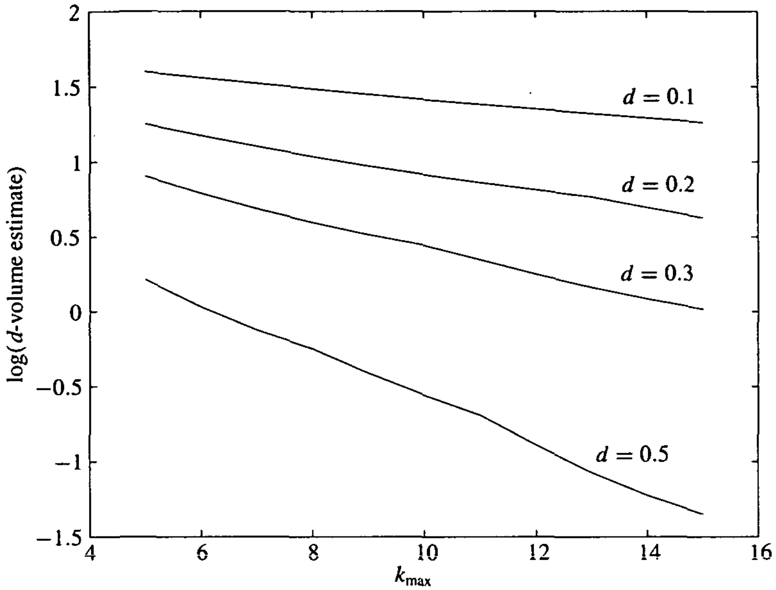

FIGURE 2. Plot of $\mathscr{H}_d(\hat{B}_1)$ estimates vs. $k_{\max}$.

increase the estimate of $\mathscr{H}_d(A)$. If $\mathscr{H}_d(A) > 0$, however, the plot of the estimates should flatten out, at least for sufficiently large values of $k_{\max}$. If $\mathscr{H}_d(A) = 0$, then there the estimates will approach zero as as $k_{\max}$ goes to infinity.

If d is the true Hausdorff–Besicovitch dimension then it would be expected that the volume vs. $k_{\max}$ graph would be fairly flat. In this case it appears that the estimates computed indicate a low Hausdorff–Besicovitch dimension, probably much less than $\approx 1/2$ as indicated by the fractal dimension estimator.

In constrast, the Cantor set has both Hausdorff and fractal dimensions equal to $\log 2/\log 3 \approx 0.6309$. This is using the standard construction where the middle third of the unit interval is deleted, then the middle third of the remaining intervals, and so on. This is a useful test, as the true Hausdorff–Besicovitch dimension is positive, and we need to understand the behaviour of these plots for $d < \dim_H C$, where C is the Cantor set.

The Cantor set was approximated by 10^5 points, generated by an iterated function system using the functions $\{x \mapsto x/3,\ x \mapsto 1 - x/3\}$ with each function chosen with probability $1/2$, and intial point zero.

The above comment that if $\mathscr{H}_d(A) > 0$, then the curves of the estimates against $k_{\max}$ flatten out, can be confirmed by the plots for the Cantor set in Figure 3. The curves in Figure 3 appear to indicate that the Hausdorff–Besicovitch dimension of the

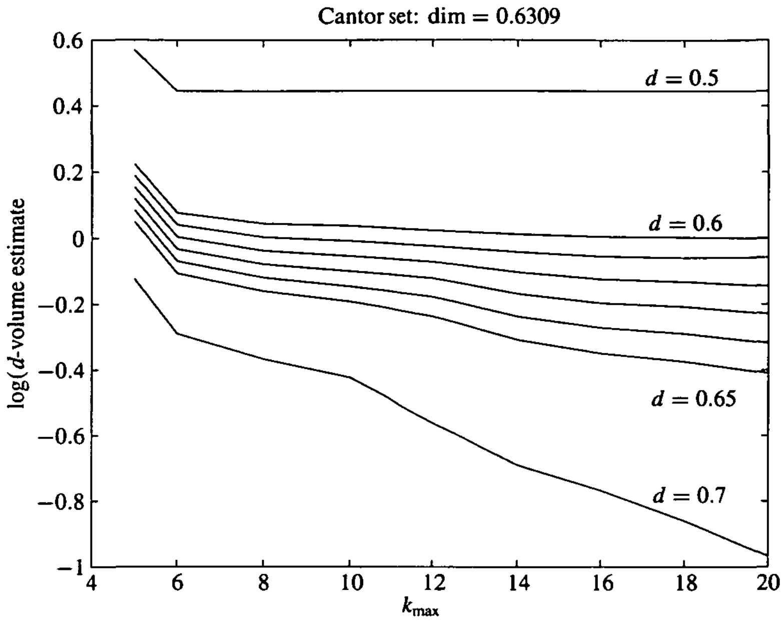

FIGURE 3. Plot of $\mathscr{H}_d(\hat{C})$ estimates vs. $k_{\max}$.

Cantor set lies between 0.60 and 0.65; the fractal dimension estimate is 0.6302. Note that the curves between those for $d = 0.6$ and $d = 0.65$ are for the values $d = 0.61$, 0.62, 0.63 and 0.64. These estimates are monotone in d as is expected. The range of 0.6 to 0.65 is still a little larger than would be preferred. However, this is still at most a range of 0.03 from the true Hausdorff–Besicovitch dimension, which is a 5% difference. More refined analysis of these plots using heuristics such as "No more than 10% drop over the range $k_{\max} = 10$ to $k_{\max} = 20$" would undoubtedly get much closer to the true value. Using such heuristics has not been pursued here as this is vulnerable to "fiddling the heuristic" to match the expected result. On the other hand, higher accuracy could be expected of more refined versions of this algorithm.

This seems to confirm the ability of the algorithm to act as a practical diagnostic tool for indicating if the Hausdorff–Besicovitch dimension and the fractal dimension are significantly different. It may even become a useful tool for estimating the Hausdorff–Besicovitch dimension directly.

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