UNIQUE CONTINUATION PROPERTY OF NON-POSITIVE WEAK SUBSOLUTIONS FOR PARABOLIC EQUATIONS OF HIGHER ORDER

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1. When L is a parabolic differential operator of second order, Nirenberg [6] proved the maximum principle for the function u which has second order continuous derivatives and satisfies $Lu \geqq 0$. Recently Friedman [2] has proved the maximum principle for the measurable function satisfying $Lu \geqq 0$ in the wide sense. This function is named a weakly L-subparabolic function. On the other hand, Littman [5] earlier than Friedman, has defined a weakly A-subharmonic function for an elliptic differential operator A of second order and has showed the maximum principle for it. When A is an elliptic operator of higher order, we defined a weak A-subsolution which coincides with a weakly A-subharmonic function if A is of second order and we proved in [4] that if a weak A-subsolution u assumes its essential supremum M over a domain $\mathfrak{D}$ almost everywhere in an open set in $\mathfrak{D}$ then $u = M$ almost everywhere in $\mathfrak{D}$. That is, unique continuation property holds for the non-positive weak A-subsolution. In this note, we define similarly a weak L-subsolution for a parabolic differential operator L of higher order and prove a unique continuation property for a non-positive weak L-subsolution under an assumption weaker than that of [4].

Up to now the following assumption was necessary for such a unique continuation property (see [1], [3] and [7]): Both u and its derivatives tend to 0 at the origin (in general, at a fixed point) faster than the finite power of the distance from the origin. However, in the present case, such an assumption of the local behavior at the origin is superfluous for derivatives of u, that is, it is sufficient to assume only for u.

2. Let $\mathfrak{D}$ be a domain in $(n+1)$-dimensional Euclidean space with coordinates $(x_1, \ldots, x_n, t)$. For simplicity, we assume that the origin O is in $\mathfrak{D}$. Denoting $(x_1, \ldots, x_n) = x$ and $(x_1, \ldots, x_n, t) = (x, t)$, we consider the follow-

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ing parabolic differential operator L,

$$L = A - \frac{\partial}{\partial t}, \quad A = \sum_{0<|\alpha|\leqq 2s} a_\alpha(x, t) D_x^\alpha,$$
$$\left(D_x^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} \right)$$

where A is an elliptic operator with coefficients $a_\alpha(x, t) \in C^{|\alpha|}(\mathfrak{D})$. If a measurable function u is essentially bounded above in $\mathfrak{D}$ and satisfies the inequality

$$\iint_{\mathfrak{D}} u(x, t) L^* v(x, t) dx dt \geqq 0$$

for all non-negative functions $v \in C^{2s}(\mathfrak{D})$ with compact support in $\mathfrak{D}$, where L^* is the adjoint operator of L, then we say that u is a weak L-subsolution in $\mathfrak{D}$. In the case when L is of second order, a weak L-subsolution is a weakly subparabolic function in the sense of Friedman [2]. We denote by $S_{\mathfrak{D}}(O)$ the set of points $(x, t) (\in \mathfrak{D})$ which can be joined with the origin O by a curve Λ in $\mathfrak{D}$ such that the t-coordinate of a point on Λ is not decreasing when the point varies from the point (x, t) to the origin along Λ. We shall prove the following

THEOREM. *If u is a weak L-subsolution and non-positive in $\mathfrak{D}$, then there exists a number N depending only on L such that u vanishes identically in $S_{\mathfrak{D}}(O)$ provided that $u/(r^{2s} + t^2)^N (r^2 = x_1^2 + \cdots + x_n^2)$ is essentially bounded in $\mathfrak{D}$.*

From this theorem follows immediately

COROLLARY. *Let u be a weak L-subsolution in $\mathfrak{D}$ and essentially bounded below in $\mathfrak{D}$. If u assumes its essential supremum M (over $S_{\mathfrak{D}}(O)$) in the intersection of the half-space $t<0$ and a neighborhood of the origin, then $u = M$ almost everywhere in $S_{\mathfrak{D}}(O)$.*

3. We denote be Ω_δ the domain surrounded by the hyperplane $t = -\delta$ $(\delta > 0)$ and the hypersurface $-r^{2s} - t = 0$, where δ is taken sufficiently small so that the closure of Ω_δ is contained in $\mathfrak{D}$. Consider the function

$$G(x, t) = \begin{cases} (t+\delta)^2 (r^{2s} - kt)^{-k_1} (-t - r^{2s})^{2s+1}, & \text{if } (x, t) \in \Omega_\delta, \\ 0 & \text{if } (x, t) \notin \Omega_\delta, \end{cases}$$

where k, $k_1 > 0$.

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Clearly $D_x^\alpha G$ ($|\alpha|\leqq 2s$) and $\partial G/\partial t$ are continuous except at the origin and they have compact supports in $\mathfrak{D}$. In addition we easily see

(1) $$|(D_t^l D_x^\alpha G)(r^{2s}-kt)^{k_1+l+|\alpha|}|<\infty \qquad (l\leqq 1,\ |\alpha|\leqq 2s).$$

We can prove the following lemma.

LEMMA. *If δ is sufficiently small and if k, k_1, and k_1/k are sufficiently large, then it holds*

$$L^*G(x,\ t)>0 \quad \textit{in } \Omega_\delta.$$

We shall give the proof of this in the next section and here, using this lemma, we prove our theorem. Let $\phi(\alpha)$ be an infinitely differentiable function such that

$$\phi(\alpha)\geqq 0, \quad \text{if } 0\leqq\alpha\leqq 1,$$
$$\phi(\alpha)=0, \quad \text{if } \alpha\geqq 1 \text{ or } \alpha\leqq 0,$$
$$\int_{-\infty}^{\infty}\phi(\alpha)d\alpha=1$$

and

$$\phi(\alpha)=O(e^{-1/\alpha}) \qquad (\alpha\to 0).$$

If we put $\phi_\varepsilon(\alpha)=\frac{1}{\varepsilon}\phi\left(\frac{\alpha}{\varepsilon}\right)$, it is obvious that $\int_{-\infty}^{\infty}\phi_\varepsilon(\alpha)d\alpha=1$. Setting $\rho=r^2$ and $\tilde{\rho}=\rho^s-kt$, we define

(2) $$G_\varepsilon(x,\ t)=G(x,\ t)\int_0^{\tilde{\rho}}\phi_\varepsilon(\alpha)d\alpha.$$

Then we see that $G_\varepsilon(x,\ t)=G(x,\ t)$ in $\tilde{\rho}\geqq\varepsilon$ and that $D_x^\alpha G_\varepsilon$ ($|\alpha|\leqq 2s$) and $\partial G_\varepsilon/\partial t$ are continuous in $\mathfrak{D}$. Hereafter, let the numbers δ, k and k_1 in $G(x,\ t)$ satisfy the condition of the lemma. By virtue of the form (2), it is clear that $L^*G_\varepsilon(x,\ t)$ is the sum of the terms $A_{\alpha,l,h}(x,\ t)\,D_x^\alpha D_t^l G\left[\frac{d^h}{d\alpha^h}\phi_\varepsilon(\alpha)\right]_{\alpha=\tilde{\rho}}$ ($|\alpha|+l\leqq 2s$, $-1\leqq h\leqq 2s-1$), where $A_{\alpha,l,h}(x,\ t)$ are bounded functions and we use the notation $\frac{d^{-1}}{d\alpha^{-1}}\phi_\varepsilon(\alpha)=\int_0^{\tilde{\rho}}\phi_\varepsilon(\alpha)d\alpha$. Noting that

$$\frac{d^h}{d\alpha^h}\phi_\varepsilon(\alpha)=\frac{1}{\varepsilon^{h+1}}\left[\frac{d^h}{d\beta^h}\phi(\beta)\right]_{\beta=\alpha/\varepsilon}$$

and $\frac{1}{\beta^{h+1}}\leqq\frac{1}{\tilde{\rho}^{2s}}$ for $\tilde{\rho}\leqq\varepsilon$, we get from (1)

$$\lim_{\varepsilon \to 0} \iint_{\tilde{\rho} \leqq \varepsilon} uL^* G_\varepsilon(x, t) dxdt = 0, \tag{3}$$

if we take N in our theorem such that $N = k_1 + 4s$. Since u is a weak L-sub-solution, it holds

$$\iint_{\tilde{\rho} \geqq \varepsilon} uL^* G dxdt = \iint_{\tilde{\rho} \geqq \varepsilon} uL^* G_\varepsilon dxdt = \iint_{\mathfrak{D}} uL^* G_\varepsilon dxdt$$
$$- \iint_{\tilde{\rho} \leqq \varepsilon} uL^* G_\varepsilon dxdt \geqq - \iint_{\tilde{\rho} \leqq \varepsilon} uL^* G_\varepsilon dxdt.$$

Let ε tend to 0 in this inequality. Then we have from (3)

$$\iint_{\mathfrak{D}} uL^* G dxdt \geqq 0.$$

Thus, our lemma implies that $u = 0$ in Ω_δ. By the standard method in proving unique continuation theorems (for example, see [2]), we arrive at the desired.

4. Now we prove our lemma. Let us put $f = (\rho^s - kt)^{-k_1}$ and $g = (-t - \rho^s)^{2s+1}$ Then, similarly as in [3], we have

$$D_x^\alpha f = \sum_{q=1}^{|\alpha|} \alpha_1! \cdots \alpha_n! \left\{ \frac{d^q}{d\rho^q} (\rho^s - kt)^{-k_1} \right\} \tag{4}$$
$$\times \Bigg(\sum_{\substack{|\beta + \gamma| = q \\ \beta + 2\gamma = \alpha}} \frac{2^{|\beta|}}{\beta_1! \cdots \beta_n! \, \gamma_1! \cdots \gamma_n!} x_1^{\beta_1} \cdots x_n^{\beta_n} \Bigg).$$

Further, setting $F(\tilde{\rho}) = \tilde{\rho}^{-k_1}$, we see

$$\frac{1}{q!} \frac{d^q}{d\rho^q} (\rho^s - kt)^{-k_1} \tag{5}$$
$$= \sum_{h=1}^{q} \sum_{(i)} \frac{1}{i_1! \cdots i_q!} \frac{d^h F}{d\tilde{\rho}^h} \left(\frac{s\rho^{s-1}}{1!} \right)^{i_1} \left(\frac{s(s-1)\rho^{s-2}}{2!} \right)^{i_2}$$
$$\cdots \left(\frac{s(s-1) \cdots (s-q+1)\rho^{s-q}}{q!} \right)^{i_q},$$

where (i) means an arbitrary real vector $(i_1, \ldots, i_q)$ such that $i_1 + \cdots + i_q = h$ and $i_1 + 2i_2 + \cdots + qi_q = q$. In addition, it is easily seen that

$$\frac{d^h F}{d\tilde{\rho}^h} = (-1)^h k_1 (k_1 + 1) \cdots (k_1 + h - 1) (\rho^s - kt)^{-k_1 - h}. \tag{6}$$

From (4), (5) and (6), we obtain

$$D_x^\alpha f = \alpha_1! \cdots \alpha_n! \sum_{q=1}^{|\alpha|} \Big(\sum_{h=1}^{q} A_{q,h} (-1)^h k_1 (k_1 + 1) \tag{7}$$
$$\cdots (k_1 + h - 1)(\rho^s - kt)^{-k_1 - h} \rho^{sh-q} \Big) P_{\alpha, q},$$

where $P_{\alpha,q}=\sum_{\substack{|\beta+\gamma|=q\\ \beta+2\gamma=\alpha}}\frac{2^{|\beta|}}{\beta_1!\cdots\beta_n!\,\gamma_1!\cdots\gamma_n!}x_1^{\beta_1}\cdots x_n^{\beta_n}$ and $A_{q,h}$ is a positive constant independent of k and k_1. Hereafter, if there is no explanation, we promise that the letters A, B, . . . and these with indices are all positive constants independent of k and k_1.

On $P_{\alpha,q}$, we see, in general,

(8) $$|P_{\alpha,q}|\leqq C_{\alpha,q}\rho^{q-(|\alpha|/2)},$$

and when $|\alpha|=q=2s$,

(9) $$P_{\alpha,q}=\frac{2^{2s}}{\alpha_1!\cdots\alpha_n!}x_1^{\alpha_1}\cdots x_n^{\alpha_n}.$$

If $q+q'=2s$, we have

(10) $$P_{\beta,q}P_{\alpha-\beta,q'}=\frac{2^{2s}}{\beta_1!\cdots\beta_n!(\alpha_1-\beta_1)!\cdots(\alpha_n-\beta_n)!}x_1^{\alpha_1}\cdots x_n^{\alpha_n}.$$

In the same manner as in the computation of $D_x^\alpha f$, we get

(11) $$D_x^\alpha g=\alpha_1!\cdots\alpha_n!\sum_{q=1}^{|\alpha|}\Big(\sum_{h=1}^{q}B_{q,h}(-1)^h(-t-\rho^s)^{2s+1-h}\rho^{sh-q}\Big)P_{\alpha,q}.$$

The Leibniz formula yields

(12) $$D_x^\alpha(f\cdot g)=\sum_\beta c_{\alpha,\beta}D_x^\beta f D_x^{\alpha-\beta}g,$$

where

(13) $$c_{\alpha,\beta}=\frac{1}{\beta_1!\cdots\beta_n!}\{\alpha_1(\alpha_1-1)\cdots(\alpha_1-\beta_1+1)\}\cdots\{\alpha_n(\alpha_n-1)\cdots(\alpha_n-\beta_n+1)\}.$$

Since the operator A is elliptic, there is a positive number c such that in Ω_δ

(14) $$\sum_{|\alpha|=2s}a_\alpha(x,t)x_1^{\alpha_1}\cdots x_n^{\alpha_n}\geqq c\rho^s.$$

We denote by L_1 the principal part of A^* which is identical with that of A and by L_2 the other part. First we estimate $L_1(f\cdot g)$ from below. Applying (12), we can write

$$L_1(f\cdot g)=\sum_{|\alpha|=2s}a_\alpha(x,t)D_x^\alpha f\cdot g+\sum_{|\alpha|=2s}a_\alpha(x,t)\Big(\sum_{\beta\neq\alpha}c_{\alpha,\beta}D_x^\beta f D_x^{\alpha-\beta}g\Big).$$

For the right hand side of this equality, let us denote by I_1 the first sum and by I_2 the second sum. Substituting (7) into I_1, we have

$$I_1=\sum_{|\alpha|=2s}a_\alpha(x,t)\alpha_1!\cdots\alpha_n!\sum_{q=1}^{2s}(\sum_{h=1}^{q}A_{q,h}(-1)^h k_1(k_1+1)$$
$$\cdots(k_1+h-1)(\rho^s-kt)^{-k_1-h}\rho^{sh-q})P_{\alpha,q}(-t-\rho^s)^{2s+1}.$$

We use (8), (9) and (14) to find

$$(15)\qquad I_1\geqq(\sum_{|\alpha|=2s}a_\alpha(x,t)2^{2s}x_1^{\alpha_1}\cdots x_n^{\alpha_n})(A_{2s,2s}k_1(k_1+1)$$
$$\cdots(k_1+2s-1)(\rho^s-kt)^{-k_1-2s}\rho^{s\cdot 2s-2s}(-t-\rho^s)^{2s+1}$$
$$+\sum_{|\alpha|=2s}a_\alpha(x,t)\alpha_1!\cdots\alpha_n!\sum_{q=1}^{2s}(\sum_{\substack{h=1,\dots,q\\ h\neq 2s}}A_{q,h}(-1)^h k_1\cdot$$
$$(k_1+1)\cdots(k_1+h-1)(\rho^s-kt)^{-k_1-h}\rho^{sh-q})P_{\alpha,q}(-t-\rho^s)^{2s+1}$$
$$\geqq Mk_1^{2s}(\rho^s-kt)^{-k_1-2s}\rho^{s(2s-1)}(-t-\rho^s)^{2s+1}$$
$$-\sum_{h=1}^{2s-1}M'k_1^h(\rho^s-kt)^{-k_1-h}\rho^{s(h-1)}(-t-\rho^s)^{2s+1}.$$

Next we estimate I_2. From (7) and (11), we can see

$$(16)\qquad I_2=\sum_{|\alpha|=2s}a_\alpha(x,t)(\sum_{\beta\neq\alpha}c_{\alpha,\beta}D_x^\beta fD_x^{\alpha-\beta}g)$$
$$=\sum_{|\alpha|=2s}a_\alpha(x,t)\sum_{\beta\neq\alpha,0}c_{\alpha,\beta}(\sum_{q=1}^{|\beta|}\sum_{q'=1}^{|\alpha-\beta|}\sum_{h=1}^{q}\sum_{h'=1}^{q'}$$
$$(-1)^{h+h'}A_{q,h}B_{q',h'}k_1(k_1+1)\cdots(k_1+h-1)(\rho^s-kt)^{-k_1-h}$$
$$\rho^{s(h+h')-(q+q')}(-t-\rho^s)^{2s+1-h'}\beta_1!\cdots\beta_n!\cdot$$
$$(\alpha_1-\beta_1)!\cdots(\alpha_n-\beta_n)!\,P_{\beta,q}P_{\alpha-\beta,q'})$$
$$+\sum_{|\alpha|=2s}a_\alpha(x,t)(\sum_{q=1}^{2s}\sum_{h=1}^{q}B_{q,h}(-1)^h(-t-\rho^s)^{2s+1-h}$$
$$\rho^{sh-q}\alpha_1!\cdots\alpha_n!P_{\alpha,q})(\rho^s-kt)^{-k_1}.$$

If $h+h'=2s$, then $h=q=|\beta|$ and $h'=q'=|\alpha-\beta|$. Hence, from (10), (13) and (14), we obtain

$$(17)\qquad \sum_{|\alpha|=2s}\sum_{|\beta|=h}a_\alpha(x,t)C_{\alpha,\beta}\beta_1!\cdots\beta_n!(\alpha_1-\beta_1)!\cdots(\alpha_n-\beta_n)!\,P_{\beta,q}\cdot P_{\alpha-\beta,q'}$$
$$\geqq\frac{2s(2s-1)\cdots(2s-h+1)}{h!}2^{2s}c\rho^s.$$

On the other hand, we see, in general,

$$(18)\qquad |P_{\beta,q}P_{\alpha-\beta,q'}|\leqq C_{\beta,q}C_{\alpha-\beta,q'}\rho^{q+q'-(|\alpha|/2)}.$$

In the right hand side of (16), we collect all the terms satisfying $h+h'=2s$ into a part and the remaining terms into the other part. Inequalities (17) and (18) give the following:

(19)
$$I_2 \geqq \sum_{h+h'=2s} Bk_1^h(\rho^s - kt)^{-k_1-h}(-t-\rho^s)^{2s+1-h'}\rho^{s(2s-1)} - \sum_{j+j'\neq 2s} B'k_1^j(\rho^s-kt)^{-k_1-j}(-t-\rho^s)^{2s+1-j'}\rho^{s(j+j'-1)},$$

where $0 \leqq h \leqq 2s$, $0 \leqq j \leqq 2s$, $1 \leqq h' \leqq 2s$ and $1 \leqq j' \leqq 2s$ respectively. It is easy to see that $\rho^s \leqq -kt$ in Ω_δ for $k \geqq 1$. Hence it holds in Ω_δ

(20)
$$\rho^s - kt \leqq 2k\rho^s + 2k(-t-\rho^s).$$

We represent each term of the right hand side of (19) by the next symbol

$$I(j, j') = K_{j,j'}k_1^j(\rho^s - kt)^{-k_1-j}(-t-\rho^s)^{2s+1-j'}\rho^{s(j+j'-1)},$$

where $K_{j,j'} = B$ for $j+j' = 2s$ and $K_{j,j'} = B'$ for $j+j' \neq 2s$. When $j+j' \neq 2s$, we apply this inequality (20) to $I(j, j')$ repeatedly. Then $I(j, j')$ is finally absorbed in $I(j_1, j_1')$ $(j_1 + j_1' = 2s)$ or $I(j_2, 0)$ $(j_2 \leqq 2s)$ provided that k/k_1 is sufficiently small. That is, the second sum in the right hand side of (19) is absorbed in the first sum or in the right hand side of (15). In the same manner as above, we have

$$|L_2(f\cdot g)| \leqq \sum_{j+j' \leqq |\alpha|,\ |\alpha|<2s} K_{\alpha,j,j'}k_1^j(\rho^s - kt)^{-k_1-j}(-t-\rho^s)^{2s+1-j'}\rho^{s(j+j')-|\alpha|}$$

from (7), (8), (11) and (12). Let us take δ sufficiently small. Then $\rho^{s(j+j')-|\alpha|} \div \rho^{s(j+j')-2s}$ is sufficiently small. Thus, from (15) and (19), we see that $L_2(f\cdot g)$ is absorbed in $L_1(f\cdot g)$. Computing $\frac{\partial}{\partial t}G(x, t)$, we have

$$\begin{aligned}\frac{\partial}{\partial t}G(x, t) &= 2(t+\delta)(\rho^s - kt)^{-k_1}(-t-\rho^s)^{2s+1} \\ &\quad + k_1k(t+\delta)^2(\rho^s - kt)^{-k_1-1}(-t-\rho^s)^{2s+1} \\ &\quad - (2s+1)(t+\delta)^2(\rho^s - kt)^{-k_1}(-t-\rho^s)^{2s}.\end{aligned}$$

If we apply (20) to the last term of the right hand side of this expression, this term is again absorbed in the second term and the right hand side of (19). Since $L^* = L_1 + L_2 + \frac{\partial}{\partial t}$, we finally obtain

$$\begin{aligned}L^*G \geqq \{&M_1k_1^{2s}(\rho^s - kt)^{-k_1-2s}\rho^{s(2s-1)}(-t-\rho^s)^{2s+1} \\ &- \sum_{h=1}^{2s-1} M_1'k_1^h(\rho^s - kt)^{-k_1-h}\rho^{s(h-1)}(-t-\rho^s)^{2s+1}\} \\ &(t+\delta)^2 + k_1k(t+\delta)^2(\rho^s - kt)^{-k_1-1}(-t-\rho^s)^{2s+1}\rho^{s(j+j')-|\alpha|}\end{aligned}$$

If we set $X = k_1\rho^s(\rho^s - kt)^{-1}$, the above inequality becomes

$$L^*G \geqq k_1(t+\delta)^2(\rho^s - kt)^{-k_1-1}(-t-\rho^s)^{2s+1}\{M_1X^{2s-1} - \sum_{h=1}^{2s-1} M_1' X^{h-1} + k\}.$$

This shows that $L^*G > 0$ in Ω_δ for sufficiently large k.

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