On a Class of Surfaces.

By Professor John Miller, D.Sc.

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§ 1. The surfaces here considered were first discussed by Monge as surfaces whose normals are tangents to given developables. Under the name general surfaces moulures, Darboux treats them as the surfaces traced out by a fixed curve on a plane which rolls on a developable. When the developable is a cylinder he gives the general coordinates in his Leçons sur la Théorie Générale des Surfaces (Volume I., page 105). This led me to take up the more general case, but I later found that Darboux had also considered this in an ingenious and elegant manner in his Leçons sur les Systèmes Orthogonaux et les Coordonnées Curvilignes (Tome 1, pages 26–34). Perhaps this quite different discussion will present some interesting points in analysis.

§ 2. The surface traced out by the curve being necessarily cut perpendicularly by the plane, and the normals to the surface intersecting, the curve is at the same time a geodesic and a line of principal curvature. Let us first find the fundamental forms of all the surfaces of which one system of the lines of curvature is geodesic.

If \( x(u, v), y(u, v), z(u, v) \) are the coordinates of a point of a surface, and if the curves \( u = \text{const.}, v = \text{const.} \) are the principal lines of curvature, the square of the element of length is given by

\[
ds^2 = Edu^2 + Gdv^2
\]

where \( E = \Sigma_{u}x_u^2, \quad G = \Sigma_{v}z_v^2.\)

The radius \( R \) of curvature of a normal section is given by

\[
\frac{ds^2}{R} = Ldu^2 + Ndv^2
\]

where

\[
L = \frac{1}{\sqrt{EG}} \begin{vmatrix} x_{uu} & x_u & x_v \\ y_{uu} & y_u & y_v \\ z_{uu} & z_u & z_v \end{vmatrix}, \quad N = \frac{1}{\sqrt{EG}} \begin{vmatrix} x_{vv} & x_v & x_u \\ y_{vv} & y_v & y_u \\ z_{vv} & z_v & z_u \end{vmatrix}.
\]
Between the quantities $E$, $G$, $L$, $N$ exist the Mainardi-Codazzi equations:—*

$$\frac{LN}{\sqrt{EG}} + \frac{\partial}{\partial u} \left( \frac{1}{\sqrt{E}} \frac{\partial \sqrt{G}}{\partial u} \right) + \frac{\partial}{\partial v} \left( \frac{1}{\sqrt{G}} \frac{\partial \sqrt{E}}{\partial v} \right) = 0;$$

$$\frac{\partial}{\partial v} \left( \frac{L}{\sqrt{E}} \right) = \frac{N}{G} \frac{\partial \sqrt{E}}{\partial v}; \quad \frac{\partial}{\partial u} \left( \frac{N}{\sqrt{G}} \right) = \frac{L}{E} \frac{\partial \sqrt{G}}{\partial u}.$$

If $E$ is a function of $u$ alone the curves $v = \text{const.}$ are geodesics. We may then, by a change of variable without loss of generality, take $E$ to be unity.

From the second of these equations $L$ is a function of $u$ alone. Let $N/\sqrt{G} = \psi$. Then from the first and the third equation

$$L\psi + \frac{\partial}{\partial u} \left( \frac{\psi}{L} \right) = 0$$

or

$$\psi_{uu} - \frac{L}{L^2} \psi + L^2 \psi = 0.$$

As $L$ from the second equation does not involve $v$, we may put $L = \frac{U'}{\sqrt{(1 - U^2)}}$ where $U$ is an arbitrary function of $u$ and $U'$ its first derivative. We are guided to this substitution by the fact that if $\psi$ in the above equation is considered known we find $L = \frac{\psi}{\sqrt{(1 - \psi^2)}}$ as a solution.

The equation then becomes

$$\psi_{uu} - \left[ \frac{U''}{U'} + \frac{UU'}{1 - U^2} \right] \psi + \frac{U'^2}{1 - U^2} \psi = 0,$$

of which one solution is evidently $\psi = U$.

We have therefore

$$\psi = \frac{U}{\rho} + \frac{U}{\tau} \int \frac{dU}{U^2 \sqrt{(1 - U^2)}}$$

$$= \frac{U}{\rho} - \frac{\sqrt{(1 - U^2)}}{\tau},$$

where $\rho$ and $\tau$ are arbitrary functions of $v$.

* (Differentialgeometrie, Bianchi. German translation, page 94).
Since \( \frac{\partial \sqrt{G}}{\partial u} = \frac{\psi_u}{L} \) we have
\[
\sqrt{G} = \frac{1}{\rho} \int \sqrt{1 - U^2} du + \frac{1}{\tau} \int U du + \chi
\]
where \( \chi \) is an arbitrary function of \( v \).

\( x, y, z \) are solutions of the equation
\[
\frac{\partial^2 \theta}{\partial u \partial v} = \frac{G_u}{2G} \frac{\partial \theta}{\partial v}.
\]

[See Salmon, Geometry of Three Dimensions, p. 441, or Bianchi, p. 111.]

\[
\therefore x = F_1(u) + \int \sqrt{1 - U^2} du \int \frac{f_1(v)}{\rho} dv + \int U du \int \frac{f_1(v)}{\tau} dv + \int \chi f_1(v) dv.
\]

\( y \) and \( z \) are obtained by replacing the suffix 1 by 2 and 3.

The three functions \( f_1, f_2, f_3 \) must be so chosen that
\[
E = \Sigma \frac{x_1^2}{\rho^2} = 1, \quad D = \Sigma \frac{x_2 x_3}{\rho^2} = 0 \quad \text{and} \quad G = \Sigma \frac{x_2^2}{\rho^2}.
\]

From the last \( f_1^2 + f_2^2 + f_3^2 = 1 \). Hence the three functions can be taken as the direction cosines (functions of \( v \)) of a line. Denote them by \( l, m, n \).

Let
\[
\int \frac{f_1}{\rho} dv = \alpha, \quad \int \frac{f_2}{\rho} dv = \beta, \quad \int \frac{f_3}{\rho} dv = \gamma,
\]
\[
\int \frac{f_1}{\tau} dv = \lambda, \quad \int \frac{f_2}{\tau} dv = \mu, \quad \int \frac{f_3}{\tau} dv = \nu.
\]

From \( \Sigma x_1 x_2 = 0 \) we have
\[
\Sigma \left( F_1'(u) + \sqrt{1 - U^2} \int \frac{f_1}{\rho} dv + U \int \frac{f_1}{\tau} dv \right) \frac{\partial f_1}{\partial v} \sqrt{G} = 0.
\]

We see that by arbitrarily varying \( u \) and \( v \), \( F_1, F_2, F_3 \) are constants, and by changing the origin we can make them zero. Also we have
\[
\alpha l + \beta m + \gamma n = 0, \quad \lambda l + \mu m + \nu n = 0.
\]

From \( \Sigma x_1^2 = 1 \) we obtain
\[
\alpha^2 + \beta^2 + \gamma^2 = 1 \quad \text{and} \quad \lambda^2 + \mu^2 + \nu^2 = 1.
\]

The nine quantities are therefore the direction cosines of three mutually rectangular lines, satisfying the equations
\[
\frac{da}{dv} = \frac{l}{\rho}, \quad \frac{d\lambda}{dv} = \frac{l}{\tau}, \quad l^2 = 1 - \alpha^2 - \lambda^2
\]
or
\[
\frac{dl}{dv} = -\frac{a}{\rho} - \frac{\lambda}{\tau} \quad \text{with two corresponding sets.}
Therefore by Frenet’s formulae $\rho$, $\tau$ are the radii of curvature and torsion of a curve of length $u$, and these nine quantities are the direction cosines of the tangent, principal normal and bi-normal.

The equations of the surface are therefore

$$x = a\int \sqrt{(1 - U^2)}du + \lambda \int Udu + \int \chi dv,$$

$$y = \beta\int \sqrt{(1 - U^2)}du + \mu \int Udu + \int \mu dv,$$

$$z = \gamma\int \sqrt{(1 - U^2)}du + \nu \int Udu + \int n dv.$$

Since $lx + my + nz = l \int \chi dv + m \int \mu dv + n \int n dv$, the curves $v = $ a constant are plane. Referred to two lines in this plane with direction cosines $a$, $\beta$, $\gamma$; $\lambda$, $\mu$, $\nu$, the equations of this curve are $\xi = \int \sqrt{(1 - U^2)}du$, $\eta = \int Udu$.

The plane of the curve is then a tangent plane to a developable. Differentiating the equation of the plane twice with regard to $v$ and solving, we find the edge of regression is given by

$$x = \int l\chi dv + \left\{a\frac{d}{dv}\left(\frac{\chi}{\tau}\right) - \lambda\frac{d}{dv}\left(\frac{\chi}{\rho}\right)\right\} + \frac{1}{\tau^2}\frac{d}{dv}\left(\frac{\tau}{\rho}\right),$$

and two similar equations for $y$ and $z$.

Making $a = \beta = \nu = n = 0$ we see that $1 = 0$ and that the plane rotates round a cylinder whose generators are parallel to the axis of $z$. If then $\lambda = \cos v$, $\mu = \sin v$, $l = \sin v$, $m = - \cos v$, we get Darboux’s formulae:—

$$x = \cos v\int Udu + \int \chi \sin v dv,$$

$$y = \sin v\int Udu - \int \chi \cos v dv,$$

$$z = \int \sqrt{(1 - U^2)}du.$$

If in the general equations $\chi = 0$, the plane rolls on a cone. If both $\chi$ and $n$ are zero, the surfaces become surfaces of revolution, because from Frenet’s formulae $\gamma$ and $\nu$ become constant.
§ 3. Suppose the position of the moving plane is given at every instant, that is, its direction cosines and its distance from the origin as functions of the time \( v \) are known.

Let its equation be

\[ lx + my + nz = F(v). \]

The direction cosines \( a, \beta, \gamma; \lambda, \mu, \nu \) have first to be found.

\[ a^2 + \lambda^2 + \mu^2 = 1. \]

Therefore on differentiating with regard to \( v \),

\[ \frac{a}{\rho} + \frac{\lambda}{\tau} = -l', \]

\[ \frac{\beta}{\rho} + \frac{\mu}{\tau} = -m', \]

\[ \frac{\gamma}{\rho} + \frac{\nu}{\tau} = -n'. \]

Squaring and adding, we have

\[ \frac{1}{\rho^2} + \frac{1}{\tau^2} = l'^2 + m'^2 + n'^2 = \frac{1}{F^2}, \text{ say.} \]

Let

\[ \frac{P}{\rho} = \cos \theta, \quad \frac{P}{\tau} = \sin \theta. \]

Then

\[ a \cos \theta + \lambda \sin \theta = -P l', \]

\[ \beta \cos \theta + \mu \sin \theta = -P m', \]

\[ \gamma \cos \theta + \nu \sin \theta = -P n'. \]

Therefore

\[ (n\beta - m\gamma) \cos \theta - (m\nu - \mu n) \sin \theta = P(mn' - m'n), \]

or

\[ \lambda \cos \theta - \nu \sin \theta = P(mn' - m'n). \]

Therefore

\[ a = -P \{ l' \cos \theta + (mn' - m'n) \sin \theta \}, \]

\[ \lambda = -P \{ l' \sin \theta - (mn' - m'n) \cos \theta \}, \]

with similar results for \( \beta, \mu \) and \( \gamma, \nu \).

Differentiating \( a \cos \theta + \lambda \sin \theta = -P l' \), we have

\[ P^2 l'' (l' l'' + m'm'' + n'n'') = P l'' + l \left( \frac{\cos \theta}{\rho} + \frac{\sin \theta}{\tau} \right) - (a \sin \theta - \lambda \cos \theta) l' \theta' \]

\[ = P l'' + \frac{l}{F} + P(mn' - m'n') \theta'. \]

Also, cyclically,

\[ P^2 m' (l' l'' + m'm'' + n'n'') = P m'' + \frac{m}{F} + P(nl' - nl') \theta'. \]
Dividing, we have

\[ \theta' P(n' l'' - n'' l' - m'm'n' + m''n) = P(l'' m' - l'm'') + \frac{1}{P}(lm' - l'm). \]

Since \( ll' + mm' + nn' = 0 \), this reduces to

\[ n\theta' = P^2(l'' m' - l'm'') + lm' - l'm. \]

Also

\[ l\theta' = P^2(m'm'n' - m'n'') + mn' - m'n, \]

\[ m\theta' = P^2(n'' l' - n'l'') + nl' - n'l. \]

Multiplying by \( n, l, m \) and adding, we obtain

\[ \theta' = \begin{vmatrix} l & m & n \\ l' & m' & n' \\ l'' & m'' & n'' \end{vmatrix} \begin{pmatrix} ll' + mm' + nn' \\ lm' - l'm \\ mn' - m'n \end{pmatrix}, \]

since \( \frac{1}{P^2} = l^2 + m^2 + n^2 = (ll' + mm' + nn''). \)

Thus \( \theta \) and the six remaining direction cosines are determined.

Since \( l \int l\chi dv + m \int m\chi dv + n \int n\chi dv = ax + by + cz, \)

we have \( l \int l\chi dv + m \int m\chi dv + n \int n\chi dv = F(v). \)

Let \( l, m, n \) be the direction cosines of the tangent to a curve of length \( v \), and let \( l_1, m_1, n_1; l_2, m_2, n_2 \) be the direction cosines of the principal normal and binormal, \( r \) and \( \sigma \) being the radii of curvature and torsion.

Differentiating and using Frenet's formulae, we obtain

\[ \sum l \int l\chi dv = (\chi' - \chi)r. \]

A second differentiation gives

\[ \sum \left( \frac{l}{r} + \frac{l_2}{\sigma} \right) \int l\chi dv = (\chi' - F''\sigma)r + (\chi - F')r'. \]

Therefore \( \sum l_2 \int l\chi dv = (\chi' - F'')r_\sigma + (\chi - F')r'r - \frac{F\sigma}{r}. \)

A third differentiation gives

\[ (F' - \chi) \frac{r}{\sigma} = \sum \frac{l_1}{\sigma} \int l\chi dv = (\chi'' - F''')r_\sigma + (\chi' - F''\sigma) r'r + (\chi' - F') (r'r_\sigma + r_\sigma') \]

\[ + (\chi - F')(r''_\sigma + r'_\sigma) - \frac{F\sigma}{r} - \frac{F_\sigma r'}{r^2} + \frac{F_\sigma' r''}{r^2}. \]
Therefore
\[ r^2\sigma \chi'' + (2rr'\sigma + r^2\sigma')\chi' + (r + r''\sigma^2 + r'\sigma') \frac{r}{\sigma} \chi \]
\[ = r^2\sigma F'' + (2rr'\sigma + r^2\sigma')F' + (r + r''\sigma^2 + r'\sigma') \frac{r}{\sigma} F' \]
\[ + F'\sigma + F\sigma - \frac{F\sigma'}{r}. \]

If the curve is spherical, so that \( R \) being the radius of spherical curvature,
\[ R^2 = r^2 + \sigma^2 \left( \frac{dr}{dv} \right)^2 \quad \text{or} \quad r + \sigma^2\nu'' + \sigma\sigma'\nu' = 0, \]
then
\[ r^2\sigma \chi'' + (2rr'\sigma + r^2\sigma')\chi' = F''r^2\sigma + (2rr'\sigma + r^2\sigma')F' \]
\[ + F'\sigma + F\sigma - \frac{F\sigma'}{r}. \]

This gives
\[ r^2\sigma \chi' = r^2\sigma F' + F\sigma - \int \frac{F\sigma'}{r} dv \]
\[ = r^2\sigma F' + \int r \frac{d}{dv} \left( \frac{F\sigma}{r} \right) dv. \]

Hence
\[ \chi = F' + \left\{ \frac{1}{r^2\sigma} \int r \frac{d}{dv} \left( \frac{F\sigma}{r} \right) dv \right\} dv. \]

If the curve is not spherical let \( ds \) be a new element of length, and choose \( v \) such a function of \( s \) that the new curve may be spherical.

From \( \frac{dl}{dv} = \frac{l_1}{r} \) we have \( \frac{dl}{ds} = \frac{l_1}{r} \frac{ds}{dv} \).

Thus we see that the direction cosines are not altered, but that the new radius of curvature \( r_1 = r \frac{ds}{dv} \), and similarly the new radius of torsion \( \sigma_1 \) is \( \frac{ds}{dv} \).

Then \( R_1 \) being the radius of the sphere, we have
\[ R_1^2 = r_1^2 + \sigma_1^2 \left( \frac{dr_1}{ds} \right)^2. \]

Therefore
\[ \sigma_1 \frac{dr_1}{ds} = \sqrt{\left\{ R_1^2 - r^2 \left( \frac{ds}{dv} \right)^2 \right\}}. \]
But \[ \frac{dr_1}{ds} = \frac{dr}{dv} + r \frac{d^2s}{dv^2} \frac{dv}{ds} \]

Therefore \[ \int \frac{dr_1}{ds} = \sigma \left( \frac{dr}{dv} \frac{ds}{dv} + r \frac{d^2s}{dv^2} \right) = \sqrt{\left( R_1^2 - r^2 \left( \frac{ds}{dv} \right)^2 \right)}, \]

or \[ \frac{d}{dv} \left( r \frac{ds}{dv} \right) \frac{1}{\sqrt{\left( R_1^2 - r^2 \left( \frac{ds}{dv} \right)^2 \right)}} = \frac{1}{\sigma}, \]

or \[ \frac{ds}{dv} = \frac{R_1}{r} \sin \int \frac{dv}{\sigma}. \]

If then in the original equation for \( \chi \) we change the independent variable from \( v \) to \( s \) and solve for \( \frac{dv}{ds} \), we get

\[
\frac{dv}{ds} = \frac{dF}{ds} + \int \left\{ \frac{1}{r_1^2 \sigma_1} \int r_1 \frac{d}{ds} \left( \frac{F_{r_1}}{r_1} \right) ds \right\} ds,
\]

or \( \chi = \frac{dF}{dv} + \frac{1}{r} \sin \int \frac{dv}{\sigma} \times \int \left[ \frac{1}{\sigma \sin^2 \int \frac{dv}{\sigma}} \int \left( \sin \int \frac{dv}{\sigma} \times \frac{d}{dv} \left( \frac{F_{r_1}}{r_1} \right) \right) dv \right]. \]

This seems complicated, but will reduce to two quadratures at most.

\[ \int \frac{dv}{\sigma} = \int \frac{ds}{r_1}, \text{ and for a spherical curve } R_1^2 = r_1^2 + \sigma_1^2 \left( \frac{dr_1}{ds} \right)^2 \]

where \( R_1 \) is constant.

Therefore \[ \int \frac{ds}{r_1} = \int \frac{dr_1}{\sqrt{(R_1^2 - r_1^2)}} = \sin^{-1} \left( \frac{r_1}{R_1} \right) + \text{constant}. \]

Hence \( \int \frac{ds}{r_1} \) for a spherical curve, and \( \int \frac{dv}{\sigma} \) for any curve of length \( v \), can be expressed in finite terms.

The equation for \( \chi \)

\[ A \int \chi Adv + B \int \chi Bdv + C \int \chi Cdv = F \]

where \( A, B, C, F \) are given functions of \( v \), could be treated in a similar way. Divide by \( \sqrt{(A^2 + B^2 + C^2)} \) and solve for \( \chi \sqrt{(A^2 + B^2 + C^2)}. \)
§ 4. Example.—Consider the developable formed by the tangents to the helix \( x = a \cos \phi, y = a \sin \phi, z = b \phi \), where \( \phi \) replaces \( v \) in the previous paragraphs.

The equation of a tangent plane is

\[
\frac{b \sin \phi}{\sqrt{a^2 + b^2}} x - \frac{b \cos \phi}{\sqrt{a^2 + b^2}} y + \frac{a}{\sqrt{a^2 + b^2}} = \frac{ab \phi}{\sqrt{a^2 + b^2}}.
\]

\[
l = \frac{b \sin \phi}{\sqrt{a^2 + b^2}}, \quad m = -\frac{b \cos \phi}{\sqrt{a^2 + b^2}}; \quad n = \frac{a}{\sqrt{a^2 + b^2}}
\]

\[
l'' + mm'' + nn'' = -\frac{b^2}{(a^2 + b^2)}.
\]

Therefore \( \theta' = \frac{1}{\sqrt{a^2 + b^2}} \begin{vmatrix} b \sin \phi & -b \cos \phi & a \\ b \cos \phi & b \sin \phi & 0 \\ -b \sin \phi & b \cos \phi & 0 \end{vmatrix} \div (-b^2) \)

\[
= -\frac{a}{\sqrt{a^2 + b^2}}.
\]

\[
\theta = -\frac{a \phi}{\sqrt{a^2 + b^2}}.
\]

\[
\frac{1}{p^2} = l^2 + m^2 + n^2 = \frac{b^2}{(a^2 + b^2)}.
\]

\[
a = -\left\{ \cos \phi \cos \frac{a \phi}{\sqrt{a^2 + b^2}} + \frac{a}{\sqrt{a^2 + b^2}} \sin \phi \sin \frac{a \phi}{\sqrt{a^2 + b^2}} \right\}.
\]

\[
\beta = -\left\{ \sin \phi \cos \frac{a \phi}{\sqrt{a^2 + b^2}} - \frac{a}{\sqrt{a^2 + b^2}} \cos \phi \sin \frac{a \phi}{\sqrt{a^2 + b^2}} \right\}.
\]

\[
\gamma = \frac{b}{\sqrt{a^2 + b^2}} \sin \frac{a \phi}{\sqrt{a^2 + b^2}}.
\]

\[
\lambda = \cos \phi \sin \frac{a \phi}{\sqrt{a^2 + b^2}} - \frac{a}{\sqrt{a^2 + b^2}} \sin \phi \cos \frac{a \phi}{\sqrt{a^2 + b^2}}.
\]

\[
\mu = \sin \phi \sin \frac{a \phi}{\sqrt{a^2 + b^2}} + \frac{a}{\sqrt{a^2 + b^2}} \cos \phi \cos \frac{a \phi}{\sqrt{a^2 + b^2}}.
\]

\[
\nu = \frac{b}{\sqrt{a^2 + b^2}} \cos \frac{a \phi}{\sqrt{a^2 + b^2}}.
\]

Then for the curve, the direction cosines of whose tangents are \( l, m, n \), we have

\[
\frac{l}{r} = \frac{b \cos \phi}{\sqrt{a^2 + b^2}}, \quad \frac{m}{r} = \frac{b \sin \phi}{\sqrt{a^2 + b^2}}, \quad \frac{1}{r} = \frac{b}{\sqrt{a^2 + b^2}}.
\]
Therefore \( l_1 = \cos \phi, \quad m_1 = \sin \phi, \quad n_1 = 0. \)

\[
- \sin \phi = -\frac{l}{r} - \frac{l_2 - l_2}{\sigma}, \quad \cos \phi = -\frac{m}{r} - \frac{m_2 - m_2}{\sigma}, \quad 0 = -\frac{n}{r} - \frac{n_2 - n_2}{\sigma},
\]

\[
\frac{1}{r^2} + \frac{1}{\sigma^2} = 1 \quad \text{and} \quad \frac{1}{\sigma} = \frac{a}{\sqrt{(a^2 + b^2)}}.
\]

The curve is not spherical and

\[
\int \frac{d\phi}{\sigma} = \frac{a \phi}{\sqrt{(a^2 + b^2)}}.
\]

\[
F = \frac{ab \phi}{\sqrt{(a^2 + b^2)}}.
\]

\[
\chi = \frac{ab}{\sqrt{(a^2 + b^2)}} + \frac{b}{\sqrt{(a^2 + b^2)}} \sin \frac{a \phi}{\sqrt{(a^2 + b^2)}} \times \int \left[ \frac{a}{\sqrt{(a^2 + b^2)}} \int \sin \frac{a \phi}{\sqrt{(a^2 + b^2)}} \times \frac{b^2 d\phi}{\sqrt{(a^2 + b^2)}} \right] d\phi
\]

\[
= \frac{ab}{\sqrt{(a^2 + b^2)}} - \frac{b^3}{(a^2 + b^2)} \sin \frac{a \phi}{\sqrt{(a^2 + b^2)}} \int \cot \frac{a \phi}{\sqrt{(a^2 + b^2)}} \csc \frac{a \phi}{\sqrt{(a^2 + b^2)}} d\phi
\]

\[
= \frac{ab}{\sqrt{(a^2 + b^2)}} + \frac{b^3}{a \sqrt{(a^2 + b^2)}}.
\]

\[
= \frac{b}{a \sqrt{(a^2 + b^2)}}.
\]