The Picard group of vertex affinoids in the first Drinfeld covering

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Abstract

Let $F$ be a finite extension of $\mathbb{Q}_p$. Let $\Omega$ be the Drinfeld upper half plane, and $\Sigma^1$ the first Drinfeld covering of $\Omega$. We study the affinoid open subset $\Sigma^1_v$ of $\Sigma^1$ above a vertex of the Bruhat–Tits tree for $GL_2(F)$. Our main result is that $\text{Pic}(\Sigma^1_v)[p] = 0$, which we establish by showing that $\text{Pic}(Y)[p] = 0$ for $Y$ the Deligne–Lusztig variety of $SL_2(\mathbb{F}_q)$. One formal consequence is a description of the representation $H^1_{\text{et}}(\Sigma^1_v, \mathbb{Z}_p(1))$ of $GL_2(\mathcal{O}_F)$ as the $p$-adic completion of $\mathcal{O}(\Sigma^1_v)^\times$.

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1. Introduction

Let $p$ be a prime, $F$ a finite extension of $\mathbb{Q}_p$, and $K$ the completion of the maximal unramified extension of $F$. Let $\mathcal{M}_0$ be the disjoint union of $\mathbb{Z}$ copies of $\Omega$, where $\Omega$ is the Drinfeld upper half plane: the rigid analytic space over $K$ defined by removing all $F$-rational points from $\mathbb{P}^{1,\text{an}}_K$. The work of Drinfeld [14] implies the existence of a tower of finite étale coverings $(\mathcal{M}_n)_{n \geq 0}$ of $\mathcal{M}_0$ equipped with compatible actions of $GL_2(F)$, which has been shown to realise both the local Langlands and Jacquet–Langlands correspondence in its étale cohomology [4, 5, 18, 19]. On the other hand, there is at present no formulated $p$-adic local Langlands correspondence for $GL_2(F)$ for general finite extensions $F$. The Drinfeld tower is expected to be of importance in yielding natural representations of $GL_2(F)$ that should appear in any such correspondence. For example, the geometric $p$-adic étale cohomology of the Drinfeld tower has been shown to encode the $p$-adic local Langlands correspondence for $F = \mathbb{Q}_p$ [7].

The preimage of the index zero piece $\Omega \hookrightarrow \mathcal{M}_0$ in the tower $(\mathcal{M}_n)_{n \geq 0}$ defines a tower $(\Sigma^n)_{n \geq 0}$ of finite étale coverings of $\Sigma^0 = \Omega$. The transition morphisms are equivariant for the action of the stabilising subgroup $GL_2(F)^+ = \{ g \in GL_2(F) \mid \det(g) \in \mathcal{O}^{\times} \}$. Let $\mathcal{T}$ be the Bruhat–Tits tree for $GL_2(F)$, $v$ the central vertex of $\mathcal{T}$, and $r : \Sigma^1 \rightarrow \Omega \rightarrow \mathcal{T}$ the retraction map. In this paper we study the open affinoid subset $\Sigma^1_v := r^{-1}(v)$ of $\Sigma^1$. This is stable under the action of $GL_2(\mathcal{O}_F)$ and after a finite extension of $K$, $\Sigma^1_v$ splits up into $q - 1$ geometrically connected components, each isomorphic to $\text{Sp}(B)$, where

$$B = A[z]/(z^{q+1} - (x^q - x)),$$

for $A = K\left\langle x, \frac{1}{x^q - x} \right\rangle$.
The group \( \text{GL}_2(F)^+ \) acts with two orbits on the set of vertices of \( T \), and one can show that for any vertex \( v \) adjacent to \( v \), \( \Sigma_w \cong \Sigma_v \). As any such \( w \) will be in the other orbit from \( v \), \( \Sigma_w \cong \Sigma_v \) for all vertices \( v \in T \), and consequently this open subset often determines global properties of \( \Sigma \). For example, the first de-Rham cohomology \( H^1_{\text{dR}}(\Sigma) \) as a representation of \( \text{GL}_2(F) \) is determined by \( H^1_{\text{dR}}(\Sigma) \) [21, theorem 6.1].

Our main result is that \( \text{Pic}(\Sigma) \) [\( p \)] = 0 (Theorem 3.2). The \( p \)-adic étale cohomology groups of Drinfeld spaces are of considerable interest [3, 6–9, 24], and one immediate consequence of Theorem 3.2 is a description of the GL_2(O)\-representation \( H^1_{\text{et}}(\Sigma, \mathbb{Z}_p(1)) \), as the \( p \)-adic completion of \( \mathcal{O}(\Sigma^1)\) (Theorem 3.4). This description is very explicit, as the unit group \( \mathcal{O}(\Sigma^1)^\times \) has been described by Junger [22, theorem 5.1].

Our main interest in Theorem 3.2 is the following. A precise statement of the \( p \)-adic local Langlands correspondence is formulated when \( F = \mathbb{Q}_p \) [10], and Dospinescu and Le Bras [13] have used this to show that for \( F = \mathbb{Q}_p \) and all \( n \geq 1 \), the representation \( \mathcal{O}(\Sigma^n) \) is naturally a coadmissible module over \( D(G, K) \), the distribution algebra of \( G \).

In an effort to remove the restriction on \( F \), Ardakov and Wadsley show in their forthcoming work [1] using \( p \)-adic \( D \)-modules that the representation \( \mathcal{O}(\Sigma^1) \) splits up naturally into a direct sum of coadmissible \( D(G, K) \)-modules. This decomposition contains \( \mathcal{O}(\Omega) \), and all other components are shown to be topologically irreducible \( D(G, K) \)-modules. The benefits of this approach over that of [13], are that it holds for general field extensions \( F \), is purely local, and establishes topological irreducibility. The obvious disadvantage is that it describes \( \mathcal{O}(\Sigma^n) \) only for \( n = 1 \). One would like to establish similar results for \( \mathcal{O}(\Sigma^n) \) for \( n \geq 2 \), where the situation is significantly more complicated. This is partially due to the fact that \( \Sigma^n \rightarrow \Sigma^{n-1} \) has degree a power of \( p \), whereas the degree of \( \Sigma^1 \rightarrow \Omega \) is coprime to \( p \). The methods of [1] use the standard result that \( \text{Pic}(\Omega) = 0 \), and in attempting to transfer these methods to \( \mathcal{O}(\Sigma^2) \), one considers the group \( \text{Pic}(\Sigma^1) \) [\( p \)] instead. Almost nothing is known about \( \text{Pic}(\Sigma^1) \) [\( p \)], which is strongly expected to be non-zero. Our result that \( \text{Pic}(\Sigma^1) \) [\( p \)] = 0 is therefore slightly surprising. It also provides the first steps towards computing \( \text{Pic}(\Sigma^1) \) [\( p \)] (by choosing an appropriate Čech cover), and allows one the possibility of using similar methods to [1] locally.

In order to prove Theorem 3.2, we consider the affine curve \( Y \) defined by,

\[ xy^q - yx^q = 1, \]

over the residue field of \( K \), where \( \mathbb{F}_q \) is the residue field of \( F \). This curve was first considered by Drinfeld, who showed that all the discrete series representations of \( \text{SL}_2(\mathbb{F}_q) \) can be realised in the cohomology of \( Y \) [2, preface]. Inspired by this, these ideas were generalised to all reductive groups \( G \) by Deligne and Lusztig in their landmark paper [12]. They introduce what are now called Deligne–Lusztig varieties, which assign to \( G(\mathbb{F}_q) \) and \( w \in W \), the Weyl group, a base space \( X(w) \) and a finite covering \( Y(w) \), and it is in the étale cohomology of \( Y(w) \) that the cuspidal representations are realised. These are spaces of considerable interest, and the Picard groups of the base spaces \( X(w) \) have been considered in [17]. Here we consider \( Y = Y(w) \) in the special case of \( G = \text{SL}_2 \), and \( w \neq 1 \). It would be interesting to study the Picard groups of \( Y(w) \) more generally.
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2. Deligne–Lusztig curves

Throughout this section, let $\mathbb{F}$ be an algebraic field extension of $\mathbb{F}_q$. We consider the affine curve,

$$Y = \text{Spec} \left( \frac{\mathbb{F}[x, y]}{xy^q - yx^q = 1} \right),$$

and its projective closure,

$$Z = \text{Proj} \left( \frac{\mathbb{F}[X, Y, Z]}{XY^q - YX^q = Z^{q+1}} \right).$$

We also consider the projective curve,

$$W = \text{Proj} \left( \frac{\mathbb{F}[U, V, W]}{UV^q + VU^q = W^{q+1}} \right).$$

We would first like to show that $\text{Pic}(Z)[p] = 0$.

**Lemma 2.1.** $Z$ is a smooth integral projective curve over $\mathbb{F}$. Furthermore, if $\mathbb{F}_q^4 \subset \mathbb{F}$, then $W \cong Z$.

**Proof.** The polynomial $P(X, Y, Z) = Z^{q+1} - (XY^q - YX^q) \in \mathbb{F}[X, Y, Z]$ is prime, which follows from Eisenstein’s criterion for $P \in \mathbb{F}[X, Y][Z]$, at the prime ideal $(X)$. Therefore $Z$ is integral. Furthermore, $Z$ is smooth, because the system $\partial_X P = \partial_Y P = \partial_Z P = 0$ has no solutions over $Z(\mathbb{F})$. For the isomorphism, let $\lambda \in \mathbb{F}_q^4$ with $\lambda q - 1 = -1$, and let $\mu \in \mathbb{F}$ with $\mu^{q+1} = \lambda^q$. The element $\mu$ lies in $\mathbb{F}_q^4$, as $\mu q^2 = (\lambda^q)q^{-1} = -\mu$,

so,

$$\mu q^4 = (-\mu)q^2 = -(\lambda^q) = \mu.$$

Then the claimed isomorphism is given by,

$$U = X, \quad V = \lambda Y, \quad W = \mu Z.$$

Indeed,

$$X(\lambda Y)^q + (\lambda Y)X^q = \lambda^q (XY^q - YX^q),$$

$$= \lambda^q Z^{q+1} = (\mu Z)^{q+1},$$

and similarly $U(\lambda^{-1} V)^q - (\lambda^{-1} V)U^q = (\mu^{-1} W)^{q+1}$.

**Proposition 2.2.** $\text{Pic}(Z)[p] = 0$.

**Proof.** By Lemma 2.1, $Z_{\overline{\mathbb{F}}} \cong W_{\overline{\mathbb{F}}}$, and thus the group $\text{Pic}(Z_{\overline{\mathbb{F}}})[p] \cong \text{Pic}(W_{\overline{\mathbb{F}}})[p] \cong J(\overline{\mathbb{F}})[p]$, where $J$ is the Jacobian of $W$. $W$ is known as the Hermitian curve, defined by affine equation $w^{q+1} = v^q + v$, and is maximal over $\mathbb{F}_q^2$ [26, lemma 6.4.4], hence $J(\overline{\mathbb{F}})[p] = 0$ by [15, corollary 2.5]. Then, because pullback induces an exact sequence $0 \to \text{Pic}(Z) \to \text{Pic}(Z_{\overline{\mathbb{F}}})$ [25, Tag 0CC5], and $p$-torsion is left exact, $\text{Pic}(Z)[p] = 0$. 

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Our next goal is to establish that $\text{Pic}(Y)[p] = 0$.

**Lemma 2.3.** $Z(F) \setminus Y(F)$ consists of the $q + 1$ points,

$\mathcal{P} := \{(a : b : 0) \mid (a : b) \in \mathbb{P}^1(F_q)\}$.

Furthermore, $\mathcal{P} = Z(F_q)$.

**Proof.** If $(a : b : c) \in Z(F)$ with $c = 0$, then $b^q a - a^q b = 0$, so $b^q a = a^q b$. If $a \neq 0$, then $(b/a)^q = b/a$, so $b/a \in F_q$, and $(a : b) \in \mathbb{P}^1(F_q)$. Similarly, if $b \neq 0$, $(a : b) \in \mathbb{P}^1(F_q)$. Thus $Z(F) \setminus Y(F) = \mathcal{P}$. To see $Z(F) \setminus Y(F) = Z(F_q)$, there are no points $(a : b : c) \in Z(F_q)$ with $c = 1$, because if so then $1 = ab^q - ba^q = ab - ba = 0$, as $a, b \in F_q$.

Therefore the closed points of $Z \setminus Y$ are $\mathcal{P}$ [16, proposition 5·4], which we enumerate by $\mathcal{P} = \{P_0, \ldots, P_q\}$. From [27, exercise 5·12 (a)] we have an exact sequence,

$$\mathbb{Z}^{q+1} \longrightarrow \text{Cl}(Z) \longrightarrow \text{Cl}(Y) \longrightarrow 0,$$

where the first map sends,

$$(m_0, \ldots, m_q) \longmapsto \sum_{i=0}^q m_i[P_i],$$

and the second sends, for $I$ a finite set of closed points of $Z$,

$$\sum_{P \in I} n_P[P] \longmapsto \sum_{P \in I \setminus \mathcal{P}} n_P[P].$$

Let $\Gamma = \langle [P_0], \ldots, [P_q] \rangle \subset \text{Cl}(Z)$ be the image of $\mathbb{Z}^{q+1}$ in $\text{Cl}(Z)$. The resulting exact sequence,

$$0 \longrightarrow \Gamma \longrightarrow \text{Cl}(Z) \longrightarrow \text{Cl}(Y) \longrightarrow 0,$$

yields the long exact sequence,

$$0 \longrightarrow \Gamma[p] \longrightarrow \text{Cl}(Z)[p] \longrightarrow \text{Cl}(Y)[p]$$

$$\longrightarrow \Gamma/p\Gamma \longrightarrow \text{Cl}(Z)/p\text{Cl}(Z) \longrightarrow \text{Cl}(Y)/p\text{Cl}(Y) \longrightarrow 0,$$

from the right derived functors of $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z}, -)$. Then from Proposition 2·2 and the above discussion we have the following.

**Proposition 2·4.** There is an exact sequence

$$0 \longrightarrow \text{Cl}(Y)[p] \longrightarrow \Gamma/p\Gamma \longrightarrow \text{Cl}(Z)/p\text{Cl}(Z),$$

where the map $\Gamma/p\Gamma \to \text{Cl}(Z)/p\text{Cl}(Z)$ is that induced by the inclusion $\Gamma \hookrightarrow \text{Cl}(Z)$.

**Remark.** We note that if $Z \setminus Y$ contained exactly one degree 1 closed point $Q$, then we could establish that $\text{Pic}(Y)[p] = 0$ almost immediately in the following way. In the exact sequence,

$$\mathbb{Z} \longrightarrow \text{Cl}(Z) \longrightarrow \text{Cl}(Y) \longrightarrow 0,$$
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the map \( \mathbb{Z} \to \text{Cl}(\mathbb{Z}) \) is actually injective and split by the degree homomorphism, hence \( \text{Cl}(\mathbb{Z}) \cong \mathbb{Z} \times \text{Cl}(\mathbb{Y}) \) so,

\[
0 = \text{Cl}(\mathbb{Z})[p] \cong \mathbb{Z}[p] \times \text{Cl}(\mathbb{Y})[p] = \text{Cl}(\mathbb{Y})[p].
\]

In particular, this can be applied to show that the class groups of affine dehomogenisations of \( \mathbb{Z} \) with respect to both \( X \) and \( Y \) both have no \( p \)-torsion.

We want to show that \( \text{Cl}(\mathbb{Y})[p] = 0 \), and so in light of Proposition 2.4, we want to show that,

\[
\Gamma/p\Gamma \longrightarrow \text{Cl}(\mathbb{Z})/p\text{Cl}(\mathbb{Z}),
\]

is injective. In order to do so, we now examine the structure of \( \Gamma \). First we compute the principal divisors of some rational functions on \( \mathbb{Z} \).

**Definition** 2.5. For \((a : b) \in \mathbb{P}^1(\mathbb{F}_q)\), we let \( P_{(a : b)} \) be the closed point of \( \mathbb{Z} \) defined by \((a : b : 0) \in \mathbb{P}^1(\mathbb{F})\).

**Lemma** 2.6. Let \((a : b), (c : d) \in \mathbb{P}^1(\mathbb{F}_q) \) with \((a : b) \neq (c : d)\). Then the rational function,

\[
f := \frac{bX - aY}{dX - cY},
\]

has associated principal divisor,

\[
(f) = (q + 1) \left[ P_{(a : b)} \right] - (q + 1) \left[ P_{(c : d)} \right].
\]

**Proof.** Consider the morphism \( \zeta : \mathbb{Z} \to \mathbb{P}^1 \) corresponding to the extension of function fields \( \mathbb{F}(\mathbb{P}^1) \to \mathbb{F}(\mathbb{Z}) \), which sends,

\[
\frac{S}{T} \rightarrow \frac{bX - aY}{dX - cY},
\]

where \( \mathbb{P}^1 = \text{Proj}(\mathbb{F}[S, T]) \), and \( \mathbb{F}(\mathbb{P}^1) = \mathbb{F}(S/T) \). On \( \mathbb{F} \)-points, \( \zeta : \mathbb{Z} \to \mathbb{P}^1 \) is given by,

\[
\zeta(x : y : z) = (bx - ay : dx - cy).
\]

This extension \( \mathbb{F}(\mathbb{P}^1) \to \mathbb{F}(\mathbb{Z}) \) has degree \( q + 1 \) because it differs by an automorphism of \( \mathbb{P}^1 \) from the extension \( \mathbb{F}(\mathbb{P}^1) \to \mathbb{F}(\mathbb{Z}) \), defined by,

\[
\frac{S}{T} \mapsto \frac{X}{Y},
\]

which clearly has degree \( q + 1 \). Let \( Q_0, Q_\infty \) be the closed points of \( \mathbb{P}^1 \) defined by \((0 : 1), (1 : 0) \in \mathbb{P}^1(\mathbb{F}) \) respectively. By [23, corollary 3.9], we have that,

\[
(f) = \zeta^*(S/T) = \zeta^*([Q_0]) - \zeta^*([Q_\infty]),
\]

and \( \deg(\zeta^*([Q_0])) = \deg(\zeta^*([Q_\infty])) = [\mathbb{F}(\mathbb{P}^1) : \mathbb{F}(\mathbb{Z})] = q + 1 \). But \( \zeta^*([Q_0]) \) is some integer multiple of \( [P_{(a : b)}] \) and \( \zeta^*([Q_\infty]) \) some integer multiple of \( [P_{(c : d)}] \), hence,

\[
(f) = (q + 1) \left[ P_{(a : b)} \right] - (q + 1) \left[ P_{(c : d)} \right].
\]
Let \( \Gamma^0 \subset \Gamma \) be the degree 0 subgroup of \( \Gamma \), and \( \text{Cl}^0(\mathbf{Z}) \subset \text{Cl}(\mathbf{Z}) \) the degree 0 subgroup of \( \text{Cl}(\mathbf{Z}) \).

**Lemma 2.7.** The function \( \phi : \mathbb{Z} \times (\mathbb{Z}/(q+1)\mathbb{Z})^q \rightarrow \Gamma \),

\[
\phi : (n_0, \ldots, n_q) \mapsto n_0[P_0] + n_1([P_1] - [P_0]) + \ldots + n_q([P_q] - [P_0]),
\]

is a surjective homomorphism. In particular, \( \Gamma^0 \) is a quotient of \( (\mathbb{Z}/(q+1)\mathbb{Z})^q \).

**Proof.** For each \( P_k \in \mathcal{P} \), we can write \( P_k = P_{(a_k : b_k)} \) for some \( a_k, b_k \in \mathbb{F}_q \). For each \( 0 \leq i \neq j \leq q \), consider the rational function,

\[
f = \frac{b_j X - a_i Y}{b_j X - a_j Y}.
\]

Taking the divisor of \( f \),

\[
0 = (f) = (q + 1)[P_i] - (q + 1)[P_j],
\]

in \( \Gamma \), by Lemma 2.6. Therefore, \( \phi \) is a well-defined homomorphism, which is surjective because \( \{[P_0], \ldots, [P_q]\} \) generate \( \Gamma \). Finally, as \( \Gamma^0 = \langle [P_1] - [P_0], \ldots, [P_q] - [P_0]\rangle \), then \( \Gamma^0 \) is a quotient of \( (\mathbb{Z}/(q+1)\mathbb{Z})^q \).

We are finally in a position to prove the main result of this section.

**Theorem 2.8.** \( \text{Pic}(Y)[p] = 0 \).

**Proof.** We can split the degree homomorphism with \( [P_0] \), as \( [P_0] \) has degree 1 and \( \langle [P_0] \rangle \) is free [27, exercise 5.12 (b)]. Then,

\[
\psi : \text{Cl}(\mathbf{Z}) \longrightarrow \text{Cl}^0(\mathbf{Z}) \times \mathbb{Z},
\]

\[
Q \mapsto (Q - \deg(Q)[P_0], \deg(Q)),
\]

is an isomorphism, which restricts to,

\[
\Gamma \cong \Gamma^0 \times \mathbb{Z}.
\]

We then obtain the following commutative diagram,

\[
\begin{array}{ccc}
\text{Cl}(\mathbf{Z}) & \cong & \text{Cl}^0(\mathbf{Z}) \\ \downarrow & & \downarrow \\ \text{Cl}(\mathbf{Z}) & \cong & \text{Cl}^0(\mathbf{Z}) \\
\end{array}
\]

Here, the vertical maps are induced from the inclusions of \( \Gamma \) into \( \text{Cl}(\mathbf{Z}) \) and of \( \Gamma^0 \) into \( \text{Cl}^0(\mathbf{Z}) \), the left horizontal maps are induced by \( \psi \), and the right horizontal maps are the standard identifications.

Now, by Lemma 2.7, \( \Gamma^0 \) is a quotient of \( (\mathbb{Z}/(q+1)\mathbb{Z})^q \), thus \( \Gamma^0/p\Gamma^0 = 0 \). Consequently, \( \Gamma/p\Gamma \rightarrow \text{Cl}(\mathbf{Z})/p\text{Cl}(\mathbf{Z}) \) is an injection. Therefore, \( \text{Cl}(Y)[p] = 0 \), by the exact sequence of Proposition 2.4.
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3. Rigid curves

Let $F$ be a finite extension of $\mathbb{Q}_p$ with uniformiser $\pi$ and residue field $\mathbb{F}_q$. Let $K$ be a complete field extension of $F$ with residue field $\mathbb{F}$, such that $\mathbb{F}$ is an algebraic extension of $\mathbb{F}_q$. Let $R$ be the ring of integers of $K$ and $\varpi \in K$ an element with $0 < |\varpi| < 1$.

Let $A$ be the affinoid algebra,

$$A = K\left\{ x, \frac{1}{x^q - x} \right\},$$

for which the associated rigid space $\text{Sp}(A)$ has admissible formal model $\text{Spf}(A_0)$, where,

$$A_0 = R\left\{ x, \frac{1}{x^q - x} \right\}.$$

Let $u := x^q - x \in A_0^\times \subset A^\times$, and let $B$ be the affinoid algebra,

$$B := A[z] / \left( z^{q+1} - u \right).$$

Consider the ring extension,

$$B_0 := A_0[z] / \left( z^{q+1} - u \right).$$

$B_0$ is $\varpi$-torsion free, and the natural map,

$$B_0 = A_0[z] / \left( z^{q+1} - u \right) \longrightarrow R\left\{ x, \frac{1}{x^q - x}, z \right\} / \left( z^{q+1} - u \right),$$

is an isomorphism (because $u \in A_0^\times$ is a unit), hence $B_0$ is an admissible $R$-algebra. The special fibre of $\text{Spf}(B_0)$ is,

$$\text{Spec}(B_0 \otimes_R \mathbb{F}) = \text{Spec}\left( \mathbb{F}[y, 1/v, t] / \left( t^{q+1} - v \right) \right),$$

where $v = y^q - y$, and the generic fibre of $\text{Spf}(B_0)$ is $\text{Sp}(B_0 \otimes_R K) = \text{Sp}(B)$.

**Lemma 3.1.** $\text{Pic}(\text{Sp}(B)) \cong \text{Pic}(Y)$.

**Proof.** First note that there is an isomorphism of $\mathbb{F}$-algebras,

$$\mathbb{F}[r, s]/(rs^q - sr^q - 1) \sim \mathbb{F}[y, 1/v, t] / \left( t^{q+1} - v \right),$$

given by $r \mapsto 1/t$, $s \mapsto y/t$, with inverse $y \mapsto s/r$, $t \mapsto 1/r$. Thus $\text{Spec}(B_0 \otimes_R \mathbb{F}) \cong Y$, and $\text{Spf}(B_0)$ is a smooth admissible formal model of $\text{Sp}(B)$. Therefore by [20, lemma 3.6], the natural maps,

$$\text{Pic}(\text{Sp}(B)) \sim \text{Pic}(\text{Sp}(B_0)) \sim \text{Pic}(\text{Spec}(B_0 \otimes_R \mathbb{F})),$$

are isomorphisms and we’re done.

We can now state our main results. If $K$ contains $\check{\mathbb{F}}$ the completion of the maximal unramified extension of $F$, then we can consider the rigid analytic space $\Sigma^1$ defined over any such $K$. For an overview of the construction and properties of $\Sigma^1$ see [22, section 2]. If $v \in T$ is

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the central vertex of the Bruhat–Tits tree, then the open affinoid subset \( \Sigma^1_v := r^{-1}(v) \subset \Sigma^1 \) has coordinate ring isomorphic to,

\[
\mathcal{O}\left( \Sigma^1_v \right) \cong A[z]/\left( z^{q^2-1} - (\pi u^{q-1}) \right),
\]

by [22, theorem 2.7].

Let \( \omega \) be a primitive \((q^2 - 1)\)st root of \( \pi \) in \( \bar{F} \). From now on we strengthen our assumption on the complete field extension \( K \) of \( F \) and assume that,

\[
K \text{ contains } \bar{F}(\omega) \text{ and } F \text{ is an algebraic extension of } \mathbb{F}_q.
\]

We note that this forces \( F \) to be an algebraic closure of \( \mathbb{F}_q \), and that this assumption holds for any complete field extension \( K \) of \( \bar{F}(\omega) \) which is contained in \( \mathbb{C}_p \).

**Theorem 3.2.** \( \text{Pic}(\Sigma^1_v)[p] = 0 \).

**Proof.** Because \( K \) contains \( \omega \),

\[
\mathcal{O}\left( \Sigma^1_v \right) \cong B^{q-1},
\]

and therefore,

\[
\text{Pic}(\Sigma^1_v) \cong \text{Pic}\left( \text{Sp}(B^{q-1}) \right) = \text{Pic}(\text{Sp}(B))^{q-1} \cong \text{Pic}(Y)^{q-1},
\]

by Lemma 3.1. But then \( \text{Pic}(\Sigma^1_v)[p] \cong \text{Pic}(Y)[p]^{q-1} \), which is zero by Theorem 2.8.

Recall that \( \Sigma^1_v = r^{-1}(v) \) is the pre-image of \( v \), the central vertex of the Bruhat–Tits tree. The vertex \( v \) is fixed by \( \text{GL}_2(\mathcal{O}_F) \), and because \( r \) is equivariant, \( \text{GL}_2(\mathcal{O}_F) \) acts on \( \Sigma^1_v \).

**Corollary 3.3.** The natural map,

\[
\mathcal{O}\left( \Sigma^1_v \right)^{\times} \otimes \mathcal{O}\left( \Sigma^1_v \right)^{\times p^n} \longrightarrow H^1_{\text{ét}}\left( \Sigma^1_v, \mu_{p^n} \right),
\]

arising from the Kummer exact sequence is an isomorphism of \( \text{GL}_2(\mathcal{O}_F) \)-modules.

**Proof.** Because \( K \) has characteristic 0, we can consider the Kummer exact sequence for rigid analytic spaces [11, section 3.2]. Then the result follows from Theorem 3.2 after taking the long exact sequence in étale cohomology, using that \( \text{Pic}(\Sigma^1_v) \cong H^1_{\text{ét}}\left( \Sigma^1_v, \mathbb{G}_m \right) \) [11, proposition 3.2-4].

As a consequence, we may now compute \( H^1_{\text{ét}}\left( \Sigma^1_v, \mathbb{Z}_p(1) \right) \) as the \( p \)-adic completion of \( \mathcal{O}(\Sigma^1_v)^{\times} \). This is completely explicit, as the group \( \mathcal{O}(\Sigma^1_v)^{\times} \) has been computed by Junger [22, theorem 5.1].

**Theorem 3.4.** There is an isomorphism of \( \mathbb{Z}_p \)-linear representations of \( \text{GL}_2(\mathcal{O}_F) \),

\[
H^1_{\text{ét}}\left( \Sigma^1_v, \mathbb{Z}_p(1) \right) \cong \lim_{\leftarrow n \geq 1} \mathcal{O}\left( \Sigma^1_v \right)^{\times} \otimes \mathcal{O}\left( \Sigma^1_v \right)^{\times p^n}.
\]
The Picard group of vertex affinoids in the first Drinfeld covering

Proof. For all \( n \geq 1 \) the diagram,

\[
\begin{align*}
\mathcal{O}(\Sigma_{v}^1) \times / \mathcal{O}(\Sigma_{v}^1)_{\times} \cong & H^1_{\text{ét}}(\Sigma_{v}^1, \mu_{p^n+1}) \\
\downarrow & \\
\mathcal{O}(\Sigma_{v}^1) \times / \mathcal{O}(\Sigma_{v}^1)_{\times} \cong & H^1_{\text{ét}}(\Sigma_{v}^1, \mu_{p^n})
\end{align*}
\]

commutes. Then by the definition of \( H^1_{\text{ét}}(\Sigma_{v}^1, \mathbb{Z}_p(1)) \) and Corollary 3.3,

\[
H^1_{\text{ét}}(\Sigma_{v}^1, \mathbb{Z}_p(1)) = \lim_{\leftarrow n \geq 1} H^1_{\text{ét}}(\Sigma_{v}^1, \mu_{p^n}) \cong \lim_{\leftarrow n \geq 1} \mathcal{O}(\Sigma_{v}^1)_{\times} / \mathcal{O}(\Sigma_{v}^1)_{\times}^{p^n}.
\]

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