DISTINGUISHED SUBMODULES

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Although there is no need for a 'distinguished' submodule to be given a formal definition in the present paper, we like to indicate the meaning attached to this concept here. Perhaps the shortest way of doing so is to say that a distinguished submodule is a (covariant idempotent) functor from the category of (left) R-modules into itself mapping each R-module into its R-submodule specified by a family of left ideals of R. If $\mathscr{K}$ is a family of left ideals of R, then all elements of an R-module M of orders belonging to $\mathscr{K}$, do not, of course, in general form a submodule of M; but, there are certain families $\bar{\mathscr{K}} \supseteqq \mathscr{K}$ such that all the elements of orders from $\bar{\mathscr{K}}$ form a submodule in any R-module (distinguished submodules defined by $\mathscr{K}$). Consequently, no particular structural properties of the R-module are involved in the definition of such submodules. In this way we can define radicals (in the sense of Kuroš [4]) of a module. In particular, we feel that an application of this method is an appropriate way in defining the (maximal) torsion submodule of a module.

Thus, the present paper is, in fact, a study of subfamilies of the family $\mathscr{L}_R$ of all left ideals of a ring R. We use this opportunity to deal with a certain duality in the set of subfamilies of $\mathscr{L}_R$; this duality relates to problems connected with the problems of dependence over modules and, in particular, with the definition of the rank of a module which will be treated elsewhere (cf. [1]).

1. Basic definitions

Throughout the paper, R stands for an (associative) ring with unity ε. The family of all proper (that is, $\neq R$) left ideals of R is denoted by $\mathscr{L}_R$, or briefly by $\mathscr{L}$. For $L \in \mathscr{L}$ and $\rho \in R$, the symbol $L : \rho$ denotes the (right) ideal-quotient of L by ρ, that is, the left ideal of all $\chi \in R$ such that $\chi\rho \in L$.

We shall consider the following three properties of a subfamily $\mathscr{K} \subseteqq \mathscr{L}$:

(Q) $\qquad K \in \mathscr{K} \wedge \rho \in R \setminus K \rightarrow K : \rho \in \mathscr{K};$

(I) $\qquad K_1 \in \mathscr{K} \wedge K_2 \in \mathscr{K} \rightarrow K_1 \cap K_2 \in \mathscr{K};$

(E) $\qquad K_1 \in \mathscr{K} \wedge K_2 \in \mathscr{K} \wedge K_1 \subseteqq K_2 \rightarrow K_2 \in \mathscr{K}.$

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We can see immediately that

1.1. (i) If $\mathscr{K}_\omega$, $\omega \in \Omega$, satisfy (Q), (I), or (E), then $\bigcap_{\omega\in\Omega} \mathscr{K}_\omega$ satisfies (Q), (I), or (E), respectively.

(ii) If $\mathscr{K}_\omega$, $\omega \in \Omega$, satisfy (Q) or (E), then $\bigcup_{\omega\in\Omega} \mathscr{K}_\omega$ satisfies (Q) or (E), respectively.

In what follows we shall be interested in the Q-families of left ideals of R, i.e. in the subfamilies of $\mathscr{L}$ satisfying (Q) and in the F-families (filters) of left ideals of R, i.e. in the subfamilies of $\mathscr{L}$ satisfying (Q), (I) and (E). Denote the set of all subfamilies of $\mathscr{L}$ by $\boldsymbol{L}$, the subset of $\boldsymbol{L}$ of all Q-families by $\boldsymbol{Q}$ and that of all F-families by $\boldsymbol{F}$. Hence, by 1.1,

1.2. The mappings $\mathfrak{c}_Q$ and $\mathfrak{c}_F$ defined on $\boldsymbol{L}$ by

$$\mathfrak{c}_Q(\mathscr{K}) = \bigcap_{\mathscr{K} \subseteq \mathscr{X} \in \boldsymbol{Q}} \mathscr{X}$$

and

$$\mathfrak{c}_F(\mathscr{K}) = \bigcap_{\mathscr{K} \subseteq \mathscr{X} \in \boldsymbol{F}} \mathscr{X}$$

are (idempotent) closure operators in $\mathscr{L}$.

M will always denote a (unital left) R-module. The order of $m \in M$ is denoted by $O(m)$; hence, $O(m) \in \mathscr{L}$ if and only if $m \neq 0$. Moreover, evidently $O(\rho m) = O(m) : \rho$ for any non-zero $m \in M$ and $\rho \in R$. Since R will be fixed throughout the paper, we shall often speak briefly about submodules, homomorphisms etc. instead of R-submodules, R-homomorphisms etc. The R-module of all cosets of R modulo L will be denoted by $R \bmod L$. A submodule N of M is said to be essential in M if it meets any non-zero submodule of M non-trivially. Specializing to $M = R \bmod \{0\}$ we get the concept of an essential left ideal in R. If $\mathscr{K}$ is a family of left ideals of R, then the subset of all elements of an R-module M whose orders belong to $\mathscr{K}^0 = \mathscr{K} \cup \{R\}$ will be denoted by $M_{\mathscr{K}} \subseteqq M$.

2. An order and a Galois connection in Q

Although our next consideration can easily be extended to $\boldsymbol{L}$, we shall, with regard to our further needs, restrict the definitions to $\boldsymbol{Q}$.

If $L \in \mathscr{L}$, then

$$\mathfrak{c}_Q(\{L\}) = \mathfrak{c}_Q(L) = \{L : \rho\}_{\rho \in R \setminus L};$$

let us call a Q-family of this type cyclic. Also, let us point out that, in view of 1.1, any family $\mathscr{K} \subseteqq \mathscr{L}$ contains the greatest Q-subfamily: the (set-theoretical) union of all cyclic Q-families contained in $\mathscr{K}$. In particular, a Q-family is the union of its cyclic Q-subfamilies.

Now, define in $\boldsymbol{Q}$ the following preorder $\ll$ by

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$$\mathscr{K}_1 \ll \mathscr{K}_2 \leftrightarrow \forall L (L \in \mathscr{L} \to \mathfrak{c}_Q(L) \notin \mathscr{K}_1 \setminus \mathscr{K}_2).$$

Clearly,

$$\mathscr{K}_1 \subseteqq \mathscr{K}_2 \to \mathscr{K}_1 \ll \mathscr{K}_2;$$

moreover, we get immediately

$$\mathscr{K}_1 \ll \mathscr{K}_2 \to (\mathscr{K}_1 \ll \mathscr{K}_1 \cap \mathscr{K}_2 \wedge \mathscr{K}_1 \cup \mathscr{K}_2 \ll \mathscr{K}_2).$$

The preorder $\ll$ yields an equivalence $\approx$ in $\boldsymbol{Q}$, namely

$$\mathscr{K}_1 \approx \mathscr{K}_2 \leftrightarrow (\mathscr{K}_1 \ll \mathscr{K}_2 \wedge \mathscr{K}_2 \ll \mathscr{K}_1).$$

Again, we can easily prove that, for $\omega \in \Omega$,

$$\mathscr{K} \approx \mathscr{K}_\omega \to \mathscr{K} \approx \bigcup_{\omega \in \Omega} \mathscr{K}_\omega$$

and, provided Ω is finite, also

$$\mathscr{K} \approx \mathscr{K}_\omega \to \mathscr{K} \approx \bigcap_{\omega \in \Omega} \mathscr{K}_\omega.$$

Hence,

2.1. There is a greatest (with respect to order by inclusion) element in each equivalence class of $\boldsymbol{Q}$. If $\boldsymbol{K} \subseteqq \boldsymbol{Q}$ is the equivalence class containing $\mathscr{K}$, then the greatest element $\mathfrak{c}(\mathscr{K})$ of $\boldsymbol{K}$ is given by

$$\mathfrak{c}(\mathscr{K}) = \bigcup_{\mathscr{X} \approx \mathscr{K}} \mathscr{X} = \bigcup_{\mathscr{X} \in \boldsymbol{K}} \mathscr{X}.$$

Also, $\mathscr{K}_1 \cap \mathscr{K}_2 \in \boldsymbol{K}$ for every $\mathscr{K}_1 \in \boldsymbol{K}$, $\mathscr{K}_2 \in \boldsymbol{K}$ and, furthermore, any $\mathscr{X} \in \boldsymbol{Q}$ such that $\mathscr{K} \subseteqq \mathscr{X} \subseteqq \mathfrak{c}(\mathscr{K})$ belongs to $\boldsymbol{K}$, as well.

Denote the set of all the greatest elements in the equivalence classes of 2.1, i.e. the set of all $\mathscr{K} \in \boldsymbol{Q}$ such that $\mathfrak{c}(\mathscr{K}) = \mathscr{K}$, by $\boldsymbol{C}$ and, furthermore, put

$$\boldsymbol{T} = \boldsymbol{C} \cap \boldsymbol{F}.$$

Evidently, $\boldsymbol{C}$ can also be characterized as the set of all those families $\mathscr{K}$ which satisfy, for any $\mathscr{X} \in \boldsymbol{L}$, the implication

$$\mathscr{X} \approx \mathscr{K} \to \mathscr{X} \subseteqq \mathscr{K}$$

(or, even the stronger implication $\mathscr{X} \ll \mathscr{K} \to \mathscr{X} \subseteqq \mathscr{K}$). Thus, we deduce that $\mathfrak{c}$ is an (idempotent) closure operator; for, if $\mathscr{K}_\omega \in \boldsymbol{C} (\omega \in \Omega)$, then evidently

$$\bigcap_{\omega \in \Omega} \mathscr{K}_\omega \in \boldsymbol{C}.$$

Also, the preorder $\ll$ induces an order in the set of all equivalence classes, and thus, it induces an order in $\boldsymbol{C}$. It turns out that this order in $\boldsymbol{C}$ induced by $\ll$ coincides with the order by inclusion.

Now, define the 'duality' map ∂ in $\boldsymbol{Q}$ as follows: For $\mathscr{K} \in \boldsymbol{Q}$, let $\partial\mathscr{K}$ be given by

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$$L \in \partial\mathscr{K} \to \mathfrak{c}_Q(L) \cap \mathscr{K} = \emptyset.$$

Clearly,

$$\partial\mathscr{K} \in \boldsymbol{Q} \text{ and } \mathscr{K} \cap \partial\mathscr{K} = \emptyset.$$

Also

$$\mathscr{K}_1 \subseteqq \mathscr{K}_2 \to \partial\mathscr{K}_1 \supseteqq \partial\mathscr{K}_2$$

and

$$\partial^2\mathscr{K} = \partial(\partial\mathscr{K}) \supseteqq \mathscr{K}.$$

Furthermore,

$$\partial\mathscr{K}_1 = \partial\mathscr{K}_2 \leftrightarrow \mathscr{K}_1 \approx \mathscr{K}_2$$

and

$$\partial^2\mathscr{K} = \mathfrak{c}(\mathscr{K}).$$

Hence,

2.2. The mapping ∂ defines for the elements of $\boldsymbol{Q}$ a Galois connection (cf. Ore [5]). The operator $\mathfrak{c}$ is the corresponding Galois closure operation and $\boldsymbol{C}$— the set of all closed elements.

In the introduction, the importance of the mapping ∂ in relation to dependence over modules was mentioned. It stems from the fact that, for any $\mathscr{K} \in \boldsymbol{Q}$, there exist maximal independent subsets consisting of elements whose orders belong to $\mathscr{K} \cup \partial\mathscr{K}$ in an arbitrary R-module (cf. [1]).

3. Distinguished submodules

The value of the concepts of an F-family and an T-family (i.e. a family belonging to $\boldsymbol{T} = \boldsymbol{C} \cap \boldsymbol{F}$) of left ideals will be apparent from the following theorems.

THEOREM 3.1. *Let $\mathscr{K} \in \mathbf{L}$. Then, in any R-module M, the subset $M_{\mathscr{K}} \subseteqq M$ is an R-submodule of M if and only if $\mathscr{K}$ is an F-family.*

PROOF. First, let M be an R-module, $\mathscr{K} \in \boldsymbol{F}$ and m_1, m_2 two non-zero elements of $M_{\mathscr{K}} \subseteqq M$. Since $O(\rho m_1) = O(m_1) : \rho$, $\rho m_1 \in M_{\mathscr{K}}$ by (Q). Also, since $O(m_1+m_2) \supseteqq O(m_1) \cap O(m_2)$, $m_1+m_2 \in M_{\mathscr{K}}$ in view of (I) and (E). Thus, if $\mathscr{K}$ is an F-family, then $M_{\mathscr{K}}$ is a submodule of M.

On the other hand, let $\mathscr{K}$ possess the property that, for any R-module M, $M_{\mathscr{K}}$ is always a submodule of M. Consider for a moment the R-module

$$M^* = R \bmod L_1 \oplus R \bmod L_2$$

with $L_i \in \mathscr{L}$ and denote by $\bar{\varepsilon}_i$ the class $\varepsilon + L_i$ of $R \bmod L_i$ $(i = 1, 2)$; thus, the general element of M^* can be written in the form $\rho_1\bar{\varepsilon}_1 + \rho_2\bar{\varepsilon}_2$. Clearly, since $O(\bar{\varepsilon}_i) = L_i$,

$$O(\rho_1\bar{\varepsilon}_1) = L_1 : \rho_1$$

and

$$O(\bar{\varepsilon}_1 + \bar{\varepsilon}_1) = L_1 \cap L_2.$$

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Also, $O(\bar{\varepsilon}_1+\bar{\varepsilon}_2) = L_1$ provided that $L_1 \subseteqq L_2$. Therefore, if $L_1 \in \mathscr{K}$, then necessarily $L_1 : \rho_1 \in \mathscr{K}^0$ $(= \mathscr{K} \cup \{R\})$ and if, moreover, $L_2 \in \mathscr{K}$, then also $L_1 \cap L_2 \in \mathscr{K}$; hence, $\mathscr{K}$ satisfies (Q) and (I). Furthermore, if $L_1 \in \mathscr{K}$ and $L_1 \subseteqq L_2$, then both $\bar{\varepsilon}_1$ and $\bar{\varepsilon}_1+\bar{\varepsilon}_2$ belong to $M_{\mathscr{K}}$ and thus

$$L_2 = O(\bar{\varepsilon}_2) = O\big((\bar{\varepsilon}_1+\bar{\varepsilon}_2)-\bar{\varepsilon}_1\big) \in \mathscr{K}.$$

We conclude that $\mathscr{K}$ satisfies (E) and consequently, it is an F-family.

The assumption in the following theorem is devised to fit the application in § 4.

THEOREM 3.2. *If $\mathscr{K} \in \mathbf{Q}$ satisfies* (E) *and, moreover, contains all proper essential left ideals of R, then $\mathfrak{c}(\mathscr{K})$ is an F-family, and thus a T-family. In particular, if $\mathscr{K}$ is an F-family containing all proper essential left ideals of R, then $\mathfrak{c}(\mathscr{K})$ is a T-family.*

PROOF. First let us prove that (I) holds for $\mathfrak{c}(\mathscr{K})$. This follows immediately from the fact that, for any $\mathscr{K} \in \mathbf{Q}$, $\partial\mathscr{K}$ always satisfies (I):

Let $L = L_1 \cap L_2$ with $L_i \in \partial\mathscr{K}$ $(i = 1, 2)$ and assume that $\rho \in R\diagdown L$ exists such that $L : \rho \in \mathscr{K}$. Since

$$L : \rho = (L_1 : \rho) \cap (L_2 : \rho),$$

$(L_2 : \rho)\diagdown(L_1 : \rho)$ is necessarily non-empty; but, for any element σ of this set

$$R \neq L_1 : \sigma\rho = (L_1 : \sigma\rho) \cap (L_2 : \sigma\rho) = L : \sigma\rho \in \mathscr{K},$$

a contradiction of $L_1 \in \partial\mathscr{K}$. Hence, $\partial\mathscr{K}$, as well as, $\mathfrak{c}(\mathscr{K}) = \partial(\partial\mathscr{K})$ satisfies (I).

Now, to complete the proof of the theorem only the validity of (E) for $\mathfrak{c}(\mathscr{K})$ need to be verified. Thus, let $K_1 \subseteqq K_2$ with $K_1 \in \mathfrak{c}(\mathscr{K})$. In order to show that $K_2 \in \mathfrak{c}(\mathscr{K})$ it is sufficient to prove that

$$\mathfrak{c}_Q(K_2) \ll \mathfrak{c}(\mathscr{K}).$$

Hence, take an arbitrary element $K_2 : \rho$ of $\mathfrak{c}_Q(K_2)$; here

$$\rho \in R\diagdown K_2 \subseteqq R\diagdown K_1.$$

Assume that $K_2 : \rho \notin \mathscr{K}$. Then, $K_2 : \rho$ is not essential and thus, there exists $\sigma_1 \in R$ such that

$$(K_2 : \rho) \cap R\sigma_1 = \{0\}.$$

Furthermore, since

$$\sigma_1 \in R\diagdown(K_2 : \rho) \subseteqq R\diagdown(K_1 : \rho), \text{ i.e. } \sigma_1\rho \in R\diagdown K_1,$$

there is $\sigma_2 \in R$ such that $K_1 : \sigma_2\sigma_1 \in \mathscr{K}$. Put $\sigma = \sigma_2\sigma_1$. Hence,

$$(K_2 : \rho) : \sigma = K_2 : \sigma\rho \neq R,$$

and since $K_2 : \sigma\rho \supseteqq K_1 : \sigma\rho$, it belongs to $\mathscr{K}$. The proof is completed.

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THEOREM 3.3. *Let $\mathscr{K}$ be a T-family. Then, for any R-module M, the quotient module $M/M_{\mathscr{K}}$ has no element of order from $\mathscr{K}$, i.e.*

$$(M/M_{\mathscr{K}})_{\mathscr{K}} = \{0\}.$$

Moreover, no other F-family equivalent to $\mathscr{K}$ possesses this property.[1]

PROOF. We present an indirect proof of the first part of the theorem. Suppose that $O(\bar{m}) \in \mathscr{K}$ for a certain $\bar{m} \in M/M_{\mathscr{K}}$ containing $m \in M$. We are going to show that, as a consequence,

$$\mathfrak{c}_Q(O(m)) \ll \mathscr{K},$$

i.e. $O(m) \in \mathscr{K}$, contradicting the assumption of $\bar{m} \neq \bar{0}$. Hence, take $L \in \mathfrak{c}_Q(O(m))$:

$$L = O(m) : \rho = O(\rho m) \text{ with } \rho m \neq 0,$$

and assume

$$O(\rho m) \nsubseteq O(\rho \bar{m});$$

otherwise $O(\rho m) = O(\rho \bar{m}) \in \mathscr{K}$. Thus, for a $\sigma \in O(\rho \bar{m}) \setminus O(\rho m)$ we get

$$L : \sigma = O(\sigma \rho m) \in \mathscr{K},$$

i.e. $\mathfrak{c}_Q(O(m)) \ll \mathscr{K}$, as required.

In order to prove the other part of the theorem, assume that there is an F-family $\mathscr{K}_1$ equivalent to $\mathscr{K}$ such that

$$(M/M_{\mathscr{K}_1})_{\mathscr{K}_1} = \{0\}$$

for any R-module M. Then, there can be no left ideal $L \in \mathscr{K} \setminus \mathscr{K}_1$; for, otherwise the R-module

$$\bar{M} = R \bmod L/(R \bmod L)_{\mathscr{K}_1}$$

would be a non-zero R-module such that

$$0 \neq \bar{m} \in \bar{M} \to O(\bar{m}) \in \mathscr{K} \setminus \mathscr{K}_1,$$

which is, because of $\mathscr{K} \approx \mathscr{K}_1$, impossible. Hence $\mathscr{K} = \mathscr{K}_1$.

The proof of the theorem is completed.

DEFINITION 3.4. Let $\mathscr{K} \in \boldsymbol{F}$. An R-module M is said to be a $\mathscr{K}$-module if $M_{\mathscr{K}} = M$; M is said to be a $\mathscr{K}$-free-module if $M_{\mathscr{K}} = \{0\}$.

Notice that $\{0\}$ is the only R-module which is simultaneously a $\mathscr{K}$-module and a $\mathscr{K}$-free-module and that the orders of non-zero elements of a $\mathscr{K}$-free-module belong to $\partial\mathscr{K}$. We shall see that the concepts of a $\mathscr{K}$- and $\mathscr{K}$-free-modules will be particularly valuable in the case when $\mathscr{K} \in \boldsymbol{T}$. Then, inspired by the radical theory of Kuroš [4] (cf. next Theorems 3.6

[1] The latter statement can be generalized (see [2]).

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and 3.7) we can speak about $\mathscr{K}$-radical and $\mathscr{K}$-semisimple modules.[2]

The following three theorems will describe some fundamental properties of $\mathscr{K}$- and $\mathscr{K}$-free-modules.

THEOREM 3.5. (a) *A submodule of a $\mathscr{K}$-module, or a $\mathscr{K}$-free-module, is a $\mathscr{K}$-module, or a $\mathscr{K}$-free-module, respectively.*

(b) *An R-module generated by a family of its $\mathscr{K}$-submodules, or $\mathscr{K}$-free-submodules, is a $\mathscr{K}$-module, or a $\mathscr{K}$-free-module, respectively.*

(c) *The direct sum of $\mathscr{K}$-modules, or $\mathscr{K}$-free-modules, is a $\mathscr{K}$-module, or a $\mathscr{K}$-free-module, respectively.*

(d) *The direct product of $\mathscr{K}$-free-modules is a $\mathscr{K}$-free-module.*

(e) *An extension of a $\mathscr{K}$-free-module by a $\mathscr{K}$-free module is a $\mathscr{K}$-free-module.*

PROOF. The statements of (a), (b), (c) and (d) are obvious. In order to prove (e), consider an R-module M with a submodule $N \subseteqq M$ such that both N and M/N are $\mathscr{K}$-free. Thus, $O(m) \notin \mathscr{K}$ for $m \in N$; also, if $m \in M \setminus N$, then $O(m) \subseteqq O(\bar{m}) \neq R$, where $\bar{m} = m+N \in M/N$, and therefore, in view of $O(\bar{m}) \notin \mathscr{K}$, $O(m) \notin \mathscr{K}$, as well.

THEOREM 3.6. (a) *In any R-module M, $M_{\mathscr{K}}$ is the (unique) maximal $\mathscr{K}$-submodule of M (and, thus, contains any other $\mathscr{K}$-submodule of M).*

(b) *Let ϕ be a homomorphism of an R-module M into an R-module M'. Then, the restriction of ϕ to $M_{\mathscr{K}}$ is a homomorphism of $M_{\mathscr{K}}$ into $M'_{\mathscr{K}}$. In particular, every homomorphic image of a $\mathscr{K}$-module is again a $\mathscr{K}$-module.*

(c) *For any R-module M, every $\mathscr{K}$-free homomorphic image of M is a homomorphic image of $M/M_{\mathscr{K}}$.*

PROOF. (a) is evident. Taking into account the obvious fact that

$$O(m) \subseteqq O(\phi(m)) \text{ for every } m \in M,$$

we deduce immediately that a homomorphic image of an element of $M_{\mathscr{K}}$ is either zero of an element or order from $\mathscr{K}$. Hence, both (b) and (c) follow.

THEOREM 3.7. *The following two equivalent statements hold if $\mathscr{K} \in \mathbf{T}$:*

(a) *An extension of a $\mathscr{K}$-module by a $\mathscr{K}$-module is a $\mathscr{K}$-module.*

(b) *For any R-module M, $M/M_{\mathscr{K}}$ is a $\mathscr{K}$-free-module.*

PROOF. First, let us prove the equivalence of the statements (a) and (b):

(a) $\to$ (b). Assume that, for a certain R-module M, $M/M_{\mathscr{K}}$ is not $\mathscr{K}$-free, i.e.

[2] Another possibility would be to define, for any $\mathscr{K} \subseteq \mathscr{L}$ and any R-module M the $\mathscr{K}$-submodule and the $\mathscr{K}$-radical of M as the $\mathfrak{c}_F(\mathscr{K})$-submodule and the $\mathfrak{c}_F(\mathscr{K})$-submodule (in an obvious meaning) of M, respectively.

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$$(M/M_{\mathscr{K}})_{\mathscr{K}} = M^*/M_{\mathscr{K}} \neq \{0\}.$$

Thus, by (a), since $M^*/M_{\mathscr{K}}$ is a $\mathscr{K}$-module, M^* is a $\mathscr{K}$-module, as well. But, on the other hand, clearly

$$(M^*)_{\mathscr{K}} = M_{\mathscr{K}} \neq M^*,$$

— a contradiction.

(b) $\rightarrow$ (a). Let M be an R-module with a $\mathscr{K}$-submodule N such that M/N is a $\mathscr{K}$-module. Here, $N \subseteqq M_{\mathscr{K}}$ and thus,

$$M/M_{\mathscr{K}} \cong (M/N)/(M_{\mathscr{K}}/N);$$

therefore, in view of Theorem 3.6(b), $M/M_{\mathscr{K}}$ is — as a homomorphic image of a $\mathscr{K}$-module — a $\mathscr{K}$-module. Also, according to (b), $M/M_{\mathscr{K}}$ is a $\mathscr{K}$-free-module. Hence, $M = M_{\mathscr{K}}$, as required.

Finally, an application of Theorem 3.3 completes the proof.

Here, before presenting an application of the obtained results, let us point out that a more detailed study of the Q-families, their equivalence classes and properties related to the 'radical' properties of modules can be found in [2].

4. Torsion and torsion-free modules

In the preceding section, a general method of defining a 'torsion' element of a module and thus, torsion and torsion-free modules was described. In what follows, we suggest a particular choice of such a definition which seems to be the most appropriate at present: a torsion element is an element of so-called maxi order. To justify the latter statement let us express our belief that the elements of essential orders should be classified as torsion elements; then, it turns out that the family $\mathscr{T}$ of all maxi ideals belongs to $\boldsymbol{T}$ and is the least family containing all proper essential ideals and satisfying the 'radical' requirements for $M_{\mathscr{T}}$. Moreover, the corresponding concepts of torsion and torsion-free modules enable us to extend some other features of the abelian group theory to modules (cf. [1]). We shall also show that $M_{\mathscr{T}}$ is, in fact, $Z_2(M)$ of Goldie introduced in [3] under some restricting conditions for R and M.

Let us start with the definition of a maxi ideal.

DEFINITION 4.1. An ideal $L \in \mathscr{L}$ is said to be maxi if, for each $\rho \in R \setminus L$, there is $\sigma \in R$ such that $L : \sigma\rho \neq R$ is essential. Also, $L \in \mathscr{L}$ is said to be a mini ideal if no quotient ideal $L : \rho \neq R$ is essential.

Denoting by $\mathscr{E}$, $\mathscr{T}$ and $\mathscr{F}$ the families of all proper essential ideals, all maxi ideals and all mini ideals, respectively, we can readily see that

$$\partial\mathscr{E} = \mathscr{F} \text{ and } \partial^2\mathscr{E} = \partial\mathscr{F} = \mathscr{T}.$$

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Thus, since evidently $\mathscr{E} \in \boldsymbol{F}$, we get by virtue of Theorem 3.2 the following

THEOREM 4.2. *The family of all maxi ideals* $\mathscr{T}$ *belongs to* $\boldsymbol{T}$ *and thus, for any R-module M,* $M_{\mathscr{T}} \subseteqq M$ *possesses the properties described in Theorems* 3.5, 3.6 *and* 3.7.

It is easy to see that $L \in \mathscr{L}$ is a maxi ideal if and only if the submodule $(R \bmod L)_{\mathscr{E}}$ is essential in $R \bmod L$. From here and the fact that an ideal $E \supseteqq L$ is essential in R if and only if the submodule $E \bmod L$ is essential in $R \bmod L$, the two statements mentioned in the introduction to this section follow immediately ((a) follows also from Theorem 3.3):

THEOREM 4.3. (a) *Every F-family* $\mathscr{K}$ *containing* $\mathscr{E}$ *such that* $M_{\mathscr{K}} \subseteqq M$ *satisfies the 'radical' properties of Theorem* 3.7 *contains* $\mathscr{T}$ *(and is a T-family).*

(b) *An ideal* $L \in \mathscr{L}$ *is a maxi ideal if and only if there is an essential ideal* E *in* R *such that* $L : \rho$ *is essential for every* $\rho \in E$.

At the end, we like to include the following two remarks:

Consider briefly the case when R is the ring of all integers, i.e. the case of abelian groups G. Of course, $G_{\mathscr{T}}$ is the maximal torsion subgroup of G. Applying the method of § 3, we can show readily that any $\mathscr{K}$-submodule satisfying 'radical' properties of Theorem 3.7 is derived from $\mathscr{K} \in \boldsymbol{T}$, and that $G_{\mathscr{T}}$ is the 'greatest radical': Denote by Π^* a subset of the set Π of all primes and by $\mathscr{P}^*$ the family of all (principal) ideals of the form

$$\langle p_1^{k_1} p_2^{k_2} \cdots p_n^{k_n} \rangle \text{ with } k_i > 0 \text{ and } p_i \in \Pi^* \qquad (1 \leqq i \leqq n).$$

Clearly, $\mathscr{P}^* \in \boldsymbol{F}$ and, in fact, $\mathscr{P}^* \in \boldsymbol{T}$. Thus, for any selection Π^* of primes there is a $\mathscr{P}^*$-radical $G_{\mathscr{P}^*}$ of G:

If $G_{\mathscr{T}} = \sum_{p \in \Pi} G_p$ is the decomposition into the p-primary components, then $G_{\mathscr{P}^*} = \sum_{p \in \Pi^*} G_p$.

It is easy to see that every 'radical' must be of this form. In particular, $G_{\mathscr{P}^*} = \{0\}$ for $\Pi^* = \emptyset$ and $G_{\mathscr{P}^*} = G_{\mathscr{T}}$ for $\Pi^* = \Pi$.

The other remark concerns the fact that for some rings R, the families $\mathscr{T}$ and $\mathscr{E}$ of all maxi and all essential left ideals of R, respectively, coincide. As a matter of fact, we can give a simple necessary and sufficient condition for the equality $\mathscr{T} = \mathscr{E}$:

THEOREM 4.4. *The equality* $\mathscr{T} = \mathscr{E}$ *holds if and only if the singular ideal* S *of* R *equals to* $\{0\}$; *here,* S *is the ideal of all elements of* R *whose left annihilators are essential in* R.

Indeed, the sufficiency (cf. [3]) follows immediately from the fact that, for any non-essential $L \in \mathscr{T}$, there is a non-zero $\rho \in R$ such that

$$L : \rho = \{0\} : \rho.$$

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Hence, there exists $\sigma \in R$ such that $L : \sigma\rho \neq R$ is essential in R; thus, $O \neq \sigma\rho \in S$, i.e. $S \neq \{0\}$.

In order to prove the necessity, assume that $S \neq \{0\}$. Denote by L a maximal left ideal of R such that $L \cap S = \{0\}$. Clearly, the left ideal E generated by L and S is essential in R. Also, for any $\rho \in E$, $\rho = \lambda + \sigma$ with $\lambda \in L$ and $\sigma \in S$,

$$L : \rho \supseteqq \{0\} : \sigma$$

is essential in R. Hence, in view of Theorem 4.3(b), $L \in \mathscr{T}$. Since $L \notin \mathscr{E}$, we get $\mathscr{T} \neq \mathscr{E}$.

In conclusion, let us note that the condition $S = \{0\}$ is satisfied in a ring R if 0 is the only nilpotent element of R and the ascending chain condition holds for the annihilator left ideals of single elements of R. For, in any ring R, the assumption that $\rho \in R$ is not nilpotent and $\{0\} : \rho$ is essential in R implies readily that

$$\{0\} : \rho^{n+1} \neq \{0\} : \rho^n \text{ for every natural } n.$$

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