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BLOWUP FOR A DEGENERATE AND SINGULAR PARABOLIC EQUATION WITH NON-LOCAL SOURCE AND ABSORPTION*

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Abstract. This paper deals with the following degenerate and singular equation

$$x^q u_t - (x^p u_x)_x = \int_0^a u^m(x, t)dx - ku^n(x, t)$$

with non-local source and absorption. The existence of a unique classical non-negative solution is established and the sufficient conditions for the solution that exists globally or blows up in finite time are obtained.

AMS (2000) *Subject Classification.* 35K55, 35K57, 35K65.

1. Introduction. In this paper, we consider the following degenerate and singular nonlinear reaction–diffusion equation

$$\begin{cases} x^q u_t - (x^p u_x)_x = \displaystyle\int_0^a u^m(x, t)dx - ku^n(x, t), & (x, t) \in (0, a) \times (0, T), \\ u(0, t) = u(a, t) = 0, & t \in (0, T), \\ u(x, 0) = u_0(x), & x \in [0, a], \end{cases} \tag{1.1}$$

where $u_0(x) \in C^{2+\alpha}([0, a])$ for some $\alpha \in (0, 1)$ are non-negative non-trivial functions. $u_0(0) = u_0(a) = 0$, $u_0(x) \geq 0$, $u_0(x)$ satisfies the compatibility condition, $T > 0$, $a > 0$, $p \in [0, 1)$, $|q| + p \neq 0$, $m,\ n > 1$, $k > 0$.

Let $D = (0, a)$ and $\Omega_t = D \times (0, t]$, $\bar{D}$ and $\bar{\Omega}_t$ are their closures, respectively. Since $|q| + p \neq 0$, the coefficients of u_t, u_x, u_{xx} may tend to 0 or ∞ as x tends to 0, we can regard the equation as degenerate and singular.

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Let us rewrite the problem (1.1) as

$$\begin{cases} u_t - x^{p-q}u_{xx} - px^{p-q-1}u_x = x^{-q}\int_0^a u^m dx - x^{-q}ku^n & (x,t) \in (0,a)\times(0,T), \\ u(0,t) = u(a,t) = 0, & t \in (0,T), \\ u(x,0) = u_0(x), & x \in [0,a], \end{cases} \tag{1.2}$$

we need more assumptions: $p - q < 2$ (for a well posedness of the Dirichlet problem at $x = 0$) and $x^{-q} \in L^1(0, a)$ (otherwise, the solution $u(x, t)$ is complete blowup at $t = 0$).

In [**18**], Ockendon studied the flow in a channel of a fluid whose viscosity depends on temperature $u(x, t)$ and gave the following model

$$xu_t = u_{xx} + e^u. \tag{1.3}$$

Floater [**11**] and Chan et al. [**4**] approximated e^u by u^p and investigated the blowup properties of the following parabolic problem

$$\begin{cases} x^q u_t - u_{xx} = u^p, & (x,t) \in (0,a)\times(0,T), \\ u(0,t) = u(a,t) = 0, & t \in (0,T), \\ u(x,0) = u_0(x), & x \in [0,a], \end{cases} \tag{1.4}$$

where $q > 0$ and $p > 1$. Under certain conditions on the initial datum $u_0(x)$, Floater proved that the solution $u(x, t)$ of (1.3) blows up at the boundary $x = 0$ for the case $1 < p \leq q + 1$. This is contrasts with one of the results in [**12**], which showed that for the case $q = 0$, the blowup set of the solution $u(x, t)$ of (1.3) is a proper compact subset of D.

Budd et al. [**2**] generalized the results in [**11**] to the following degenerate quasi-linear parabolic equation

$$x^q u_t = (u^m)_{xx} + u^p, \tag{1.5}$$

with homogeneous Dirichlet conditions in the critical exponent $q = (p - 1)/m$, where $q > 0$, $m \geq 1$ and $p > 1$. They pointed out that the general classification of blowup solution for the degenerate equation (1.5) stays the same for the quasi-linear equation (see [**2, 22**])

$$u_t = (u^m)_{xx} + u^p. \tag{1.6}$$

In [**5**], Chen et al. considered the following degenerate nonlinear reaction–diffusion equation with non-local source

$$\begin{cases} x^q u_t - (x^\gamma u_x)_x = \int_0^a u^p dx, & (x,t) \in (0,a)\times(0,T), \\ u(0,t) = u(a,t) = 0, & t \in (0,T), \\ u(x,0) = u_0(x), & x \in [0,a], \end{cases} \tag{1.7}$$

they established the local existence and unique solution of a classical solution. Under approximate hypotheses, they also got some sufficient conditions for the global

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existence and blowup of a positive solution. Furthermore, under certain condition, it is proved that the blowup set of the solution is the whole domain.

In [**7**], Chen et al. considered the following more general problem

$$\begin{cases} x^q u_t - (x^\gamma u_x)_x = \int_0^a f(u)dx, & (x,t) \in (0,a) \times (0,T), \\ u(0,t) = u(a,t) = 0, & t \in (0,T), \\ u(x,0) = u_0(x), & x \in [0,a], \end{cases} \tag{1.8}$$

they established the local existence and uniqueness of a classical solution. Under approximate hypotheses, they obtain some sufficient conditions for the global existence and blowup of a positive solution.

In [**17**], Liu et al. considered the following degenerate parabolic equation with non-local source

$$\begin{cases} u_t - x^\alpha (u^m)_{xx} = \int_0^a u^p dx - ku^q, & (x,t) \in (0,a) \times (0,T), \\ u(0,t) = u(a,t) = 0, & t \in (0,T), \\ u(x,0) = u_0(x), & x \in [0,a], \end{cases} \tag{1.9}$$

they established the local existence and uniqueness of a classical solution. Under approximate hypotheses, they also get some sufficient conditions for blowup of a positive solution. Furthermore, under certain conditions, it is proved that the blowup set of the solution is the whole domain.

In [**25**], we considered the following parabolic system

$$\begin{cases} x^{q_1} u_t - (x^{r_1} u_x)_x = \int_0^a v^{p_1} dx, & (x,t) \in (0,a) \times (0,T), \\ x^{q_2} v_t - (x^{r_2} v_x)_x = \int_0^a u^{p_2} dx, & (x,t) \in (0,a) \times (0,T), \\ u(0,t) = u(a,t) = v(0,t) = v(a,t) = 0, & t \in (0,T), \\ u(x,0) = u_0(x), \ v(x,0) = v_0(x), & x \in [0,a], \end{cases} \tag{1.10}$$

under certain conditions, we proved that the blowup set of the solution is the whole domain. The existence of a unique classical non-negative solution is established and the sufficient conditions for solution that exists globally or blows up in finite time are obtained.

In [**16**], Li et al. considered the following degenerate and singular nonlinear reaction-diffusion equations with localized sources

$$\begin{cases} x^{q_1} u_t - (x^{r_1} u_x)_x = v^{p_1}(x_0,t), & (x,t) \in (0,a) \times (0,T), \\ x^{q_2} v_t - (x^{r_2} v_x)_x = u^{p_2}(x_0,t), & (x,t) \in (0,a) \times (0,T), \\ u(0,t) = u(a,t) = v(0,t) = v(a,t) = 0, & t \in (0,T), \\ u(x,0) = u_0(x), \ v(x,0) = v_0(x), & x \in [0,a], \end{cases} \tag{1.11}$$

under certain conditions, the existence of a unique classical non-negative solution is established and the sufficient conditions for the solution exists globally and blows up in finite time are obtained. Furthermore, under certain conditions, it is proved that the blowup set of the solution is the whole domain.

Motivated by the above cited papers, we consider the problem (1.1) and before leaving this section, we should remark the blowup properties for non-local semi-linear parabolic equations (see [**10**, **19**, **20**, **23**, **24**] and the references therein) and the degenerate parabolic equations without non-local terms (for example, see [**1**, **4**, **5**, **8**, **13**, **17**, **21**] and the references therein).

This paper is organized as follows: in the next section, we establish the comparison principle to problem (1.1). In Section 3, we show the existence of a unique classical solution. In Section 4, we give some criteria for the solution to exist globally or blow up in finite time.

THEOREM 1.1. *Let $u(x, t)$ be the non-negative solution of* (1.1). *Let us assume that a non-negative function $w(x, t) \in C(\bar{\Omega}_r) \cap C^{2,1}(\Omega_r)$ satisfies*

$$\begin{cases} x^q w_t - (x^p w_x)_x \geq (\leq) \displaystyle\int_0^a w^m(x, t)dx - kw^n(x, t), & (x, t) \in \Omega_r, \\ w(0, t) \geq (=)0, \quad w(a, t) \geq (=)0, & t \in (0, r), \\ w(x, 0) \geq (\leq)u_0(x), & x \in [0, a]. \end{cases} \tag{1.12}$$

Then, $w(x, t) \geq (\leq)u(x, t)$ in $[0, a] \times [0, r)$.

THEOREM 1.2. *There exists $t_0(< T)$ such that* (1.1) *has a unique non-negative solution $u(x, t) \in C(\bar{\Omega}_{t_0}) \cap C^{2,1}(\Omega_{t_0})$.*

THEOREM 1.3. *Let T be the supremum over t_0 for which there is a unique non-negative solution $u(x, t) \in C(\bar{\Omega}_{t_0}) \cap C^{2,1}(\Omega_{t_0})$ of* (1.1). *Then* (1.1) *has a unique non-negative solution $u(x, t) \in C([0, a] \times [0, T)) \cap C^{2,1}((0, a) \times (0, T))$. If $T < +\infty$, we have $\limsup_{t\to T} \max_{x\in[0,a]} |u(x, t)| = +\infty$.*

THEOREM 1.4. (1) *Assume $m < n$, then the solution of* (1.1) *$u(x, t)$ exists globally.*
(2) *Assume $m > n > 1$, then the solution of* (1.1) *$u(x, t)$ exists globally if $u_0(x) \leq (k/a)^{\frac{1}{m-n}}$.*
(3) *Let $u(x, t)$ be the solution of* (1.1). *If $u_0(x) \leq a_1\varphi(x)$, then $u(x, t)$ exists globally, where,*

$$\begin{aligned} \varphi(x) &= \frac{a^{2-p}}{2-p}\left(\frac{x}{a}\right)^{1-p}\left(1 - \frac{x}{a}\right), \\ a_1 &= \left(\frac{a^{(2-p)m+1}\int_0^1 x^{m(1-p)}(1-x)^m dx}{(2-p)^m}\right)^{1/(1-m)}. \end{aligned} \tag{1.13}$$

To give the blowup results, we assume the the initial data $u_0(x)$ satisfying the following assumption
(H) *For any $x \in (0, a)$, the initial data satisfies $(x^p u_{0x})_x + \int_0^a u_0^m dx - ku_0^n \geq 0$.*

Furthermore, we need the following result, which is proved in [**24**].

LEMMA 1.5. *Let $m > 1$ and $a > k$, then there exists a constant $\delta \in (0, 1)$ and a function $\omega(x) \in C_0^\infty(0, a)$, such that*

$$\int_0^a \omega(x)dx = 1, \quad 0 \le k\omega(x) \le 1 - \delta, \quad \int_0^a w^m(x) - kw^m(x) \ge \delta, \quad \textit{for } x \in (0, a). \tag{1.14}$$

THEOREM 1.6. (1) *Assume $m = n > 1$, $a > k$ and the assumption (H) holds. Then the solution of problem* (1.1) *blow up in finite time if $u_0(x) \ge a_0\omega(x)$, where a_0 is sufficient large such that*

$$a_0 \ge \max\left\{ \left(\frac{\lambda_0}{\delta}\right)^{\frac{1}{m-1}}, \ \frac{a\lambda_0^{\frac{1}{m-1}}}{\int_0^a \omega(x)dx} \right\},$$

$\lambda_0 = \max_{x\in[0,a]} -(x^p\omega_x)_x$ *and $\omega(x)$, δ are given by Lemma* 1.5.
(2) *Assume $m > n > 1$, $q > p - 1$ and the assumption (H) holds. Then the solution of problem* (1.1) *blow up in finite time if $u_0(x) \ge a_1\varphi(x)$, where*

$$\varphi(x) = k_1 x^{(1-p)/2} J_{(1-p)/(q+2-p)}\left(\frac{2\sqrt{\mu}}{q+2-p} x^{(q+2-p)/2}\right),$$

$J_{(1-p)/(q+2-p)}$ denote the Bessel function of the first kind of order $(1-p)/(q+2-p)$, μ is first root of $J_{(1-p)/(q+2-p)}\left(\frac{2\sqrt{\lambda}}{q+2-p} a^{(q+2-p)/2}\right)$ and a_1 is sufficient large such that the following inequalities hold

$$a_1^m \int_0^a \varphi^m dx - ka_1^n\varphi^n \ge a_1\mu,$$

$$a_1^{m-1} \int_0^a \varphi dx \left(\int_0^a \varphi^m dx\right)^{\frac{m-1}{m}} \ge 4\mu a^{\frac{m-1}{m}},$$

$$a_1^{m-n} \int_0^a \varphi dx \left(\int_0^a \varphi^{n+1} dx\right)^{\frac{m-n}{n}} \ge 2k\left(\int_0^a \varphi^{\frac{m}{m-n}} dx\right)^{\frac{m-n}{n}}.$$

2. Comparison principle. In this section, we prove the comparison principle, i.e., Theorem 1.1. We start with the following lemma.

LEMMA 2.1. *Let $b_1(x, t)$ and $b_2(x, t)$ be continuous functions defined on $[0, a] \times [0, r]$ for any $r \in (0, T)$, $b_2(x, t) \ge 0$ in Ω_r. Let $u(x, t) \in C(\bar{\Omega}_r) \cap C^{2,1}(\Omega_r)$ satisfies*

$$\begin{cases} x^q u_t - (x^p u_x)_x \ge b_1(x, t)u(x, t) + \int_0^a b_2(x, t)u(x, t)dx, & (x, t) \in \Omega_r, \\ u(0, t) \ge 0, \quad u(a, t) \ge 0, & t \in (0, r), \\ u(x, 0) \ge 0, & x \in [0, a], \end{cases} \tag{2.1}$$

then, $u(x, t) \ge 0$ in $[0, a] \times [0, T)$.

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Proof. At first, similar to the proof of Lemma 2.1 in [**24**], by using Lemma 2.2.1 of [**19**], we can obtain the following conclusion

If $w(x,t) \in C(\bar{\Omega}_r) \cap C^{2,1}(\Omega_r)$ satisfies

$$\begin{cases} x^q w_t - (x^p w_x)_x \geq b_1(x,t)w(x,t) + \int_0^a b_2(x,t)w(x,t)dx, & (x,t) \in \Omega_r, \\ w(0,t) > 0, \quad w(a,t) \geq 0, & t \in (0,r), \\ w(x,0) \geq 0, & x \in [0,a], \end{cases} \tag{2.2}$$

then, $w(x,t) > 0$, $(x,t) \in \Omega_r$.

Next, let $p' \in (p,1)$ be positive constant and

$$w(x,t) = u(x,t) + \eta\left(1 + x^{p'-p}\right)e^{ct}, \tag{2.3}$$

where $\eta > 0$ is sufficiently small and c is a positive constant to be determined. Then $w(x,t) > 0$ on the parabolic boundary of Ω_r and

$$\begin{aligned} & x^q w_t - (x^p w_x)_x - b_1(x,t)w(x,t) - \int_0^a b_2(x,t)w(x,t)dx \\ & \geq x^q \eta\left(1 + x^{p'-p}\right)ce^{ct} + \frac{(p'-p)(1-p')\eta e^{ct}}{x^{2-p'}} - b_1(x,t)\eta\left(1 + x^{p'-p}\right)e^{ct} \\ & \quad - \int_0^a b_2(x,t)\eta\left(1 + x^{p'-p}\right)e^{ct}dx \\ & \geq \eta e^{ct}\left(cx^q + \frac{(p'-p)(1-p')}{x^{2-p'}} - \left(1 + a^{p'-p}\right)\max_{(x,t)\in\bar{\Omega}_r} b_1(x,t)\right. \\ & \quad \left. - a\left(1 + a^{p'-p}\right)\max_{(x,t)\in\bar{\Omega}_r} b_2(x,t)\right) \\ & \geq \eta e^{ct}\left(cx^q + \frac{(p'-p)(1-p')}{x^{2-p'}}\right. \\ & \quad \left. - (1+a)\left(1 + a^{p'-p}\right)\max\left\{\max_{(x,t)\in\bar{\Omega}_r} b_1(x,t), \max_{(x,t)\in\bar{\Omega}_r} b_2(x,t)\right\}\right). \end{aligned} \tag{2.4}$$

Case 1. If

$$\max\left\{\max_{(x,t)\in\bar{\Omega}_r} b_1(x,t), \max_{(x,t)\in\bar{\Omega}_r} b_2(x,t)\right\} \leq \frac{(p'-p)(1-p')}{a^{2-p}(1+a)(1+a^{p'-p})}. \tag{2.5}$$

It is obvious that

$$x^q w_t - (x^p w_x)_x - b_1(x,t)w(x,t) - \int_0^a b_2(x,t)w(x,t)dx \geq 0. \tag{2.6}$$

Case 2. If

$$\max\left\{\max_{(x,t)\in\bar{\Omega}_r} b_1(x,t), \max_{(x,t)\in\bar{\Omega}_r} b_2(x,t)\right\} > \frac{(p'-p)(1-p')}{a^{2-p}(1+a)(1+a^{p'-p})}. \tag{2.7}$$

Let x_0 be the root of the algebraic equaiton

$$(1+a)\left(1+a^{p'-p}\right)\max\left\{\max_{(x,t)\in\bar{\Omega}_r} b_1(x,t),\ \max_{(x,t)\in\bar{\Omega}_r} b_2(x,t)\right\} = \frac{(p'-p)(1-p')}{x^{2-p'}}. \tag{2.8}$$

let c be sufficiently large that

$$c > \left(\max\left\{\max_{(x,t)\in\bar{\Omega}_r} b_1(x,t),\ \max_{(x,t)\in\bar{\Omega}_r} b_2(x,t)\right\}(1+a)(1+a^{p'-p})\right) \Big/ x_0^q. \tag{2.9}$$

Then we have

$$\begin{aligned}
&x^q w_t - (x^p w_x)_x - b_1(x,t)w(x,t) - \int_0^a b_2(x,t)w(x,t)dx \\
&\geq \begin{cases} \eta e^{ct}\left(\dfrac{(p'-p)(1-p')}{x^{2-p'}} - (1+a)\left(1+a^{p'-p}\right)\max\left\{\max\limits_{(x,t)\in\bar{\Omega}_r} b_1(x,t),\ \max\limits_{(x,t)\in\bar{\Omega}_r} b_2(x,t)\right\}\right), \\ \hfill \text{for } x \leq x_0, \\ \eta e^{ct}\left(cx^q - (1+a)\left(1+a^{p'-p}\right)\max\left\{\max\limits_{(x,t)\in\bar{\Omega}_r} b_1(x,t),\ \max\limits_{(x,t)\in\bar{\Omega}_r} b_2(x,t)\right\}\right), \\ \hfill \text{for } x > x_0, \end{cases} \\
&\geq 0.
\end{aligned} \tag{2.10}$$

From the above two cases, we know that $w(x,t) > 0$ in $[0,a]\times[0,r]$. Letting $\eta \to 0^+$, we have $u(x,t) \geq 0$ in $[0,a]\times[0,r]$. By the arbitrariness of $r \in (0,T)$, we complete the proof of Lemma 2.1. □

Proof of Theorem 1.1. We only consider the case '$\geq$' (as for the other case '$\leq$', the proof is similar). Let $\varphi(x,t) = w(x,t) - u(x,t)$, then $\varphi(x,t)$ satisfies

$$x^q\varphi_t - (x^p\varphi_x)_x \geq \int_0^a (w^m - u^m)\,dx - k\,(w^n - u^n)\,. \tag{2.11}$$

By using the mean value theorem, we get

$$w^m - u^m = m\eta_1^{m-1}(w-u), \quad w^n - u^n = n\eta_2^{n-1}(w-u), \tag{2.12}$$

where η_1, η_2 are some intermediate values between w and u. Since $m \geq n > 1$, we have $m\eta_1^{m-1} \geq 0,\ n\eta_2^{n-1} \geq 0$. So, the function $\varphi(x,t)$ satisfies

$$\begin{cases} x^q\varphi_t - (x^p\varphi_x)_x \geq \displaystyle\int_0^a m\eta_1^{m-1}\varphi(x,t)dx - kn\eta_2^{n-1}\varphi(x,t), & (x,t)\in\Omega_r, \\ \varphi(0,t) \geq 0, \quad \varphi(a,t) \geq 0, & t \in (0,r), \\ \varphi(x,0) \geq 0, & x \in [0,a]. \end{cases} \tag{2.13}$$

Lemma 2.1 ensures that $\varphi(x,t) \geq 0$, that is $w(x,t) \geq u(x,t)$ in $[0,a]\times[0,r)$. We complete the proof of Theorem 1.1. □

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3. Local existence. In this section, we will establish the existence of a unique classical non-negative solution of problem (1.1) and prove Theorems 1.2 and 1.3. We start with the definition of a supersolution of problem (1.1)

DEFINITION. A non-negative function $\bar{u}(x,t)$ is called an supersolution of (1.1). If $\bar{u}(x,t)\in C([0,a]\times[0,T))\cap C^{2,1}((0,a)\times(0,T))$ and satisfies

$$\begin{cases} x^q\bar{u}_t-(x^p\bar{u}_x)_x\geq\int_0^a\bar{u}^m(x,t)dx-k\bar{u}^n(x,t), & (x,t)\in(0,a)\times(0,T),\\ \bar{u}(0,t)\geq 0,\quad \bar{u}(a,t)\geq 0, & t\in(0,T),\\ \bar{u}(x,0)\geq u_0(x), & x\in[0,a]. \end{cases} \tag{3.1}$$

Similarly, $\underline{u}(x,t)\in C([0,a]\times[0,T))\cap C^{2,1}((0,a)\times(0,T))$ is called a subsolution of (1.1) if it satisfies all the reversed inequalities in (3.1).

REMARK. Let $u(x,t)$ be the solution of (1.1), $\bar{u}(x,t)$, $\underline{u}(x,t)$ be the supersolution and subsolution of (1.1), respectively. Then, $\underline{u}(x,t)\leq u(x,t)\leq\bar{u}(x,t)$.

Obviously, $\underline{u}(x,t)=0$ is a subsolution of (1.1), we need to construct a supersolution of (1.1).

LEMMA 3.1. *There exists a positive constant t_0 ($t_0<T$) such that the problem (1.1) has an supersoution $h(x,t)\in C(\bar{\Omega}_{t_0})\cap C^{2,1}(\Omega_{t_0})$.*

Proof. Let $\psi(x)=(\frac{x}{a})^{1-p}(1-\frac{x}{a})+(\frac{x}{a})^{(1-p)/2}(1-\frac{x}{a})^{1/2}$ and k_0 be a positive constant such that $k_0\psi(x)\geq u_0(x)$. Denote the positive constant $\int_0^1(s^{1-p}(1-s)+s^{(1-p)/2}(1-s)^{1/2})^m ds$ by b_0. Let $k_1\in(0,(1-p)/(2-p))$ be positive constant such that

$$k_1\leq\min\left\{\left(b_02^{m+1}a^{3-p}k_0^{m-1}\right)^{-2/(1-p)},\ \left(b_02^{m+1}a^{3-p}k_0^{m-1}\right)^{-2}\right\}. \tag{3.2}$$

Let $k(t)$ be the solution of the following initial value problem

$$\begin{cases} k'(t)=\begin{cases}\dfrac{b_0k^m(t)}{a^{q-1}k_0^q\left(k_0(1-k_0)^{1-p}+k_0^{1/2}(1-k_0)^{(1-p)/2}\right)}, & q\geq 0,\\ \dfrac{b_0k^m(t)}{a^{q-1}(1-k_0)^q\left(k_0(1-k_0)^{1-p}+k_0^{1/2}(1-k_0)^{(1-p)/2}\right)}, & q<0,\end{cases}\\ k(0)=k_0. \end{cases} \tag{3.2}$$

Then the solution is given by

$$k(t)=\begin{cases}\left(k_0^{1-m}+\dfrac{b_0(1-m)t}{a^{q-1}k_0^q\left(k_0(1-k_0)^{1-p}+k_0^{1/2}(1-k_0)^{(1-p)/2}\right)}\right)^{\frac{1}{1-m}}, & q\geq 0,\\ \left(k_0^{1-m}+\dfrac{b_0(1-m)t}{a^{q-1}(1-k_0)^q\left(k_0(1-k_0)^{1-p}+k_0^{1/2}(1-k_0)^{(1-p)/2}\right)}\right)^{\frac{1}{1-m}}, & q<0.\end{cases} \tag{3.3}$$

Since $k(t)$ is an increasing function, we can choose $t_0>0$ such that $k(t)\leq 2k_0$ for all $t\in[0,t_0]$. Let $h(x,t)=k(t)\psi(x)$, then $h(x,t)\geq 0$ in $\bar{\Omega}_{t_0}$. We would like to show that

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$h(x,t)$ is a supersolution of (1.1) in Ω_{t_0}. To do this, let us construct a function J by

$$J = x^q h_t - (x^p h_x)_x - \int_0^a h^m dx + kh^n, \quad (x,t) \in \Omega_{t_0}. \tag{3.4}$$

Then,

$$\begin{aligned}
J &= x^q h_t - (x^p h_x)_x - \int_0^a h^m dx + kh^n \\
&\geq x^q h_t - (x^p h_x)_x - \int_0^a h^m dx \\
&\geq x^q k'(t)\psi(x) + \left(\frac{2-p}{a^{2-p}} + \left(\frac{(1-p)^2}{4} x^{(p-3)/2}(a-x)^{1/2} + \frac{1}{2} x^{(p-1)/2}(a-x)^{-1/2}\right.\right. \\
&\quad \left.\left. + \frac{1}{4} x^{(1+p)/2}(a-x)^{-3/2}\right) \times \frac{1}{a^{1-p/2}}\right) k(t) - ab_0 k^m(t) \\
&\geq x^q k'(t)\psi(x) + x^{(p-1)/2}(a-x)^{-1/2}\frac{k(t)}{2a^{1-p/2}} - ab_0 k^m(t).
\end{aligned} \tag{3.5}$$

For $(x,t) \in (0, ak_1) \times (0, t_0)$,

$$\begin{aligned}
J &\geq x^{(p-1)/2}(a-x)^{-1/2}\frac{k(t)}{2a^{1-p/2}} - ab_0 k^m(t) \\
&\geq \frac{k_0}{2a^{1-p/2}} a^{-1/2}(ak_1)^{(p-1)/2} - ab_0(2k_0)^m \\
&= \frac{k_0 k_1^{(p-1)/2}}{2a^{2-p}} - ab_0(2k_0)^m \\
&\geq 0.
\end{aligned} \tag{3.6}$$

For $(x,t) \in [ak_1, a(1-k_1)] \times (0, t_0)$,

$$\begin{aligned}
J &\geq x^q k'(t)\psi(x) - ab_0 k^m(t) \\
&\geq \begin{cases} a^q k_1^q k'(t)\left(k_1(1-k_1)^{1-p} + k_1^{1/2}(1-k_1)^{(1-p)/2}\right) - ab_0 k^m(t), & q \geq 0, \\ a^q(1-k_1)^q k'(t)\left(k_1(1-k_1)^{1-p} + k_1^{1/2}(1-k_1)^{(1-p)/2}\right) - ab_0 k^m(t), & q < 0, \end{cases} \\
&\geq 0.
\end{aligned} \tag{3.7}$$

For $(x,t) \in (a(1-k_1), a) \times (0, t_0)$,

$$\begin{aligned}
J &\geq x^{(p-1)/2}(a-x)^{-1/2}\frac{k(t)}{2a^{1-p/2}} - ab_0 k^m(t) \\
&\geq \frac{k_0}{2a^{1-p/2}} a^{(p-1)/2}(a - a(1-k_1))^{-1/2} - ab_0(2k_0)^m \\
&= \frac{k_0 k_1^{-1/2}}{2a^{2-p}} - ab_0(2k_0)^m \\
&\geq 0.
\end{aligned} \tag{3.8}$$

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Thus, $J(x,t) \geq 0$ in Ω_{t_0}. It follows from $h(0,t) = h(a,t) = 0$ and $h(x,0) = k_0\psi(x) \geq u_0(x)$ that $h(x,t)$ is a supersolution of (1.1) in Ω_{t_0}. We complete the proof of Lemma 3.1. □

To show the existence of the classical solution $u(x,t)$ of (1.1). Let us introduce a cut-off function $\rho(x)$. By Dunford and Schwartz **[10]**, there exists a undecreasing function $\rho(x) \in C^3(\mathbb{R})$ such that $\rho(x) = 0$ if $x \leq 0$ and $\rho(x) = 1$ if $x \geq 1$. Let $0 < \delta < \frac{1-p}{2-p}a$,

$$\rho_\delta(x) = \begin{cases} 0, & x \leq \delta, \\ \rho\left(\dfrac{x}{\delta} - 1\right), & \delta < x < 2\delta, \\ 1, & x \geq \delta, \end{cases} \tag{3.9}$$

and $u_{0\delta}(x) = \rho_\delta(x)u_0(x)$. We note that

$$\frac{\partial u_{0\delta}(x)}{\partial \delta} = \begin{cases} 0, & x \leq \delta, \\ -\dfrac{x}{\delta^2}\rho'\left(\dfrac{x}{\delta} - 1\right)u_0(x), & \delta < x < 2\delta, \\ 0, & x \geq \delta. \end{cases} \tag{3.10}$$

Since ρ is non-decreasing, we have $\frac{\partial u_{0\delta}(x)}{\partial \delta} \leq 0$. From $0 \leq \rho(x) \leq 1$, we have $u_0(x) \geq u_{0\delta}(x)$ and $\lim_{\delta\to 0} u_{0\delta}(x) = u_0(x)$.

Let $d_\delta = (\delta, a)$, let $w_\delta = d_\delta \times (0, t_0]$, let $\bar{d}_\delta$ and $\bar{w}_\delta$ be their respective closures, and let $s_\delta = \{\delta, a\} \times (0, t_0]$. We consider the following regularized problem

$$\begin{cases} x^q u_{\delta t} - (x^p u_{\delta x})_x = \displaystyle\int_0^a u_\delta^m(x,t)dx - ku_\delta^n(x,t), & (x,t) \in w_\delta, \\ u_\delta(0,t) = u_\delta(a,t) = 0, & t \in (0, t_0], \\ u_\delta(x,0) = u_{0\delta}(x), & x \in \bar{d}_\delta. \end{cases} \tag{3.11}$$

By using Schauder's fixed point theorem, we have the following lemma.

LEMMA 3.2. *Problem* (3.11) *admits a unique non-negative solution* $u_\delta(x,t) \in C^{2+\alpha,1+\alpha/2}(\bar{w}_\delta)$, $\alpha \in (0,1)$. *Moreover,* $0 \leq u_\delta(x,t) \leq h(x,t)$, $(x,t) \in \bar{w}_\delta$, *where* $h(x,t)$ *is given in Lemma* 3.1.

Proof. The proof is very similar to **[5]**, we omit the details. □

Proof of Theorem 1.2. By Lemma 3.2, the problem (3.11) has a unique non-negative solution $u_\delta(x,t) \in C^{2+\alpha,1+\alpha/2}(\bar{w}_\delta)$. It follows from Lemma 2.1 that $u_{\delta 1}(x,t) \leq u_{\delta 2}(x,t)$ if $\delta 1 > \delta 2$. Therefore, $\lim_{\delta\to 0} u_\delta(x,t)$ exists for all $(x,t) \in (0,a] \times [0,t_0]$. Let $u(x,t) = \lim_{\delta\to 0} u_\delta(x,t)$ and define $u(0,t) = 0$, $t \in [0,t_0]$. We would like to show that $u(x,t)$ is a classical solution of (1.1) in Ω_{t_0}. For any $(x_1,t_1) \in \Omega_{t_0}$, there exist domains $Q' = (a_1', a_2') \times (t_2', t_3']$, $Q'' = (a_1'', a_2'') \times (t_2'', t_3'']$ and $Q''' = (a_1''', a_2''') \times (t_2''', t_3''']$ such that $(x_1,t_1) \in Q' \subset Q'' \subset Q''' \subset (0,a) \times (0,t_0]$ with $0 < a_1''' < a_1'' < a_1' < x_1 < a_2' < a_2'' < a_2''' < a, 0 \leq t_2''' \leq t_2'' \leq t_2' < t_1 < t_3' \leq t_3'' \leq t_3''' \leq t_0$. Since $u_\delta(x,t) \leq h(x,t)$ in

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Q''' and $h(x,t)$ is finite on $\bar{Q}'''$, for any $\tilde{q} > 1$ and some constants k_3, k_4, we have

$$\begin{aligned} &\text{(i) } ||u_\delta||_{L_{\tilde{q}}(Q''')} \leq ||h||_{L_{\tilde{q}}(Q''')} \leq k_3, \\ &\text{(ii) } \left|\left| x^{-q}\int_\delta^a u_\delta^m dx \right|\right|_{L_{\tilde{q}}(Q''')} \leq (a_1^*)^{-q} \left|\left| \int_0^a h^m dx \right|\right|_{L_{\tilde{q}}(Q''')} \leq k_4, \\ &\text{(iii) } ||x^{-q}u_\delta^n||_{L_{\tilde{q}}(Q''')} \leq (a_1^*)^{-q}\, ||h^n||_{L_{\tilde{q}}(Q''')} \leq k_4, \end{aligned} \tag{3.12}$$

where $a_1^* = a_1'''$ if $q \geq 0$, $a_1^* = a_2'''$ if $q < 0$.

By the local L_p estimate of Ladyěnskaja et al. [**14**], $u_\delta \in W_{\tilde{q}}^{2,1}(Q'')$. By the embedding theorem in [**14**], $W_{\tilde{q}}^{2,1}(Q'') \hookrightarrow H^{\alpha,\alpha/2}(Q'')$ if we choose $\tilde{q} > 2/(1-\alpha)$. Then $||u_\delta||_{H^{\alpha,\alpha/2}(Q'')} \leq k_5$ for some positive constant k_5 and we have

$$\begin{aligned} &\left|\left| x^{-q}\int_\delta^a u_\delta^m dx \right|\right|_{H^{\alpha,\alpha/2}(Q'')} \\ &\leq (a_1^*)^{-q}\left|\left| \int_\delta^a h^m dx \right|\right|_\infty + \sup_{(x,t)\in Q'',(\tilde{x},t)\in Q''} \frac{\left|\int_\delta^a u_\delta^m dx\right| \cdot \left|x^{-q} - \tilde{x}^{-q}\right|}{|x-\tilde{x}|^\alpha} \\ &\quad + \sup_{(\tilde{x},t)\in Q'',(\tilde{x},\tilde{t})\in Q''} \frac{\left|\tilde{x}^{-q}\right| \cdot \left|\int_\delta^a m(u_\delta(x,\tilde{t})+\tau(u_\delta(x,t)-u_\delta(x,\tilde{t})))^{m-1}(u_\delta(x,t)-u_\delta(x,\tilde{t}))dx\right|}{|t-\tilde{t}|^{\alpha/2}} \\ &\leq (a_1^*)^{-q}\left|\left| \int_0^a h^m dx \right|\right|_\infty + \left|\left| \int_0^a h^m dx \right|\right|_\infty \cdot ||x^{-q}||_{H^\alpha(a_1'',a_2'')} \\ &\quad + (a_1^*)^{-q}\left|\left| \int_0^a m h^{m-1} dx \right|\right|_\infty \cdot ||u_\delta||_{H^{\alpha,\alpha/2}(Q'')} \\ &\leq k_6, \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} &||x^{-q}u_\delta^n||_{H^{\alpha,\alpha/2}(Q'')} \\ &\leq (a_1^*)^{-q}\,||h^n||_\infty + \sup_{(x,t)\in Q'',(\tilde{x},t)\in Q''} \frac{\left|x^{-q}u_\delta^n(x,t) - \tilde{x}^{-q}u_\delta^n(\tilde{x},t)\right|}{|x-\tilde{x}|^\alpha} \\ &\quad + \sup_{(\tilde{x},t)\in Q'',(\tilde{x},\tilde{t})\in Q''} \frac{|\tilde{x}^{-q}| \cdot \left|u_\delta^n(\tilde{x},t) - u_\delta^n(\tilde{x},\tilde{t})\right|}{|t-\tilde{t}|^{\alpha/2}} \\ &\leq (a_1^*)^{-q}\,||h^n||_\infty \\ &\quad + \sup_{(x,t)\in Q'',(\tilde{x},t)\in Q''} \frac{|x^{-q}| \cdot \left|n\,(u_\delta(\tilde{x},t)+\tau_1(u_\delta(x,t)-u_\delta(\tilde{x},t)))^{n-1}\right| \cdot |u_\delta(x,t)-u_\delta(\tilde{x},t)|}{|x-\tilde{x}|^\alpha} \\ &\quad + \sup_{(x,t)\in Q'',(\tilde{x},t)\in Q''} \frac{\left|u_\delta^n(\tilde{x},t)\right| \cdot \left|x^{-q} - \tilde{x}^{-q}\right|}{|x-\tilde{x}|^\alpha} \\ &\quad + \sup_{(\tilde{x},t)\in Q'',(\tilde{x},\tilde{t})\in Q''} \frac{|\tilde{x}^{-q}| \cdot \left|n(u_\delta(\tilde{x},\tilde{t})+\tau_2(u_\delta(\tilde{x},t)-u_\delta(\tilde{x},\tilde{t})))^{n-1}\right| \cdot |u_\delta(\tilde{x},t)-u_\delta(\tilde{x},\tilde{t})|}{|t-\tilde{t}|^{\alpha/2}} \\ &\leq (a_1^*)^{-q}\,||h^n||_\infty + (a_1^*)^{-q}\,||nh^{n-1}||_\infty \cdot ||u_\delta||_{H^{\alpha,\alpha/2}(Q'')} + ||h^n||_\infty \cdot ||x^{-q}||_{H^\alpha(a_1'',a_2'')} \\ &\quad + (a_1^*)^{-q}\,||nh^{n-1}||_\infty \cdot ||u_\delta||_{H^{\alpha,\alpha/2}(Q'')} \\ &\leq k_6, \end{aligned} \tag{3.14}$$

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for some constant $k_6 > 0$, which is independent of δ, where $\tau, \tau_1, \tau_2 \in (0, 1)$. By Ladyěnskaja et al. [**14**], we have

$$||u_\delta||_{H^{2+\alpha,1+\alpha/2}(Q')} \le k_7, \tag{3.15}$$

for some positive constant independent of δ. This implies that u_δ, $u_{\delta t}$, $u_{\delta x}$ and $u_{\delta xx}$ are equicontinuous in Q'. By the Ascoli–Arela theorem, we know that

$$||u||_{H^{2+\alpha',1+\alpha'/2}(Q')} \le k_8, \tag{3.16}$$

for some $\alpha' \in (0, \alpha)$ and some positive constant k_8 independent of δ, and that the derivatives of u are uniform limits of the corresponding partial derivatives of u_δ. Hence, $u(x, t)$ satisfies (1.1) and $\lim_{t\to 0} u(x, t) = \lim_{t\to 0} \lim_{\delta\to 0} u_\delta(x, t) = \lim_{\delta\to 0} u_{0\delta}(x) = u_0(x)$. It follows from $0 \le u(x, t) \le h(x, t)$ and $\lim_{x\to 0} h(x, t) = \lim_{x\to a} h(x, t) = 0$ that $\lim_{x\to 0} u(x, t) = \lim_{x\to a} u(x, t) = 0$. Thus, $u(x, t) \in C(\bar{\Omega}_{t_0}) \cap C^{2,1}(\Omega_{t_0})$ is a solution of problem (1.1). We complete the proof of Theorem 1.2. □

By using Lemma 2.1, there exists at most one non-negative solution of (1.1), similar to the proof [**11**], we can get Theorem 1.3.

4. Existence and no-existence of the global solution. In this section, we prove Theorems 1.4 and 1.6.

Proof of Theorem 1.4. (1) Since $u_0 \in C([0, a])$ and $m > n$, we can choose M large enough such that

$$M \ge \max\left\{\max_{x\in[0,a]} u_0(x), \left(\frac{k}{a}\right)^{\frac{1}{m-n}}\right\}.$$

Then, it is easy to see $\overline{u} = M$ is a supersolution solution of problem (1.1) and the result follows by Theorem 1.1.

(2) Since $\overline{u} = (k/a)^{\frac{1}{m-n}}$ is a supersolution solution of problem (1.1) and the result follows by Theorem 1.1.

(3) Obviously, $\varphi(x)$ defined in (1.13) is the solution of the following elliptic boundary problem

$$\begin{cases} -\left(x^p\varphi'(x)\right)' = 1, & x \in (0, a), \\ \varphi(0) = \varphi(a) = 0. \end{cases} \tag{4.1}$$

Let $\bar{u}(x, t) = a_1\varphi(x)$ and Beta function $B(l, m) = \int_0^1 x^{l-1}(1 - x)^{m-1}dx$, then we have

$$\begin{aligned} & x^q\bar{u}_t - (x^p\bar{u}_x)_x \\ & \quad = -\left(x^p a_1\varphi'(x)\right)' \\ & \quad = a_1 = \left(\frac{a^{m(2-p)+1}B(m(1-p) + 1, m + 1)}{m(2 - p)}\right)^{1/(1-m)} \\ & \quad = \int_0^a a_1^m\varphi^m dx = \int_0^a \bar{u}^m dx \ge \int_0^a \bar{u}^m dx - k\bar{u}^n, && (x, t) \in (0, a) \times (0, T), \\ & \bar{u}(0, t) = \bar{u}(a, t) = 0, && t \in (0, T), \\ & \bar{u}(x, 0) = a_1\varphi(x) \ge u_0(x), && x \in [0, a]. \end{aligned}$$

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That is to say $\bar{u}(x, t) = a_1\varphi(x)$ is a supersolution of (1.1). By, Theorem 1.3, we know that $u(x, t)$ exists globally. We complete the proof of Theorem 1.4. □

To give the proof of Theorem 1.6, we need the following lemma.

LEMMA 4.1. *If* $(x^p u_{0x})_x + \int_0^a u_0^m dx - k u_0^n \geq 0$, *then the solution of problem (1.1)* $u(x, t)$ *is non-decreasing in* t.

Proof. Let $v = u_t$, then the function v satisfies

$$\begin{cases} x^q v_t - (x^p v_x)_x = \displaystyle\int_0^a m u^{m-1} v(x, t) dx - k n u^{n-1} v(x, t), & (x, t) \in (0, a) \times (0, T), \\ v(0, t) = v(a, t) = 0, & t \in (0, T), \\ v(x, 0) = x^{-q}\left((x^p u_{0x})_x + \displaystyle\int_0^a u_0^m dx - k u_0^n\right) \geq 0, & x \in [0, a], \end{cases} \tag{4.2}$$

then, Lemma 2.1 tells us that $v \geq 0$ for $t \in [0, T)$, i.e., u is non-decreasing in t for $t \in [0, T)$. We complete the proof of Lemma 4.1. □

Proof of Theorem 1.6. Let $\phi(x, t)$ be the solution of (1.1) with initial data $\phi_0(x) = a_0\omega(x)$ where a_0 and $\omega(x)$ are given in Theorem 1.6 and Lemma 1.5 respectively. Since $\lambda_0 \geq -(x^p\omega(x))_x$ and $a_0\lambda_0 \leq \delta a_0^m$, it is easy to show that $(x^p\phi_{0x})_x + \int_0^a \phi_0^m dx - k\phi_0^n \geq 0$. So $\phi(x, t)$ is non-decreasing is t by Lemma 4.1.

Setting $J(t) = \int_0^a x^q\phi(x, t)\omega(x)dx$ and using Jensen's inequality, we have

$$\begin{aligned} J'(t) &= \int_0^a x^q \phi_t(x, t)\omega(x) dx \\ &= \int_0^a (x^p\omega_x)_x \phi dx + \int_0^a \phi^m dx - k\int_0^a \phi^m \omega dx \\ &\geq -\lambda_0 \int_0^a \phi dx + \int_0^a \phi^m dx + (1-\delta)\int_0^a \phi^m dx \\ &\geq \int_0^a \phi dx \left(a^{1-m}\left(\int_0^a \phi dx\right)^{m-1} - \lambda_0\right) + (1-\delta)\int_0^a \phi^m dx \\ &\geq \int_0^a \phi dx \left(a^{1-m}\left(\int_0^a \phi_0 dx\right)^{m-1} - \lambda_0\right) + (1-\delta)\int_0^a \phi^m dx. \end{aligned} \tag{4.3}$$

Since $a_0 \geq \frac{a\lambda_0^{\frac{1}{m-1}}}{\int_0^a \omega(x)dx}$, it is easy to obtain

$$a^{1-m}\left(\int_0^a \phi_0 dx\right)^{m-1} - \lambda_0 \geq 0.$$

So, we get form (4.3) that

$$J'(t) \geq (1-\delta)\int_0^a \phi^m dx. \tag{4.4}$$

Using Hölder's inequality we obtain

$$\int_0^a x^q\phi\omega dx \le \left(\int_0^a \phi^m dx\right)^{\frac{1}{m}} \left(\int_0^a (x^q\omega)^{\frac{m}{m-1}} dx\right)^{\frac{m-1}{m}}. \tag{4.5}$$

Combining (4.4) and (4.5), we get

$$\begin{aligned} J'(t) &\ge (1-\delta)\left(\int_0^a (x^q\omega)^{\frac{m}{m-1}} dx\right)^{1-m} \left(\int_0^a x^q\phi\omega dx\right) \\ &= (1-\delta)\left(\int_0^a (x^q\omega)^{\frac{m}{m-1}} dx\right)^{1-m} \times J^m(t). \end{aligned} \tag{4.6}$$

Since $m > 1$ and $J(0) \ge 0$, we know that $\phi(x,t)$ blows up in finite time. So $u(x,t)$ blows up in finite time for $\phi(x,t)$ is a subsolution of problem (1.1).

(2) First, the following eigenvalue problem

$$\begin{cases} -\left(x^p\varphi'(x)\right)' = \lambda x^q\varphi(x), & x \in (0,a), \\ \varphi(0) = \varphi(a) = 0. \end{cases} \tag{4.7}$$

is given by

$$\varphi(x) = k_1 x^{(1-p)/2} J_{(1-p)/(q+2-p)}\left(\frac{2\sqrt{\mu}}{q+2-p}x^{(q+2-p)/2}\right), \tag{4.8}$$

which is positive for $x \in (0,a)$, where $J_{(1-p)/(q+2-p)}$ denote the Bessel function of the first kind of order $(1-p)/(q+2-p)$, μ is first root of $J_{(1-p)/(q+2-p)}(\frac{2\sqrt{\lambda}}{q+2-p}a^{(q+2-p)/2})$. It is obvious that μ is the first eigenvalue of problem (4.7). Since $q > p-1$, we can choose $k_1 > 0$ such that $\max_{x\in[0,a]} x^q\varphi(x) = 1$ (see [**5**]).

Let $\phi(x,t)$ be the solution of (1.1) with initial data $\phi_0(x) = a_1\varphi(x)$ where a_0 and $\omega(x)$ are given in Theorem 1.6. Since $a_1\mu \le a_1^m \int_0^a \varphi^m dx - ka_1^n\varphi^n$, it is easy to show that $(x^p\phi_{0x})_x + \int_0^a \phi_0^m dx - k\phi_0^n \ge 0$. So $\phi(x,t)$ is non-decreasing is t by Lemma 4.1. Set $J(t) = \int_0^a x^q\phi(x,t)\varphi(x)dx$, we have

$$\begin{aligned} J'(t) &= \int_0^a x^q\phi_t(x,t)\varphi(x)dx \\ &= \int_0^a (x^p\varphi_x)_x\,\phi dx + \int_0^a \varphi dx \int_0^a \phi^m dx - k\int_0^a \phi^n\varphi dx \\ &\ge -\mu\int_0^a \phi dx \\ &\quad + \int_0^a \varphi dx\left(\frac{1}{2}\int_0^a \phi^m dx + \frac{A}{2}\left(\int_0^a \phi^n\varphi dx\right)^{\frac{m}{n}}\right) - k\int_0^a \phi^n\varphi dx, \end{aligned} \tag{4.9}$$

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where $A=(\int_0^a \varphi^{\frac{m}{m-n}}dx)^{\frac{n-m}{n}}$. Since $a_1^{m-n}A\int_0^a \varphi dx(\int_0^a \varphi^{n+1}dx)^{\frac{m-n}{n}} \geq 2k$ and $\phi(x,t)$ is non-decreasing is t, it is easy to show that

$$\begin{aligned}
&\frac{A}{2}\int_0^a \varphi dx\left(\int_0^a \phi^n\varphi dx\right)^{\frac{m}{n}} - k\int_0^a \phi^n\varphi dx \\
&\quad = \int_0^a \phi^n\varphi dx\left(\frac{A}{2}\int_0^a \varphi dx\left(\int_0^a \phi^n\varphi dx\right)^{\frac{m-n}{n}} - k\right) \\
&\quad \geq \int_0^a \phi^n\varphi dx\left(\frac{A}{2}\int_0^a \varphi dx\left(\int_0^a \phi_0^n\varphi dx\right)^{\frac{m-n}{n}} - k\right) \\
&\quad = \int_0^a \phi^n\varphi dx\left(\frac{A}{2}a_1^{m-n}\int_0^a \varphi dx\left(\int_0^a \varphi^{n+1}dx\right)^{\frac{m-n}{n}} - k\right) \\
&\quad \geq 0.
\end{aligned} \tag{4.10}$$

Combining (4.9), (4.10) and using Hölder inequality, we get

$$\begin{aligned}
J'(t) &\geq -\mu a^{\frac{m-1}{m}}\left(\int_0^a \phi^m dx\right)^{\frac{1}{m}} + \frac{1}{2}\int_0^a \varphi dx\int_0^a \phi^m dx \\
&\geq \frac{1}{4}\int_0^a \varphi dx\int_0^a \phi^m dx \\
&\quad + \left(\int_0^a \phi^m dx\right)^{\frac{1}{m}}\left(\frac{1}{4}\int_0^a \varphi dx\left(\int_0^a \phi^m dx\right)^{\frac{m-1}{m}} - \mu a^{\frac{m-1}{m}}\right).
\end{aligned} \tag{4.11}$$

Since $a_1^{m-1}\int_0^a \varphi dx\left(\int_0^a \varphi^m dx\right)^{\frac{m-1}{m}} \geq 4\mu a^{\frac{m-1}{m}}$ and $\phi(x,t)$ is non-decreasing is t, it is easy to show that

$$\begin{aligned}
&\frac{1}{4}\int_0^a \varphi dx\left(\int_0^a \phi^m dx\right)^{\frac{m-1}{m}} - \mu a^{\frac{m-1}{m}} \\
&\quad \geq \frac{1}{4}\int_0^a \varphi dx\left(\int_0^a \phi_0^m dx\right)^{\frac{m-1}{m}} - \mu a^{\frac{m-1}{m}} \\
&\quad = \frac{1}{4}a_1^{m-1}\int_0^a \varphi dx\left(\int_0^a \varphi^m dx\right)^{\frac{m-1}{m}} - \mu a^{\frac{m-1}{m}} \\
&\quad \geq 0.
\end{aligned} \tag{4.12}$$

Using Hölder's inequality we obtain

$$\int_0^a x^q\phi\varphi dx \leq \left(\int_0^a \phi^m dx\right)^{\frac{1}{m}}\left(\int_0^a (x^q\varphi)^{\frac{m}{m-1}}dx\right)^{\frac{m-1}{m}}. \tag{4.13}$$

Combining (4.11)–(4.13), we obtain

$$\begin{aligned}
J'(t) &\geq \frac{1}{4}\int_0^a \varphi dx\int_0^a \phi^m dx \\
&\geq \frac{1}{4}\left(\int_0^a (x^q\varphi)^{\frac{m}{m-1}}dx\right)^{1-m}\left(\int_0^a x^q\phi\varphi dx\right)^m \\
&\geq \frac{1}{4}\left(\int_0^a (x^q\varphi)^{\frac{m}{m-1}}dx\right)^{1-m} \times J^m(t).
\end{aligned} \tag{4.14}$$

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Since $m > 1$ and $J(0) \geq 0$, we know that $\phi(x, t)$ blows up in finite time. So $u(x, t)$ blows up in finite time for $\phi(x, t)$ is a subsolution of problem (1.1). The proof of Theorem 1.6 is complete. □

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