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REGULARIZERS OF CLOSED OPERATORS

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1. **Introduction.** Let X and Y be two Banach spaces and let $B(X, Y)$ denote the set of bounded linear operators with domain X and range in Y. For $T \in B(X, Y)$, let $N(T)$ denote the null space and $R(T)$ the range of T. J. I. Nieto [5, p. 64] has proved the following two interesting results. An operator $T \in B(X, Y)$ has a left regularizer, i.e., there exists a $Q \in B(Y, X)$ such that $QT=I+A$, where I is the identity on X and $A \in B(X, X)$ is a compact operator, if and only if dim $N(T)<\infty$ and $R(T)$ has a closed complement. Also, T has a right regularizer, i.e., $TQ=I+A$, where $A \in B(Y, Y)$ is compact, if and only if dim $Y/R(T)<\infty$ and $N(T)$ has a closed complement. Incidentally, we note that if $R(T)$ has a closed complement (in particular dim $Y/R(T)<\infty$), then $R(T)$ is closed. This is true even if T is a closed operator with domain $D(T) \subseteq X$ [2, p. 100]. With a different approach the same assertions have been proved by B. Yood [6, p. 609]. In particular, he has shown the following characterizations:

$$c^{-1}(G^l) = \{T \in B(X, X) : T \text{ has a left regularizer}\}$$

and

$$c^{-1}(G^r) = \{T \in B(X, X) : T \text{ has a right regularizer}\},$$

where c is the canonical homomorphism of the Banach algebra $B(X, X)$ onto the Banach algebra $B(X, X)/K(X, X)$, $K(X, X)$ is the closed two-sided ideal of compact operators on X and G^l (resp. G^r) denote the set of left (resp. right) invertible elements in $B(X, X)/K(X, X)$.

The purpose of this note is to consider different types of regularizations for an unbounded operator T with $D(T) \subseteq X$ and to characterize T in terms of its regularizers.

2. **Regularizers of closed operators.** Let $C(X, Y)$ denote the set of closed linear operators with domain contained in X and range in Y. For $T \in C(X, Y)$, if there exists an $S \in B(Y, X)$ such that

$$ST = I-A \text{ on } D(T) \text{ with } R(S) \subseteq D(T)$$

$$(\text{resp. } TS = I-A \text{ on } Y \text{ and } R(S) \text{ is closed}),$$

where I is the identity on X (resp. Y) and A is a strictly singular operator on $D(A)$, $X \supseteq D(A) \supseteq D(T)$, into $D(T)$ (resp. in $B(Y, Y)$). Then S is said to be a left (resp. right) s-regularizer of T. In particular, if $A \in B(X, X)$ (resp. $B(Y, Y)$)

is compact, degenerate or degenerate projection (for definitions of these operators see, e.g., [2, 3, 4]), S is said to be a left (resp. right) *c*-, *d*- or *dp*-regularizer of T respectively.

THEOREM 1. *For $T \in C(X, Y)$, the following statements are equivalent.*

(1) dim $N(T)=\alpha(T)<\infty$ *and $R(T)$ has a closed complement.*

(2) *T has a left dp-regularizer.*

(3) *T has a left d-regularizer.*

(4) *T has a left c-regularizer.*

(5) *T has a left s-regularizer and $R(T)$ has a closed complement.*

(6) *$T+K$ has a left s-regularizer for any strictly singular operator K from X into Y with $D(T)\subseteq D(K)$, and $R(T)$ has a closed complement.*

(7) *There exists an $S \in B(Y, X)$ with $R(S)\subseteq D(T)$ such that $\alpha(ST)=$* dim $D(T)/R(ST)<\infty$.

(8) *T is decomposible in the form $T=E+J$ on $D(T)$, where $E \in C(X, Y)$, $D(E)=D(T)$, $R(T)\subseteq R(E)$, $N(E)\subseteq N(T)$ and $J \in B(X, Y)$ is degenerate. Moreover, E has a left dp-regularizer.*

(9) *Same as* (8), *but where J is compact.*

Proof. (1)$\Rightarrow$(2). We have that $R(T)$ is closed, $X=N(T)\oplus X_0$ and $Y=R(T)\oplus Y_0$ where X_0 and Y_0 are some closed subspaces of X and Y respectively. $D(T)=N(T)\oplus(X_0 \cap D(T))$. Let $T_0=T \mid (X_0 \cap D(T))$, then $T_0 \in C(X, Y)$ which is one-to-one with the closed range $R(T)$ or, equivalently, T_0 has a bounded inverse T_0^{-1} by the closed-graph theorem. Let Q be the projection of Y onto $R(T)$ and $S=T_0^{-1}Q$. Also let A be the projection of X onto $N(T)$. Then A is degenerate and $ST=I-A$ on both $N(T)$ and $X_0 \cap D(T)$, and hence on $D(T)$.

(2)$\Rightarrow$(3)$\Rightarrow$(4) trivially.

(4)$\Rightarrow$(5): The first part is clear. Now, $ST=I-A$ on $D(T)$ and A is compact, hence $\alpha(ST)$ and dim $D(T)/R(ST)$ are finite by the theory of F. Riesz [1, p. 315]. $N(S)$ is closed since $S \in B(Y, X)$, dim $T(N(ST))<\infty$ and $T(N(ST))\subseteq N(S)$, so we may let Y_1 be a closed subspace of $N(S)$ such that $N(S)=T(N(ST))\oplus Y_1$. Also let M be a closed subspace of X such that $X=N(ST)\oplus M$. That $T(N(ST)) \cap T(M \cap D(T))=\{0\}=Y_1 \cap T(M \cap D(T))$ is easily verified. Let

$$Y_0 = (T(N(ST)) \oplus T(M \cap D(T))) \oplus Y_1 = R(T) \oplus Y_1,$$

and Y_2 be a subspace of Y such that $Y=Y_0\oplus Y_2$. Since $N(S)\subseteq Y_0$,

$$D(T) \supseteq R(S) = S(Y_0) \oplus S(Y_2) = R(ST) \oplus S(Y_2).$$

On Y_2 the operator S is one-to-one, and dim $S(Y_2)<\infty$ since dim $D(T)/R(ST)<\infty$, it follows that dim $Y_2<\infty$ and hence $Y_1\oplus Y_2$ is closed in Y. The relation $Y=R(T)\oplus(Y_1\oplus Y_2)$ implies the result.

(5)$\Rightarrow$(6): That $T+K \in C(X, Y)$ is easily verified. Since $ST=I-A$ on $D(T)$, $S(T+K)=I-(A-SK)$ on $D(T)$ and $A-SK$ is strictly singular [3, p. 286].

(6)⇒(1): Take $K=0$, then $ST=I-A$ on $D(T)$. $N(ST)=\{x \in D(T): Ax=x\}$, hence $\|Ax\|=\|x\|$ for $x \in N(ST)$, i.e., the strictly singular operator A has a bounded inverse on $N(ST)$ and thus $\alpha(ST)<\infty$. $\alpha(T)<\infty$ since $N(T) \subseteq N(ST)$.

(1)⇒(7): We see from "(1)⇒(2)" that $R(ST)=T_0^{-1}Q(R(T))=X_0 \cap D(T)$ and $N(ST)=N(T_0^{-1}T)=N(T)$. Hence $\alpha(ST)=\alpha(T)=\dim D(T)/R(ST)$ which is finite.

(7)⇒(1): It remains to show that $R(T)$ has a closed complement, but this follows exactly the same as (4)⇒(5).

(1)⇒(8): Notation as in "(1)⇒(2)". By using a known method we may construct a bounded linear operator G on the finite dimensional space $N(T)$ into Y_0. Say,

$$G(x)=\sum_{i=1}^{n} f_i(x)y_i,$$

where f_i is a bounded linear functional on X such that $f_i(x_j)=\delta_{ij}$ and $\{x_1, \ldots, x_n\}$ is a basis of $N(T)$, and $\{y_1, \ldots, y_n\}$ is a linearly independent subset (resp. a set of n arbitrary elements) of Y_0 if $\dim Y_0 \geq n = \dim N(T)$ (resp. $\dim Y_0<n$). On $D(T)$ let $E=T-J$ and $J=GA$, then $R(T)=T\,|\,(X_0 \cap D(T))=E\,|\,(X_0 \cap D(T)) \subseteq R(E)$, and if $x \in N(E)$, $Tx=GAx \in Y_0$, $Tx=0$ and hence $N(E) \subseteq N(T)$. Since $SE=ST-SJ=ST-T_0^{-1}QGA=ST=I-A$ on $D(T)$, the last part follows.

(8)⇒(4): If $S \in B(Y, X)$ is a left dp-regularizer of E, $SE=I-A$, then $ST=SE-SJ=I-(SJ+A)$ on $D(T)$.

Now, that (1)⇒(9)⇒(4) is clear. Q.E.D.

THEOREM 2. *For $T \in C(X, Y)$, the following statements are equivalent.*

(1) $\dim Y/R(T)=\beta(T)<\infty$ *and $D(T)$ is a direct sum of $N(T)$ and a closed subspace of X.*

(2) *T has a right dp-regularizer.*

(3) *T has a right d-regularizer.*

(4) *T has a right c-regularizer.*

(5) *T has a right s-regularizer.*

(6) *$T+K$ has a right s-regularizer for any strictly singular operator K from X into Y with $D(T) \subseteq D(K)$.*

(7) *There exists an $S \in B(Y, X)$ with the closed range $R(S) \subseteq D(T)$ such that $\beta(TS)=\alpha(TS)<\infty$.*

(8) *T is decomposible in the form $T=E+J$ on $D(T)$, where $E \in C(X, Y)$, $D(E)=D(T)$, $R(T) \subseteq R(E)$, $N(E) \subseteq N(T)$ and $J \in B(X, Y)$ is degenerate. Moreover, E has a right dp-regularizer.*

(9) *Same as* (8), *but where J is strictly singular.*

Proof. (1)⇒(2). $D(T)=N(T) \oplus X_0$ and $Y=R(T) \oplus Y_0$, where X_0 is some closed subspace of X and $\dim Y_0<\infty$ by assumption. Note that $R(T)$ is closed. If $T_0=T\,|\,X_0$, then $T_0 \in C(X, Y)$ which is bounded as well, and it has a bounded inverse

T_0^{-1}. Let Q and S be as in Theorem 1 "(1)⇒(2)" and A be the projection of Y onto Y_0, then $R(S)=X_0$ and $TS=I-A$ on Y.

(2)⇒(3)⇒(4)⇒(5) trivially.

(5)⇒(6) similarly as in Theorem 1.

(6)⇒(7): Take $K=0$, then $TS=I-A$ on Y, and $\beta(TS)=\alpha(TS)<\infty$ by either the Riesz-Schauder theorem for a strictly singular operator [3, p. 321] or a stability theorem perturbed by a strictly singular operator [2, p. 117].

(7)⇒(1): Since $R(TS)\subseteq R(T)$, $\beta(T)\subseteq\beta(TS)<\infty$. Let $Y=N(TS)\oplus M$, where M is some closed subspace of Y, then $R(S)=S(N(TS))\oplus S(M)$ since $N(S)\subseteq N(TS)$. Since $R(S)$ is closed and dim $S(N(TS))<\infty$, $S(M)$ is closed by a remark in the section 1. Obviously $N(T)\cap S(M)=\{0\}$, so let $X_0=N(T)\oplus S(M)\subseteq D(T)$ and X_1 be a subspace of X such that $X=X_0\oplus X_1$. Then $D(T)=N(T)\oplus S(M)\oplus (X_1\cap D(T))$ and hence

$$Y\supseteq R(T) = TS(M)\oplus T(X_1\cap D(T)) = R(TS)\oplus T(X_1\cap D(T)).$$

But on $X_1\cap D(T)$ the operator T is one-to-one and $\beta(TS)<\infty$, so dim $(X_1\cap D(T))<\infty$ and hence $S(M)\oplus(X_1\cap D(T))$ is closed in X.

(1)⇒(8): Let X_1 be a finite dimensional subspace of $N(T)$ and let $X=X_1\oplus X_2$, where X_2 is some closed subspace. Let P be the projection of X onto X_1. Notation as in "(1)⇒(2)", as before we may construct a bounded linear operator G on X_1 into Y_0. On $D(T)$ let $E=T-J$ and $J=GP$. Then the desired result follows as in Theorem 1.

(8)⇒(5) as in Theorem 1 "(8)⇒(4)" and that (1)⇒(9)⇒(5) is easily seen. Q.E.D.

3. Remarks. Let us consider operators in $B(X, Y)$, now, the condition that $R(T)$ has a closed complement in Y in (5) and (6) of Theorem 1 may be omitted, because in this case $\alpha(ST)=\beta(ST)<\infty$ by a remark in the proof (6)⇒(7) of Theorem 2. Accordingly, the operator J in (9) of Theorem 1 may be strictly singular. The closedness of $R(S)$ in the definition of a right regularizer may also be omitted, since we may regard T as a left regularizer of S and hence $R(S)$ is closed. In the proof (7)⇒(1) of Theorem 2 we need the closedness condition of $R(S)$, however, from (7) we see that S has a left regularizer and hence $R(S)$ is automatically closed.

Finally, we note that if both X and Y are Hilbert spaces and $T\in B(X, Y)$, the statement (5) of Theorem 1 and 2 is superfluous, since T is compact if and only if T is strictly singular [3, p. 287]. Also, the second condition in (1) of Theorem 1 is equivalent to the closedness of $R(T)$, and that of Theorem 2 is superfluous.

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