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On the absolute Nörlund summability factors of a Fourier series and its conjugate series at a point

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The object of this paper is to give generalizations of Okuyama's Theorem [*Bull. Austral. Math. Soc.* 12 (1975), 9-21, *Tôhoku Math. J.* (2) 28 (1976), 563-581] on the absolute Nörlund summability factors of a Fourier series and its conjugate series.

Our theorems imply many results proved by other authors: especially Theorem 1 includes the results of Bhatt and Kishor [*Indian J. Math.* 9 (1967), 259-267 (1968)], Dikshit [*Pacific J. Math.* 63 (1976), 371-379], and Lal [*Publ. Inst. Math. (Beograd)* 20 (34) (1976), 169-178], and we can easily deduce Lal's result [*Indian J. Math.* 16 (1974), 1-22] from our Corollary 2.

1. Notations and theorems

Let $\sum a_n$ be a given infinite series with the sequence of partial sums $\{s_n\}$. Let $\{p_n\}$ be a given sequence of constants, real or complex, such that $P_n = \sum_{k=0}^{n} p_k \neq 0$ for $n \geq 0$ and $p_n = P_n = 0$ for $n < 0$.

The sequence $\{t_n\}$ given by

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$$(1.1) \qquad t_n = \frac{1}{P_n}\sum_{k=0}^{n} p_{n-k}s_k = \frac{1}{P_n}\sum_{k=0}^{n} P_k a_{n-k}$$

defines the Nörlund means of the sequence $\{s_n\}$ generated by the sequence $\{p_n\}$. The series $\sum a_n$ is said to be absolutely summable (N, p_n), or summable $|N, p_n|$, if the sequence $\{t_n\}$ is of bounded variation, that is, if

$$(1.2) \qquad \sum_{n=1}^{\infty} |t_n - t_{n-1}|$$

is convergent.

Let $f(t)$ be a periodic function with period 2π and integrable over $(-\pi, \pi)$. Without any loss of generality we may assume that the constant term in the Fourier series of $f(t)$ is zero, so that

$$(1.3) \qquad f(t) \sim \sum_{n=1}^{\infty} \left(a_n \cos nt + b_n \sin nt\right) \equiv \sum_{n=1}^{\infty} A_n(t)$$

and

$$\int_{-\pi}^{\pi} f(t)dt = 0 .$$

The conjugate series to series (1.3) is

$$\sum_{n=1}^{\infty} \left(b_n \cos nt - a_n \sin nt\right) = \sum_{n=1}^{\infty} B_n(t) .$$

In what follows, we use the following notations:

$$\varphi_x(t) = \varphi(t) = \tfrac{1}{2}\{f(x+t)+f(x-t)\} ;$$

$$\Phi_\alpha(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-u)^{\alpha-1}\varphi(u)du \quad (\alpha > 0) ;$$

$$\Phi_0(t) = \varphi(t) ;$$

$$h(u) = \int_u^{\pi} (t-u)^{-\alpha} \cos ktdt ;$$

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$$P(n, k) = \frac{p_{n-k}}{P_n} - \frac{p_{n-k-1}}{P_{n-1}} ;$$

$$G_{\beta-\alpha}(n; j, l, t) = \sum_{k=j}^{l} P(n, k)\lambda_k\mu_k k^{\beta-\alpha} \cos kt ;$$

$$H_{\beta-\alpha}(n; j, l, u) = \frac{2}{\pi\Gamma(1-\alpha)} \int_u^\pi (t-u)^{-\alpha} G_{\beta-\alpha}(n; j, l, t)dt ;$$

$$I(n; j, l, u) = \int_0^u v^\beta \frac{d}{dv} H_{\beta-\alpha}(n; j, l, v)dv .$$

Moreover we write

$$\psi_x(t) = \psi(t) = \tfrac{1}{2}\{f(x+t)-f(x-t)\} ;$$

$$\tilde{G}_{\beta-\alpha}(n; j, l, t) = \sum_{k=j}^{l} P(n, k)\lambda_k\mu_k k^{\beta-\alpha} \sin kt .$$

We employ $\Psi_\alpha(t)$, $\tilde{H}_{\beta-\alpha}(n; j, l, u)$ with meanings similar to the above notations. Throughout the present paper we denote by $\mu(t)$ a positive bounded function, $\lambda(t)$ a positive non-decreasing function, and $\{p_n\}$ a non-negative, non-increasing sequence.

Given a function $\omega(t)$, we write for $n = 1, 2, \ldots$,

$$\omega(n) = \omega_n , \quad \Delta\omega_n = \omega_n - \omega_{n+1} .$$

Let $[x]$ denote the greatest integer not greater than x ; in particular we write $m = [n/2]$ and $\tau = \left[\tfrac{1}{2}\left((2\pi/u)-1\right)\right]$; and A denotes a positive constant which is not necessarily the same at each occurrence.

The purpose of this paper is to establish some generalizations of Okuyama's results [*10*], [*11*].

THEOREM 1. *Let the sequence* $\left\{\lambda_n\mu_n/(n+1)^{1-\beta}\right\}$ $(0 \le \beta \le 1)$ *be non-increasing.*

If the conditions

(1.4) $$\sum_{k=n}^{\infty} \frac{\lambda_k\mu_k}{k^{1-\beta}P_k} = O\left(\frac{\lambda_n n^\beta}{P_n}\right) \quad (n = 1, 2, \ldots) ,$$

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(1.5) $$\sum_{k=1}^{\infty} \frac{\lambda_k \mu_k}{k} \left|\cos \frac{\pi}{2} (\beta-\alpha+1)\right| < \infty ,$$

and

(1.6) $$\int_0^{\pi} \lambda(2\pi/t) \left| d\left\{t^{-\beta} \Phi_\alpha(t)\right\}\right| < \infty$$

hold for $0 \le \alpha \le \beta \le 1$ *, then the series*

$$\sum_{n=1}^{\infty} \lambda_n \mu_n n^{\beta-\alpha} A_n(t)$$

is summable $|N, p_n|$ *at* $t = x$.

If $P_n \le A\lambda_n n^{\beta}$ *, the right-hand side of condition* (1.4) *is replaced by* $O(1)$.

This theorem has wider applications than the results of Bhatt and Kishore [1], Dikshit [2], and Lal [5], [6]. As special cases of Theorem 1 we obtain the results of Matsumoto [7].

THEOREM 2. *Let* $\{\Delta p_n\}$ *be non-negative non-increasing. Assume that* $n\lambda_n \mu_n$, $n^{1-\beta}/\lambda_n \mu_n$ *, and* $\lambda_n \mu_n / p_n$ *are all non-decreasing, where* $0 \le \beta \le 1$. *If the conditions* (1.4),

(1.7) $$\int_0^{\pi} \frac{\lambda(2\pi/t)\mu(2\pi/t)}{t^{1+\beta}} |\Psi_\alpha(t)| dt < \infty ,$$

and

(1.8) $$\int_0^{\pi} \frac{\lambda(2\pi/t)}{t^{\beta}} |d\Psi_\alpha(t)| < \infty$$

hold for $0 \le \alpha \le \beta \le 1$ *, then the series*

$$\sum_{n=1}^{\infty} \lambda_n \mu_n n^{\beta-\alpha} B_n(t)$$

is summable $|N, p_n|$ *at* $t = x$.

THEOREM 3. *Let* $\left\{\lambda_n \mu_n / n^{1-\beta}\right\}$ *be a non-increasing sequence. If the*

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conditions (1.4),

$$\int_0^\pi \frac{\lambda(2\pi/t)}{t^\beta} |d\Psi_\alpha(t)| < \infty \text{ , and } \Psi_\alpha(+0) = 0 \tag{1.9}$$

hold for $0 \le \alpha \le \beta \le 1$ *, then the series*

$$\sum_{n=1}^{\infty} \lambda_n \mu_n n^{\beta-\alpha} B_n(t)$$

is summable $|N, p_n|$ *at* $t = x$ *.*

In the two theorems above, if $P_n \le A\lambda_n n^\beta$, then the right-hand side of (1.4) is replaced by $O(1)$.

If the property

$$\frac{d}{dt} \lambda(2\pi/t) = A \frac{\lambda(2\pi/t)\mu(2\pi/t)}{t}$$

holds for a suitable constant A , it is easy to see that (1.9) implies (1.7).

2. Proof and corollaries of Theorem 1

We need some lemmas for the proof of Theorem 1.

LEMMA 1. *Let* $\{a_n\}$ *be a given sequence; then for any* x *, we have*

$$(1-x) \sum_{k=r}^{s} a_k x^k = a_r x^r - a_s x^{s+1} - \sum_{k=r}^{s-1} \Delta a_k x^{k+1} ,$$

where r *and* s *are integers such that* $s \ge r \ge 0$.

LEMMA 2. *For* $0 \le a \le b \le \infty$ *and any* n *,*

$$\left| \sum_{k=a}^{b} p_k \exp i(n-k)u \right| \le AP_\tau$$

uniformly in $0 \le u \le \pi$.

LEMMA 3. *For all* $k \ge 0$ *and* $1 \le a \le b \le \infty$ *,*

$$\sum_{n=a}^{b} P(n, k) = \sum_{n=a}^{b} \left(\frac{P_{n-k}}{P_n} - \frac{P_{n-k-1}}{P_{n-1}} \right) \le 1 .$$

The proofs of the above lemmas are quite easy and we omit them.

LEMMA 4. *For* $0 < t \le \pi$, *and* $0 \le \alpha \le \beta \le 1$,

$$\int_0^t u^{\beta-\alpha} \cos kudu = \frac{t^{\beta-\alpha}\sin kt}{k} + O\left(1/k^{\beta-\alpha+1}\right)$$

and

$$\int_t^\pi u^{\beta-\alpha} \cos kudu = \frac{\pi\Gamma(\beta-\alpha+1)}{2k^{\beta-\alpha+1}} \cos \frac{\pi}{2}(\beta-\alpha+1) - \frac{t^{\beta-\alpha}\sin kt}{k} + O\left(1/k^{\beta-\alpha+1}\right),$$

where if $\alpha = \beta$ *we may obviously omit the last term in both cases.*

Proof. By integration by parts, we have

$$\begin{aligned}\int_0^t u^{\beta-\alpha} \cos kudu &= \left[\frac{\sin ku}{k} u^{\beta-\alpha}\right]_0^t + \frac{\alpha+1-\beta}{k}\left\{\int_0^{\pi/k} u^{\beta-\alpha} \frac{\sin ku}{u} du + \int_{\pi/k}^t u^{\beta-\alpha-1} \sin kudu\right\}\\ &= \frac{t^{\beta-\alpha}\sin kt}{k} + \frac{\alpha+1-\beta}{k}\left\{\left(\frac{\pi}{k}\right)^{\beta-\alpha}\int_\xi^{\pi/k} \frac{\sin ku}{u} du + \left(\frac{\pi}{k}\right)^{\beta-\alpha-1}\int_{\pi/k}^\eta \sin kudu\right\}\\ &= \frac{t^{\beta-\alpha}\sin kt}{k} + O\left(1/k^{\beta-\alpha+1}\right),\end{aligned}$$

where $0 \le \xi \le \pi/k \le \eta \le t$. The second formula is obvious by

$$\frac{2}{\pi}\int_0^\pi u^{\beta-\alpha} \cos kudu = \frac{\Gamma(\beta-\alpha+1)}{k^{\beta-\alpha+1}} \cos \frac{\pi}{2}(\beta-\alpha+1).$$

Proof of Theorem 1. We suppose $0 < \alpha < 1$, because we can treat $\alpha = 0$ or 1 more easily (see [3], [*10*]). Since

$$\begin{aligned}A_k(x) &= \frac{2}{\pi}\int_0^\pi \varphi(t) \cos ktdt\\ &= \frac{2}{\pi\Gamma(1-\alpha)}\int_0^\pi \cos kt \int_0^t (t-u)^{-\alpha} d\Phi_\alpha(u)dt\\ &= \int_0^\pi \frac{2}{\pi\Gamma(1-\alpha)}\int_u^\pi (t-u)^{-\alpha} \cos ktdtd\Phi_\alpha(u),\end{aligned}$$

we have

$$\sum_{k=j}^{l}\left\{\frac{P_{n-k}}{P_n}-\frac{P_{n-k-1}}{P_{n-1}}\right\}\lambda_k\mu_k k^{\beta-\alpha}A_k(x) \quad (1 \le j \le l \le n)$$

$$= \int_0^\pi d\Phi_\alpha(u)\,\frac{2}{\pi\Gamma(1-\alpha)}\int_u^\pi (t-u)^{-\alpha}\left\{\sum_{k=j}^{l} P(n,\, k)\lambda_k\mu_k k^{\beta-\alpha}\cos kt\right\}dt$$

$$= \int_0^\pi d\Phi_\alpha(u)\,\frac{2}{\pi\Gamma(1-\alpha)}\int_u^\pi (t-u)^{-\alpha}G_{\beta-\alpha}(n;\, j,\, l,\, t)dt$$

$$= \int_0^\pi H_{\beta-\alpha}(n;\, j,\, l,\, u)d\Phi_\alpha(u)$$

$$= \left[H_{\beta-\alpha}(n;\, j,\, l,\, u)\Phi_\alpha(u)\right]_0^\pi - \int_0^\pi u^\beta \frac{d}{du}H_{\beta-\alpha}(n;\, j,\, l,\, u)\left\{u^{-\beta}\Phi_\alpha(u)\right\}du$$

$$= -\left[u^{-\beta}\Phi_\alpha(u)\int_0^u v^\beta\frac{d}{dv}H_{\beta-\alpha}(n;\, j,\, l,\, v)dv\right]_0^\pi$$

$$+ \int_0^\pi\int_0^u v^\beta\frac{d}{dv}H_{\beta-\alpha}(n;\, j,\, l,\, v)dv\,d\left\{u^{-\beta}\Phi_\alpha(u)\right\}$$

$$= -\pi^{-\beta}\Phi_\alpha(\pi)I(n;\, j,\, l,\, \pi) + \int_0^\pi I(n;\, j,\, l,\, u)d\left\{u^{-\beta}\Phi_\alpha(u)\right\} .$$

If, in particular, we suppose that $\varphi(t) = t^{\beta-\alpha}$, in which case

$$\Phi_\alpha(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-u)^{\alpha-1}u^{\beta-\alpha}du = \frac{\Gamma(\beta-\alpha+1)}{\Gamma(\beta+1)}\,t^\beta$$

and

$$A_k(x) = \frac{\Gamma(\beta-\alpha+1)}{k^{\beta-\alpha+1}}\cos\frac{\pi}{2}(\beta-\alpha+1) ,$$

we get $d\left\{t^{-\beta}\Phi_\alpha(t)\right\} = 0$ and

$$I(n;\, j,\, l,\, \pi) = A\sum_{k=j}^{l} P(n,\, k)\,\frac{\lambda_k\mu_k}{k}\cos\frac{\pi}{2}(\beta-\alpha+1) .$$

If t_n denotes the nth $(N,\, p_n)$ mean of $\sum_{n=1}^{\infty}\lambda_n\mu_n n^{\beta-\alpha}A_n(x)$, then by (1.1) and the above calculations

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$$t_n - t_{n-1} = \sum_{k=1}^{n} \left(\frac{P_{n-k}}{P_n} - \frac{P_{n-k-1}}{P_{n-1}}\right) \lambda_k \mu_k k^{\beta-\alpha} A_k(x)$$

$$= A \sum_{k=1}^{n} P(n, k) \frac{\lambda_k \mu_k}{k} \cos \frac{\pi}{2} (\beta-\alpha+1) + \int_0^\pi I(n;\ l,\ n,\ u) d\left\{u^{-\beta}\Phi_\alpha(u)\right\} .$$

Since, by (1.5) and Lemma 3,

$$\sum_{n=1}^{\infty} \left| \sum_{k=1}^{n} P(n, k) \frac{\lambda_k \mu_k}{k} \cos \frac{\pi}{2} (\beta-\alpha+1) \right|$$

$$\le \sum_{k=1}^{\infty} \frac{\lambda_k \mu_k}{k} \left|\cos \frac{\pi}{2} (\beta-\alpha+1)\right| \sum_{n=k}^{\infty} P(n, k) < A ,$$

to prove our theorem it is enough to show that

$$\sum_{n=1}^{\infty} |t_n - t_{n-1}| < A + \int_0^\pi \sum_{n=1}^{\infty} |I(n;\ l,\ n,\ u)| \left| d\left\{u^{-\beta}\Phi_\alpha(u)\right\}\right| = O(1) .$$

Thus it suffices for our purpose to prove that, uniformly in $0 < u \le \pi$,

$$J = \sum_{n=1}^{\infty} |I(n;\ l,\ n,\ u)| = O\big(\lambda(2\pi/u)\big) . \tag{2.1}$$

We divide J into the following three parts:

$$J = \sum_{n=1}^{2\tau+1} |I(n;\ l,\ n,\ u)| + \sum_{n=2\tau+2}^{\infty} |I(n;\ l,\ \tau,\ u)| + \sum_{n=2\tau+2}^{\infty} |I(n,\ \tau+1,\ n,\ u)| \tag{2.2}$$

$$= J_1 + J_2 + J_3 ,$$

say.

Using the first and second mean value theorems, we have

(2.3) $$h(u) = \left\{\int_u^{u+\pi/k} + \int_{u+\pi/k}^{\pi}\right\}(t-u)^{-\alpha}\cos ktdt$$

$$= \cos k\xi \int_u^{u+\pi/k} (t-u)^{-\alpha}dt + (k/\pi)^{\alpha}\int_{u+\pi/k}^{\eta}\cos ktdt$$

$$(u \le \xi \le u+\pi/k \le \eta \le \pi)$$

$$= \frac{k^{\alpha-1}}{(1-\alpha)\pi^{\alpha-1}}\cos k\xi + \frac{k^{\alpha-1}}{\pi^{\alpha}}[\sin kt]_{u+\pi/k}^{\eta} .$$

Thus

(2.4) $$H_{\beta-\alpha}(n;\, j,\, l,\, u)$$

$$= \frac{2}{\pi^{1+\alpha}\Gamma(1-\alpha)}\sum_{k=j}^{l} P(n,\, k)\lambda_k\mu_k k^{\beta-1}\left\{\frac{\pi\cos k\xi}{1-\alpha} + \sin k\eta + \sin ku\right\} .$$

Moreover, for some ζ $(0 \le \zeta \le u)$,

$$I(n;\, j,\, l,\, u) = \int_0^u v^{\beta}\frac{d}{dv}H_{\beta-\alpha}(n;\, j,\, l,\, v)dv$$

$$= u^{\beta}\int_{\zeta}^{u}\frac{d}{dv}H_{\beta-\alpha}(n;\, j,\, l,\, v)dv$$

$$= u^{\beta}[H_{\beta-\alpha}(n;\, j,\, l,\, v)]_{\zeta}^{u} .$$

Hence we get

(2.5) $$|I(n;\, j,\, l,\, u)| \le Au^{\beta}\sum_{k=j}^{l} P(n,\, k)\lambda_k\mu_k k^{\beta-1} .$$

Using (2.5),

(2.6) $$J_1 \le Au^{\beta}\sum_{n=1}^{2\tau+1}\sum_{k=1}^{n} P(n,\, k)\lambda_k\mu_k k^{\beta-1}$$

$$= Au^{\beta}\sum_{k=1}^{2\tau+1}\lambda_k\mu_k k^{\beta-1}\sum_{n=k}^{2\tau+1} P(n,\, k)$$

$$\le Au^{\beta}\lambda_{2\tau+1}\sum_{k=1}^{2\tau+1} k^{\beta-1} = O(\lambda(2\pi/u)) ,$$

by our assumptions and Lemma 3. For the same reasons,

(2.7)
$$J_2 \le Au^\beta \sum_{n=2\tau+2}^{\infty} \sum_{k=1}^{\tau} P(n,\ k)\lambda_k\mu_k k^{\beta-1}$$
$$= Au^\beta \sum_{k=1}^{\tau} \lambda_k\mu_k k^{\beta-1} \sum_{n=2\tau+2}^{\infty} P(n,\ k)$$
$$\le Au^\beta \lambda_\tau \tau^\beta = O\big(\lambda(2\pi/u)\big)\ .$$

For the calculation of J_3 , we see

$$I(n;\ j,\ l,\ u) = I(n;\ j,\ l,\ \pi) - \int_u^\pi v^\beta \frac{d}{dv} H_{\beta-\alpha}(n;\ j,\ l,\ v)dv$$
$$= I(n;\ j,\ l,\ \pi) - \Big[v^\beta H_{\beta-\alpha}(n;\ j,\ l,\ v)\Big]_u^\pi$$
$$+ \beta \int_u^\pi v^{\beta-1}dv \frac{2}{\pi\Gamma(1-\alpha)} \int_v^\pi (t-v)^{-\alpha} G_{\beta-\alpha}(n;\ j,\ l,\ t)dt$$
$$= I(n;\ j,\ l,\ \pi) + u^\beta H_{\beta-\alpha}(n;\ j,\ l,\ u)$$
$$+ \frac{2\beta}{\pi\Gamma(1-\alpha)} \int_u^\pi G_{\beta-\alpha}(n;\ j,\ l,\ t)dt \int_u^t v^{\beta-1}(t-v)^{-\alpha}dv$$
$$= I(n;\ j,\ l,\ \pi) + u^\beta H_{\beta-\alpha}(n;\ j,\ l,\ u)$$
$$+ \frac{2\beta}{\pi\Gamma(1-\alpha)} \int_{u/\pi}^1 w^{\beta-1}(1-w)^{-\alpha}dw \int_\theta^\pi t^{\beta-\alpha} G_{\beta-\alpha}(n;\ j,\ l,\ t)dt$$

(by the second mean value theorem, where $u \le \theta \le \pi$)

$$= A \sum_{k=j}^{l} P(n,\ k) \frac{\lambda_k\mu_k}{k} \cos\frac{\pi}{2}(\beta-\alpha+1) + u^\beta H_{\beta-\alpha}(n;\ j,\ l,\ u)$$
$$+ O\left\{\sum_{k=j}^{l} P(n,\ k)\lambda_k\mu_k k^{\beta-\alpha-1}\theta^{\beta-\alpha} \sin k\theta\right\} \quad \text{(by Lemma 4).}$$

Thus, using (1.5) and (2.4), we have

(2.8)
$$J_3 \le A \sum_{n=2\tau+2}^{\infty} u^\beta\{|G_{\beta-1}(n;\ \tau+1,\ n,\ \xi)| + |\tilde{G}_{\beta-1}(n;\ \tau+1,\ n,\ \eta)|$$
$$+ |\tilde{G}_{\beta-1}(n;\ \tau+1,\ n,\ u)|\} + A \sum_{n=2\tau+2}^{\infty} \theta^{\beta-\alpha}|\tilde{G}_{\beta-\alpha-1}(n;\ \tau+1,\ n,\ \theta)| + A\ .$$

Now we put

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$$G_\gamma^*(n;\ j,\ l,\ \omega) = \sum_{k=j}^{l} P(n,\ k)\lambda_k\mu_k k^\gamma e^{ik\omega}\ ,$$

where $\gamma = \beta - \varepsilon - 1$ $(\varepsilon = 0$ or $\alpha)$ and $u \le \omega \le \pi$. Then, by (2.8), in order to prove that $J_3 = O\big(\lambda(2\pi/u)\big)$, it is enough to prove that

$$(2.9)\quad \omega^{\gamma+1} \sum_{n=2\tau+2}^{N} |G_\gamma^*(n,\ \tau+1,\ n,\ \omega)|$$

$$\le \omega^{\gamma+1}\left\{\sum_{n=2\tau+2}^{N} |G_\gamma^*(n,\ \tau+1,\ m,\ \omega)| + \sum_{n=2\tau+2}^{N} |G_\gamma^*(n;\ m+1,\ n,\ \omega)|\right\}$$

$$= K_1 + K_2 = O\big(\lambda(2\pi/u)\big)\ ,\ \text{as}\quad N \to \infty\ .$$

Now using Lemma 1, we get

$$|G^*(n;\ j,\ l,\ \omega)| \le A\omega^{-1}\Bigg[P(n,\ j)\lambda_j\mu_j j^\gamma + P(n,\ l)\lambda_l\mu_l l^\gamma + \sum_{k=j}^{l-1}\left\{\frac{P_{n-k-1}}{P_{n-1}} - \frac{P_{n-k}}{P_n}\right\}\lambda_{k+1}\mu_{k+1}(k+1)^\gamma + \sum_{k=j}^{l-1} P(n,\ k)\Delta\left(\frac{\lambda_k\mu_k}{k^{-\gamma}}\right)\Bigg]\ .$$

Hence

$$K_1$$

$$\le A\omega^\gamma\Bigg[\sum_{n=2\tau+2}^{N} P(n,\ \tau+1)\lambda_{\tau+1}\mu_{\tau+1}(\tau+1)^\gamma + \sum_{n=2\tau+2}^{N} \frac{P_n p_{n-m} - p_n P_{n-m}}{P_n P_{n-1}}\lambda_m\mu_m m^\gamma$$
$$+ \sum_{n=2\tau+2}^{N}\sum_{k=\tau+1}^{m}\left\{\frac{P_{n-k-1}}{P_{n-1}} - \frac{P_{n-k}}{P_n}\right\}\lambda_{k+1}\mu_{k+1}(k+1)^\gamma$$
$$+ \sum_{n=2\tau+2}^{N}\sum_{k=\tau+1}^{m} P(n,\ k)\Delta\left(\frac{\lambda_k\mu_k}{k^{-\gamma}}\right)\Bigg]$$

$$\le A\omega^\gamma\Bigg[\lambda_{\tau+1}\tau^\gamma \sum_{n=2\tau+2}^{N} P(n,\ \tau+1) + \sum_{n=2\tau+2}^{N} \frac{p_{n-m}}{P_{n-1}}\lambda_m\mu_m m^\gamma$$
$$+ \sum_{k=\tau+1}^{[N/2]} \lambda_{k+1}\mu_{k+1}(k+1)^\gamma \sum_{n=2k}^{N}\left\{\frac{P_{n-k-1}}{P_{n-1}} - \frac{P_{n-k}}{P_n}\right\}$$
$$+ \sum_{k=\tau+1}^{[N/2]} \Delta\left(\frac{\lambda_k\mu_k}{k^{-\gamma}}\right) \sum_{n=2k}^{N} P(n,\ k)\Bigg] \le$$

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$$\le A\lambda_{2\tau} + A\omega^{\gamma}\left[\sum_{n=2\tau+2}^{N} \frac{p_m}{P_m} \lambda_m \mu_m m^{\gamma} + \sum_{k=\tau+1}^{N} \lambda_{k+1}\mu_{k+1}(k+1)^{\gamma} \frac{p_{k-1}}{P_{2k-1}} + \sum_{k=\tau+1}^{N} \Delta\left(\frac{\lambda_k \mu_k}{k^{-\gamma}}\right)\right]$$

$$\le A\lambda_{2\tau} + A\omega^{\gamma}\left[p_\tau \tau^{-\varepsilon} \sum_{n=\tau}^{\infty} \frac{n^{\gamma+\varepsilon}\lambda_n \mu_n}{P_n} + p_\tau \tau^{-\varepsilon} \sum_{k=\tau}^{\infty} \frac{k^{\gamma+\varepsilon}\lambda_k \mu_k}{P_k} + \lambda_{\tau+1}(\lambda+1)^{\gamma}\right]$$

(because $\{p_n\}$ and $\{\lambda_k \mu_k / k^{-\gamma}\}$ are non-increasing)

$$\le A\lambda_{2\tau} + A\omega^{\gamma}\left(\tau^{\gamma+1}\lambda_\tau p_\tau / P_\tau + \tau^{\gamma}\lambda_{2\tau}\right)$$

$$= O\left(\lambda(2\pi/u)\right) \quad \left(\text{by } (1.4) \text{ and } \tau p_\tau \le P_\tau\right).$$

Finally, by $P(n, k) = \left(P_n p_{n-k} - p_n P_{n-k}\right)/P_n P_{n-1}$, we get

$$K_2 \le A\omega^{\gamma+1} \sum_{n=2\tau+2}^{N} \left|G_\gamma^*(n, m+1, n, \omega)\right|$$

$$= A\omega^{\gamma+1} \sum_{n=2\tau+2}^{N} \left|\sum_{k=m+1}^{n} \left(\frac{p_{n-k}}{P_{n-1}} - \frac{p_n P_{n-k}}{P_n P_{n-1}}\right)\lambda_k \mu_k k^{\gamma} e^{ik\omega}\right|$$

$$\le A\omega^{\gamma+1}\left[\sum_{n=2\tau+2}^{N} \frac{1}{P_{n-1}} \left|\sum_{k=m+1}^{n} p_{n-k}\lambda_k \mu_k k^{\gamma} e^{ik\omega}\right| + \sum_{n=2\tau+2}^{N} \frac{p_n}{P_n P_{n-1}} \left|\sum_{k=m+1}^{n} P_{n-k}\lambda_k \mu_k k^{\gamma} e^{ik\omega}\right|\right]$$

$$= A\omega^{\gamma+1}\left[K_{21} + K_{22}\right],$$

say. Since $\{\lambda_k \mu_k k^{\gamma}\}$ is non-increasing,

$$K_{21} \le \sum_{n=2\nu}^{N} \frac{\lambda_m \mu_m m^{\gamma}}{P_{n-1}} \max_{m+1 \le l \le n} \left|\sum_{k=m+1}^{l} p_{n-k} e^{ik\omega}\right|$$

$$\le AP_\nu \nu^{-\varepsilon} \sum_{n=\nu}^{\infty} \frac{\lambda_n \mu_n n^{\gamma+\varepsilon}}{P_n} \le AP_\nu \frac{\lambda_\nu \nu^{\gamma+1}}{P_\nu} = O\left(\nu^{\gamma+1}\lambda_\tau\right),$$

by Lemma 2 and Condition (1.4), where $\nu = [1/\omega]$. Similarly,

$$K_{22} \le \sum_{n=2\nu+2}^{N} \frac{p_n}{P_n P_{n-1}} \frac{P_{n-m}}{p_{n-m}} \lambda_m \mu_m m^{\gamma} \max_{m+1 \le l \le n} \left|\sum_{k=m+1}^{l} p_{n-k} e^{ik\omega}\right|$$

$$\le AP_\nu \sum_{n=2\nu+2}^{\infty} \frac{\lambda_m \mu_m m^{\gamma}}{P_m} \le AP_\nu \cdot \lambda_\nu \nu^{\gamma+1}/P_\nu = O\left(\nu^{\gamma+1}\lambda_\tau\right).$$

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For the case $P_n \le An^{\beta}\lambda_n$, the above estimations of K_1, K_2 also hold because, for example $\omega^{\gamma}\tau^{-\varepsilon}p_{\tau} \le \omega^{\gamma}\tau^{-1-\varepsilon}P_{\tau} \le A(\omega\tau)^{\gamma}\lambda_{\tau} \le A\lambda_{\tau}$. Collecting these estimations, we obtain (2.9).

Summing (2.2), (2.6), (2.7), and (2.9), we obtain (2.1).

This terminates the proof of Theorem 1.

Now we consider some applications of Theorem 1.

COROLLARY 1. *If* $0 < \alpha \le \beta < 1$, $\gamma \ge 0$, *and*

$$\int_0^{\pi} \left(\log \frac{2\pi}{t}\right)^{\gamma} \left|d\left\{t^{-\beta}\Phi_{\alpha}(t)\right\}\right| < \infty ,$$

then the series $\sum_{n=1}^{\infty} n^{\beta-\alpha}(\log(n+1))^{\gamma-\delta}A_n(t)$ *is summable* $|N, (\log(n+1))^{\lambda}/(n+1)^{1-\beta}|$ *at* $t = x$, *where* $\lambda \ge 0$, $\delta-\gamma > 1$ *for* $\alpha \ne \beta$ *and* $\lambda+\delta-\gamma > 1$ *for* $\alpha = \beta$.

We can restate Theorem 1 in the following form.

COROLLARY 2. *Suppose that* $\{w_n\}$ *is a positive sequence such that* $P_n w_n/n$ *is non-increasing,* $P_n w_n/n^{\beta}\lambda_n$ *is bounded and*

$$\sum_{k=n}^{\infty} \frac{w_k}{k} = O\left(\frac{n^{\beta}\lambda_n}{P_n}\right), \tag{2.10}$$

$$\sum_{k=1}^{\infty} \frac{w_k P_k}{k^{1+\beta}} \left|\cos \frac{\pi}{2}(\beta-\alpha+1)\right| < \infty \quad (0 \le \alpha \le \beta \le 1)$$

hold. Then under the condition (1.6) *we conclude that* $\sum_{n=1}^{\infty} \frac{P_n w_n}{n^{\beta}} A_n(t)$ *is summable* $|N, p_n|$ *at* $t = x$. *If* $P_n \le An^{\beta}\lambda_n$, *the right-hand side of* (2.10) *is replaced* $O(1)$.

The next theorem is a special case of Corollary 2.

COROLLARY 3. *Let* $\{w_n\}$ *be a positive sequence such that* $w_n/n^{1-\alpha}$ *is non-increasing and* $\sum_{k=1}^{\infty} \frac{w_k}{k}$ *is convergent. If* $t^{-\alpha}\Phi_{\alpha}(t)$ $(0 \le \alpha \le 1)$

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is a function of bounded variation in $(0, \pi)$ *, then the series* $\sum_{n=1}^{\infty} w_n A_n(t)$ *is summable* $|C, \alpha|$ *at* $t = x$.

It is worth while to compare the results of Corollaries 2 and 3 with those due to Dikshit [2], Kishore [4], Lal [5], Mehrotra [8], Mohapatra, Das, and Srivastava [9], Prasad and Bhatt [12], and Vershney [13].

3. Proofs and corollaries of Theorems 2 and 3

We need the following lemmas, which are generalizations of Lemmas 3, 4, and 6 of Okuyama [11].

LEMMA 5. $\left\{\left(P_n-P_{n-k}\right)/k^{1-\alpha}\right\}$ $(k = 1, 2, \ldots, n)$ *is a non-decreasing sequence for* $0 \le \alpha \le 1$.

Proof.

$$
\begin{aligned}
k^{1-\alpha}\left(P_n-P_{n-k-1}\right) - (k+1)^{1-\alpha}\left(P_n-P_{n-k}\right) &= \left\{k^{1-\alpha}-(k+1)^{1-\alpha}\right\}P_n - k^{1-\alpha}P_{n-k-1} + (k+1)^{1-\alpha}\left(P_{n-k}+P_{n-k-1}\right) \\
&\ge \left\{k^{1-\alpha}-(k+1)^{1-\alpha}\right\}\left(P_n-P_{n-k-1}\right) + (k+1)^{-\alpha}\left(P_n-P_{n-k-1}\right) \\
&= \left(P_n-P_{n-k-1}\right)(k+1)^{-\alpha}k\left\{(k+1)/k^{\alpha}-1\right\} .
\end{aligned}
$$

Hence we have

$$
\frac{P_n-P_{n-k-1}}{(k+1)^{1-\alpha}} - \frac{P_n-P_{n-k}}{k^{1-\alpha}} \ge \left(P_n-P_{n-k-1}\right)\left\{(k+1)^{\alpha}-k^{\alpha}\right\}/(k+1) \ge 0 .
$$

LEMMA 6. *Let* $\{\Delta p_n\}$ *be non-negative and non-increasing; then the sequence* $\left\{\left(p_{n-k}-p_n\right)/k^{1-\alpha}\right\}$ $(k = 1, 2, \ldots, n)$ *is non-decreasing for* $0 \le \alpha \le 1$.

This follows similarly to Lemma 5.

LEMMA 7. *Suppose that* $t^{1-\alpha}/\lambda(t)\mu(t)$ *and* $t\lambda(t)\mu(t)$ *are non-decreasing for* $0 \le \alpha \le 1$. *If* $g(t)$ *is a function of bounded variation and*

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$$\int_0^{\pi} \frac{\lambda(C/t)\mu(C/t)}{t^{1+\alpha}} |g(t)|dt < \infty$$

for some constant $C > \pi$, *then the series*

$$\sum_{n=2}^{\infty} \frac{\lambda_n \mu_n |g(\theta_n)|}{n^{1-\alpha}}$$

converges, where θ_n *is a continuity point of* $g(t)$ *such that* $\pi/(n+1) \le \theta_n < \pi/n$, $n \ge 2$.

The proof runs similarly to that of Lemma 6 of Okuyama [11].

Proof of Theorem 2. We may confine ourselves to the case $0 < \alpha = \beta < 1$. When $\alpha \ne \beta$, we only use $\Lambda(t) = t^{\beta-\alpha}\lambda(t)$ instead of $\lambda(t)$. Since

$$\begin{aligned} B_k(x) &= \frac{2}{\pi} \int_0^{\pi} \psi(t) \sin ktdt \\ &= \frac{2}{\pi} \left[\Psi_1(t) \sin kt\right]_0^{\pi} - \frac{2}{\pi} \int_0^{\pi} \Psi_1(t)k \cos ktdt \\ &= -\frac{2}{\pi} \int_0^{\pi} \cos ktdt \frac{k}{\Gamma(1-\alpha)} \int_u^t (t-u)^{-\alpha}\Psi_\alpha(u)du \\ &= -\frac{2}{\pi\Gamma(1-\alpha)} \int_0^{\pi} \Psi_\alpha(u)du \int_0^{\pi} (t-u)^{-\alpha}k \cos ktdt , \end{aligned}$$

we get for the nth (N, p_n) mean $\tilde{t}_n$ of $\sum_{n=1}^{\infty} \lambda_n\mu_n B_n(x)$,

$$\begin{aligned} \tilde{t}_n - \tilde{t}_{n-1} &= \sum_{k=1}^{n} P(n, k)\lambda_k\mu_k B_k(x) \\ &= -\int_0^{\pi} \Psi_\alpha(u)du \frac{2}{\pi\Gamma(1-\alpha)} \int_u^{\pi} (t-u)^{-\alpha}G_1(n; l, n, t)dt \\ &= -\int_0^{\pi} H_1(n; l, n, u)\Psi_\alpha(u)du . \end{aligned}$$

Let θ_n be a continuity point of $\Psi_\alpha(u)$ such that $\pi/(n+1) \le \theta_n < \pi/n$ for $n \ge 1$. Then

$$(3.1)\qquad \sum_{n=1}^{\infty} |\tilde{t}_n - \tilde{t}_{n-1}| < A \sum_{n=1}^{\infty} \left\{ \left| \int_0^{\theta_n} \right| + \left| \int_{\theta_n}^{\pi} \right| \right\} H_1(n;\, l,\, n,\, u)\Psi_\alpha(u)du$$
$$= L_1 + L_2 ,$$

say.

We define

$$Q_n(u) = \begin{cases} H_1(n;\, l,\, n,\, u) & \text{for} \quad 0 \le u < \theta_n , \\ 0 & \text{for} \quad \theta_n \le u \le \pi . \end{cases}$$

Since $Q_n(u) = 0$ for $n \ge 2\tau+1 \ge \pi/u$, we have by (2.3),

$$\begin{aligned} \sum_{n=1}^{\infty} |Q_n(u)| &\le \sum_{n=1}^{2\tau+1} |Q_n(u)| \\ &\le A \sum_{n=1}^{2\tau+1} \sum_{k=1}^{n} P(n,\, k)\lambda_k\mu_k k^\alpha \\ &\le A \sum_{k=1}^{2\tau+1} k\lambda_k\mu_k k^{\alpha-1} \sum_{n=k}^{2\tau+1} P(n,\, k) \\ &\le A\tau\lambda_{2\tau+1}\mu_{2\tau+1} \sum_{k=1}^{2\tau+1} k^{\alpha-1} \\ &= O\big(\tau^{\alpha+1}\lambda(2\pi/u)\mu(2\pi/u)\big) , \end{aligned}$$

by Lemma 3 and the non-decreasing property of $\{k\lambda_k\mu_k\}$. Thus, by (1.7), we have

$$(3.2)\qquad L_1 \le \int_0^{\pi} |\Psi_\alpha(u)| \sum_{n=1}^{\infty} |Q_n(u)| du$$
$$\le A \int_0^{\pi} \frac{\lambda(2\pi/u)\mu(2\pi/u)}{u^{1+\alpha}} |\Psi_\alpha(u)| du < \infty .$$

Observing that

$$\int_{\theta_n}^{u} h(v)dv = \int_{\theta_n}^{u} dv \int_{\theta_n}^{\pi} (t-v)^{-\alpha} \cos ktdt - \int_{\theta_n}^{u} dv \int_{\theta_n}^{v} (t-v)^{-\alpha} \cos ktdt$$

$$= \int_{\theta_n}^{\pi} \cos ktdt \int_{\theta_n}^{u} (t-v)^{-\alpha}dv - \int_{\theta_n}^{u} \cos ktdt \int_{t}^{u} (t-v)^{-\alpha}dv$$

$$= -\frac{1}{1-\alpha} \int_{\theta_n}^{\pi} (t-u)^{1-\alpha} \cos ktdt + \frac{1}{1-\alpha} \int_{\theta_n}^{\pi} \left(t-\theta_n\right)^{1-\alpha} \cos ktdt$$

$$+ \frac{1}{1-\alpha} \int_{\theta_n}^{u} (t-u)^{1-\alpha} \cos ktdt ,$$

we get, by integration by parts and simple calculations

$$-\frac{2k}{\pi\Gamma(1-\alpha)} \int_{\theta_n}^{\pi} h(u)\Psi_\alpha(u)du = \frac{2k}{\pi(1-\alpha)\Gamma(1-\alpha)} \left\{- \int_{\theta_n}^{\pi} d\Psi_\alpha(u) \int_{u}^{\pi} (t-u)^{1-\alpha} \cos ktdt\right.$$

$$\left. - \Psi_\alpha\left(\theta_n\right) \int_{\theta_n}^{\pi} \left(t-\theta_n\right)^{1-\alpha} \cos ktdt\right\}$$

$$= \frac{2}{\pi\Gamma(1-\alpha)} \left\{\int_{\theta_n}^{\pi} d\Psi_\alpha(u) \int_{u}^{\pi} (t-u)^{-\alpha} \sin ktdt\right.$$

$$\left. + \Psi_\alpha\left(\theta_n\right) \int_{\theta_n}^{\pi} \left(t-\theta_n\right)^{-\alpha} \sin ktdt\right\} .$$

Hence

$$(3.3) \quad L_2 \le \sum_{n=1}^{\infty} \left|\int_{\theta_n}^{\pi} \tilde{H}_0(n;\, l,\, n,\, u)d\Psi_\alpha(u)\right| + \sum_{n=1}^{\infty} \left|\Psi_\alpha\left(\theta_n\right)\right| \left|\tilde{H}_0\left(n;\, l,\, n,\, \theta_n\right)\right|$$

$$= L_{21} + L_{22} ,$$

say. Considering

$$(3.4) \quad \tilde{H}_0(n;\, j,\, l,\, u) = \frac{2}{\pi^{1+\alpha}\Gamma(1-\alpha)} \sum_{k=j}^{l} P(n,\, k)\lambda_k\mu_k k^{\alpha-1}\left\{\frac{\pi}{1-\alpha} \sin k\xi\right.$$

$$\left. - \cos k\eta - \cos ku\right\} \quad \left(\text{see } (2.3)\right),$$

we have

$$\tilde{H}_0(n;\ j,\ l,\ u) = O\left(\sum_{k=j}^{l} P(n,\ k)\lambda_k\mu_k k^{\alpha-1}\right) .$$

Thus, using Lemmas 5, 6, and 7, we get by the same calculations as those of I_{21} of Okuyama [11],

$$(3.5) \qquad L_{22} \le A \sum_{n=1}^{\infty} |\Psi_\alpha(\theta_n)| \sum_{k=1}^{n} P(n,\ k)\lambda_k\mu_k k^{\alpha-1}$$
$$\le A \sum_{n=2}^{\infty} \frac{\lambda_n\mu_n|\Psi_\alpha(\theta_n)|}{n^{1-\alpha}} < \infty .$$

Next, we shall estimate L_{21} . We define

$$R_n(u) = \begin{cases} 0 & \text{for} \quad 0 \le u < \theta_n , \\ \tilde{H}_0(n;\ l,\ n,\ u) & \text{for} \quad \theta_n \le u \le \pi . \end{cases}$$

Since $R_n(u) = 0$ for $n \le \tau-1 < \pi/u - 1$, we have

$$L_{21} \le \sum_{n=1}^{\infty} \int_0^{\pi} |R_n(u)||d\Psi_\alpha(u)|$$
$$\le \int_0^{\pi} \sum_{n=\tau}^{\infty} |R_n(u)||d\Psi_\alpha(u)| .$$

Hence, by (1.7), to prove the finiteness of L_{21} , it suffices to show that, uniformly in $0 < t \le \pi$,

$$(3.6) \qquad M = \sum_{n=\tau}^{\infty} |R_n(u)|$$
$$= \sum_{n=\tau}^{\infty} |\tilde{H}_0(n;\ l,\ n,\ u)| = O\left(u^{-\alpha}\lambda(2\pi/u)\right) .$$

We divide M in the following form:

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$$M \le \sum_{n=\tau}^{2\tau+1} |\tilde{H}_0(n;\ l,\ n,\ u)| + \sum_{n=2\tau+2}^{\infty} |\tilde{H}_0(n;\ l,\ \tau,\ u)| + \sum_{n=2\tau+2}^{\infty} |\tilde{H}_0(n;\ \tau+1,\ n,\ u)|$$

$$= M_1 + M_2 + M_3 ,$$

say. Then

$$M_1 \le A \sum_{n=\tau}^{2\tau+1} \sum_{k=1}^{n} P(n,\ k)\lambda_k \mu_k k^{\alpha-1}$$

$$= A \sum_{k=1}^{\tau} \frac{\lambda_k \mu_k}{k^{1-\alpha}} \sum_{n=\tau}^{2\tau+1} P(n,\ k) + A \sum_{k=\tau}^{2\tau+1} \frac{\lambda_k \mu_k}{k^{1-\alpha}} \sum_{n=k}^{2\tau+1} P(n,\ k)$$

$$\le A\lambda_\tau \sum_{k=1}^{\tau} k^{\alpha-1} + A\lambda_{2\tau+1} \sum_{k=\tau}^{2\tau+1} k^{\alpha-1} = O\left(\tau^{\alpha}\lambda_{2\tau}\right) .$$

Using (3.4) instead of (2.4), we may treat M_2 and M_3 by easier methods than those used for J_2 and J_3 in §2. Thus we have (3.6). Combining (3.1), (3.2), (3.3), (3.5), and (3.6), Theorem 2 is completely proved.

Since the calculations to prove Theorem 3 are similar to those for Theorem 1, we omit them.

Using Theorems 2 and 3, we obtain several corollaries which are parallel to those of §2 or Okuyama's paper [11].

We shall show one of them.

COROLLARY 4. *If* $0 < \alpha \le \beta < 1$,

$$\int_0^{\pi} t^{-1-\beta}\left(\log \frac{2\pi}{t}\right)^{\gamma} |\Psi_\alpha(t)| dt < \infty \quad \left(\text{or} \quad \Psi_\alpha(+0) = 0\right)$$

and

$$\int_0^{\pi} t^{-\beta}\left(\log \frac{2\pi}{t}\right)^{\gamma+1} |d\Psi_\alpha(t)| < \infty ,$$

then the series $\sum_{n=2}^{\infty} n^{\beta-\alpha}(\log n)^{\gamma} B_n(t)$ *is summable* $|\bar{N},\ (\log n)^{\lambda}/n^{1-\beta}|$ *at* $t = x$, *where* $0 < \gamma+1 < \lambda$.

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