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# INVERSE SEMIGROUPS ALL OF WHOSE PROPER HOMOMORPHIC IMAGES ARE GROUPS

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We characterise those inverse semigroups whose proper (non-isomorphic) homomorphic images are all groups. We also show that the bicyclic semigroup is the only such semigroup in certain cases.

## 1. INTRODUCTION

The *bicyclic semigroup* is defined as $C = \langle p, q \mid pq = 1 \rangle$. It is well known [**2**, Corollary 3.2], that every proper (non-isomorphic) homomorphic image of the bicyclic semigroup $C$ is a group. (In fact, every proper homomorphic image of $C$ is a cyclic group; however, we shall not use the cyclic property.) We shall refer to inverse semigroups all of whose proper homomorphic images are groups as *h-groups*; to eliminate certain trivial cases, we shall require that an $h$-group $S$ does have homomorphic images other than itself and the one-element semigroup, that $S$ is not a group, and that $S$ has more than two elements. In this paper we characterise $h$-groups in general. We also show that the bicyclic semigroup is the only $h$-group in certain cases.

There seem to be few published results on $h$-groups. Apparently Tamura [**14**] was the first person to ask about the structure of $h$-groups in his review of [**1**]. Bernstein has shown that $h$-groups are simple, and that they contain a copy of $C$; see [**1**, Theorem 1.1, Theorem 1.2, Corollary 1.3]. Fortunatov [**3**, Corollary 3] has given examples of $h$-groups which are similar to the bicyclic semigroup. Reilly [**12**, Lemma 3.3] has characterised the full inverse subsemigroups of $T_E$ which are $h$-groups, where $E$ is a semilattice which is a dense tree without zero and $T_E$ is the semigroup of all order isomorphisms between principal order ideals of $E$. Such semigroups have a modular lattice of congruences. Goberstein [**4**, Corollary 4.15] has proved that if an inverse semigroup has no idempotent-separating congruences and certain order relations are trivial, then that semigroup is an $h$-group. In a different vein, Munn [**8**] has characterised inverse semigroups which have no congruences except for the identity congruence and the universal congruence; these semigroups are known as congruence-free semigroups. As we shall see, $h$-groups are a natural generalisation of congruence-free inverse semigroups.

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More generally, several authors have studied semigroups whose congruences all share some property. Bernstein [**1**], Putcha [**11**], and Tamura and Hamilton [**15**] have studied semigroups such that every homomorphic image which is contained in a group is itself a group. Jensen [**5**] and Schein [**13**] have worked on semigroups $S$ whose non-trivial homomorphic images are all isomorphic to $S$. Zhu [**17**] has looked at semigroups whose non-identity congruences are all Rees congruences. Kowol and Mitsch [**6**] have studied finite Clifford semigroups whose non-trivial homomorphic images all have a non-trivial centre.

A semigroup $S$ is *regular* if for every $s \in S$ there is $t \in S$ such that $s = sts$. An element $t \in S$ is an *inverse of* $s$ if $s = sts$ and $t = tst$. A semigroup $S$ is *inverse* if $S$ is regular and every element of $S$ has a unique inverse. It is known [**2**, Theorem 1.17] that a regular semigroup is inverse if and only if its idempotents commute with each other, which implies that the product of two idempotents is an idempotent. If $s \in S$ then we denote the inverse of $s$ by $s^{-1}$. It is easy to see that the bicyclic semigroup is inverse. In what follows every semigroup will be inverse. The set of idempotents is denoted $E$, and the cardinality of this set is denoted $|E|$. Note that, if $|E| = 1$, then $S$ is a group. An inverse semigroup admits a partial order as follows: if $s, t \in S$, then $s \leqslant t$ if $s = et$ for some $e \in E$.

A semigroup $S$ is *congruence-free* if every congruence on $S$ is either the universal congruence or the identity congruence. A *group congruence* is a congruence $\rho$ such that $S/\rho$ is a group. An *idempotent separating congruence* is a congruence $\rho$ such that $\rho \cap (E \times E)$ is the identity congruence on $E$. Every inverse semigroup has a unique maximal idempotent-separating congruence $\mu$ and a unique minimal group congruence $\sigma$; see [**9**, p. 131, p. 142].

If $\rho$ is a congruence, then the *kernel of* $\rho$ is the set $\ker \rho = \{s \in S \mid s\rho e$ for some $e \in E\}$. The *trace of* $\rho$ is the restriction of $\rho$ to the set of idempotents, denoted $\operatorname{tr} \rho$. The *centraliser of the set of idempotents* is the set $C(E) = \{s \in S \mid se = es$ for all $a \in E\}$.

## 2. The General Case

An inverse semigroup $S$ is an *h-group* if $S$ is not congruence-free, $S$ is not a group, $|S| > 2$, and every proper homomorphic image of $S$ is a group.

Note that every semigroup with two elements has the one-element group as its only proper homomorphic image; for this reason, we eliminate this case from the definition. We need the next result to characterise $h$-groups.

**Theorem 1.** *If $S$ is an inverse semigroup then* $\ker \mu = C(E)$.

Proof: This is found in [**7**, Proposition 3, p. 139] or [**9**, Theorem III.3.5]. ∎

Theorem 2 is based on the characterisation of congruence-free inverse semigroups due to Munn in [**8**]. The proof of Theorem 2 follows that of [**9**, Theorem IV.3.3] almost exactly.

**THEOREM 2.** *Let $S$ be an inverse semigroup such that $|E| > 1$; that is, $S$ is not a group. Then $S$ is an $h$-group if and only if $S$ satisfies the following conditions:*

(a) $C(E) = E$;

(b) *for any $e, f, g, h \in E$ with $e > f$ and $g > h$, there exist $t_1, t_2, \ldots, t_n \in S$ such that $g = t_1^{-1}et_1$, $t_i^{-1}ft_i = t_{i+1}^{-1}et_{i+1}$ for $1 \leqslant i < n$, and $t_n^{-1}ft_n \leqslant h$.*

PROOF: Let $S$ be an $h$-group. By supposition there exist $e \neq f$ in $S$; hence $e\mu \neq f\mu$ and so $S/\mu$ cannot be a group. Therefore, since $S$ is an $h$-group, we have that $\mu$ is the identity congruence, and hence $\ker\mu = C(E) = E$ by Theorem 1. This proves (a).

To prove (b), let $e, f, g, h \in E$ be such that $e > f$ and $g > h$. Let $\tau = \{e,\ f\} \times \{e,\ f\}$. Define $\tau^+$ on $E$ as follows: if $a, b \in E$, then $a\tau^+b$ if and only if $a = b$ or $a = s_1^{-1}u_1s_1$, $s_i^{-1}v_is_i = s_{i+1}^{-1}u_{i+1}s_{i+1}$ for $1 \leqslant i < n$, $s_n^{-1}v_ns_n = b$, where the $s_i$'s are in $S$ and $u_i, v_i \in E$ such that either $u_i\tau v_i$ or $v_i\tau u_i$ for $1 \leqslant i \leqslant n$. Note that, by [**9**, Lemma IV.3.2], the relation $\tau^+$ is a normal congruence on $E$ which contains $\tau$. Then by [**9**, Theorem III.2.5] there is a congruence $\rho$ on $S$ such that the restriction of $\rho$ to $E$ is $\tau^+$. Since $\tau$ is not the identity congruence on $E$, then $\rho$ is not the identity congruence on $S$, and hence by hypothesis $\tau^+$ must be the universal congruence on $E$. Hence, substituting $g, h$ for $a, b$, respectively, we have that there exist $s_i \in S$ and $u_i, v_i \in \{e,\ f\}$ such that

$$(1) \qquad g = s_1^{-1}u_1s_1,\ s_i^{-1}v_is_i = s_{i+1}^{-1}u_{i+1}s_{i+1} \quad \text{for all} \quad 1 \leqslant i < n,\ s_n^{-1}v_ns_n = h.$$

Without loss of generality we can assume that $u_i \neq v_i$. For $i = 1, \ldots, n$, we define $p_i = (s_i^{-1}u_is_i)(s_{i-1}^{-1}u_{i-1}s_{i-1}) \cdots (s_1^{-1}u_1s_1)$, $t_i = s_ip_i$. Note that each $u_i$, $s_i^{-1}u_is_i$, and $p_i$ is an idempotent. Further, as in the proof of [**9**, Theorem IV.3.3],

$$(2) \qquad t_i^{-1}u_it_i = p_i \quad \text{for} \quad 1 \leqslant i \leqslant n.$$

In particular, $g = p_1 = t_1^{-1}u_1t_1$, and by (1) and (2), also $t_i^{-1}v_it_i = t_{i+1}^{-1}u_{i+1}t_{i+1}$ for all $1 \leqslant i < n$, and $t_n^{-1}v_nt_n = (s_n^{-1}v_ns_n)p_n = hp_n \leqslant h$. We have shown that

$$(3) \qquad g = t_1^{-1}u_1t_1,\ t_i^{-1}v_it_i = t_{i+1}^{-1}u_{i+1}t_{i+1} \quad \text{for all} \quad 1 \leqslant i < n,\ t_n^{-1}v_nt_n \leqslant h.$$

By (2) we also have

$$(4) \qquad t_i^{-1}u_it_i = p_i \geqslant p_i(s_i^{-1}v_is_i)p_i = t_i^{-1}v_it_i \quad \text{for all} \quad 1 \leqslant i \leqslant n.$$

When $(u_i, v_i) = (f, e)$ it follows from $f < e$ that $t_i^{-1}u_it_i \leqslant t_i^{-1}v_it_i$, which by (4) yields that $t_i^{-1}u_it_i = t_i^{-1}v_it_i$. Thus we may delete all terms involving $u_i$ and $v_i$ whenever $(u_i, v_i) = (f, e)$. Therefore, (b) is satisfied.

Conversely, suppose that $S$ satisfies (a) and (b). Let $\rho$ be a congruence on $S$. If $\operatorname{tr}\rho$ is the identity congruence on $E$, then $\rho$ is idempotent separating, so $\rho \subseteq \mu$ and thus $\rho$ is the identity congruence on $S$ because $C(E) = E$. Hence, assume that $\operatorname{tr}\rho$ is not the identity congruence on $E$. Then there are distinct $e, f \in E$ such that $e\rho f$. Then $e\rho ef \rho f$, so we may assume that $e > f$. Let $g, h \in E$ be such that $g > h$. By (b), we have

$$g = t_1^{-1}et_1\rho t_1^{-1}ft_1 = t_2^{-1}et_2\rho \cdots \rho t_n^{-1}ft_n = t \tag{5}$$

so that $g\rho t$ and $t \leqslant h$. This implies that $h = hg\rho ht = t$, so that $g\rho h$. On the other hand, if $g$ and $h$ are not comparable, then $gh < g$; hence $g\rho gh$ by what has just been shown. Similarly, $h\rho gh$. Therefore, again we have that $g\rho h$. Hence $\operatorname{tr}\rho$ is the universal congruence on $E$, which makes $\rho$ a group congruence. ∎

Munn [8] has characterised congruence-free inverse semigroups as those inverse semigroups which satisfy the conditions of Theorem 2 and also have no group congruences. Hence, if we take the conditions for a congruence-free inverse semigroup $S$ and remove the condition that $S$ has no group congruences, then we obtain an $h$-group. For this reason, it is natural to think of $h$-groups as generalisations of congruence-free inverse semigroups.

## 3. Special Cases

In this section we show that the bicyclic semigroup is the only $h$-group in certain classes of semigroups.

Bruck semigroups. Recall that a *Bruck semigroup over a monoid $T$* is defined as follows: Let $T$ be a monoid, $\alpha$ be a homomorphism of $T$ into its group of units, and $N$ be the set of all non-negative integers. On $S = N \times T \times N$ define a multiplication by

$$(m, a, n)(p, b, q) = (m + n - r, (a\alpha^{p-r})(b\alpha^{n-r}), n + q - r)$$

where $r = \min\{n, p\}$ and $\alpha^0$ is the identity map on $T$. If $S$ is both a Bruck semigroup and an $h$-group, we say that $S$ is a *Bruck $h$-group*.

**Theorem 3.** *The bicyclic semigroup is the only Bruck $h$-group.*

Proof: Note that the bicyclic semigroup is the Bruck semigroup with $T$ the one-element monoid. Conversely, suppose that $S$ is a Bruck $h$-group. Take the homomorphism $\phi : (m, a, n) \to (m, 1, n)$ where 1 is the identity of $T$. Then $\phi(S) \cong C$. Since $S$ is an $h$-group, the map $\phi$ must be 1-1, so that $S \cong C$. ∎

$\omega$-$h$-groups. An inverse semigroup $S$ is an *$\omega$-semigroup* if the set of idempotents is linearly ordered with ordering the opposite of that of the non-negative integers. If $S$ is an $\omega$-semigroup and an $h$-group, we say that $S$ is an *$\omega$-$h$-group*.

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**Lemma 4.** [**1**, Theorem 1.1] *If $S$ is an $h$-group, then $S$ is simple.*

**Theorem 5.** *A semigroup $S$ is a simple $\omega$-semigroup if and only if it is a Bruck semigroup with $T$ being a chain of groups.*

Proof: This follows from [**10**, Structure Theorem, p. 89]. ∎

**Theorem 6.** *The bicyclic semigroup is the only $\omega$-$h$-group.*

Proof: Clearly, the bicyclic semigroup is an $\omega$-$h$-group. Conversely, if $S$ is an $\omega$-$h$-group, then $S$ is simple by Lemma 4, and hence $S$ is a Bruck semigroup by Theorem 5. The result now follows from Theorem 3. ∎

Recall that a *Reilly semigroup* is a bisimple $\omega$-semigroup. Hence, we have the following result.

**Corollary 7.** *The bicyclic semigroup is the only Reilly $h$-group.*

Proof: By [**2**, Theorem 2.53] the bicyclic semigroup is bisimple, and hence is a Reilly $h$-group. Conversely, a Reilly $h$-group is an $\omega$-$h$-group by definition. The result now follows from Theorem 6. ∎

$\omega^n$-bisimple semigroups. Let $N$ be the set of non-negative integers and let $n$ be any positive integer. Define the reverse lexicographical order on $N^n$ as follows: $(a_1, a_2, \ldots, a_n) < (b_1, b_2, \ldots, b_n)$ if and only if $a_1 > b_1$, or there is some index $k$, where $1 < k < n$, such that $a_1 = b_1$, $a_2 = b_2, \ldots, a_{k-1} = b_{k-1}$, and $a_k > b_k$. An *$\omega^n$-bisimple semigroup* [**16**] is a bisimple semigroup whose idempotents are order isomorphic to the set $N^n$ under reverse lexicographical order.

**Theorem 8.** *Let $S$ be an $\omega^n$-bisimple semigroup, and let $k < n$. Then $S$ has a congruence $\rho$ such that $S/\rho$ is an $\omega^k$-bisimple semigroup.*

Proof: This is [**16**, Corollary 1.1]. ∎

**Theorem 9.** *A semigroup $S$ is an $\omega^n$-bisimple $h$-group if and only if $n = 1$ and $S \cong \mathcal{C}$.*

Proof: By [**2**, Theorem 2.53] the bicyclic semigroup $\mathcal{C}$ is a bisimple $\omega^n$-semigroup, where $n = 1$. By Theorem 6 $\mathcal{C}$ is an $h$-group. Hence $\mathcal{C}$ is an $\omega^n$-bisimple $h$-group.

Conversely, suppose that $S$ is an $\omega^n$-bisimple $h$-group. By Theorem 8 we have $n = 1$. Hence $S$ is an $\omega$-$h$-group. The result now follows from Theorem 6. ∎

In light of previous results, it is natural to ask if an $h$-group is uniquely determined by its maximal group homomorphic image. We show that this is not the case. Recall that the maximal group homomorphic image of the bicyclic semigroup is $\mathbb{Z}$, the group of integers, and that the map $\phi : \mathcal{C} \to \mathbb{Z}$ is given by $\phi(q^m p^n) = m - n$; see [**7**, Section 3.4, Theorem 5].

**Example 10.** *There exists an h-group $S$ whose maximal homomorphic image is $\mathbb{Z}$, but $S \not\cong \mathcal{C}$.*

Let $\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_n, \dots$ be a countably infinite collection of bicyclic semigroups. We can consider each $\mathcal{C}_j$ to be a proper subsemigroup of $\mathcal{C}_{j+1}$ by [**2**, Theorem 2.54]. Let $S = \bigcup_{j=1}^{\infty} \mathcal{C}_j$, with the natural operation.

To see that $S$ is an $h$-group, let $\rho$ be a non-trivial congruence on $S$, let $a, b$ be distinct elements of $S$ such that $a\rho b$, and let $e, f$ be distinct idempotents of $S$. Find $j$ such that $a, b, e, f \in \mathcal{C}_j$. Since $\mathcal{C}_j$ is bicyclic, we get that $e\rho f$. Since $e, f$ are arbitrary idempotents, we have that $S/\rho$ is a group.

We can show that the maximal homomorphic image of $S$ is $\mathbb{Z}$ by using the same argument as that in the proof of [**7**, Section 3.4, Theorem 5]. Finally, $S \not\cong \mathcal{C}$ because $S$ has an infinite ascending chain of idempotents.

## 4. Open Problems

Problem 1. An inverse semigroup $S$ is *E-unitary* if, when $s \in S$ and $e \in E$, then $es \in E$ implies that $s \in E$. The congruences on $E$-unitary semigroups have been characterised; see [**9**, Theorem VII.2.1]. Use this characterisation to sharpen Theorem 2 for $h$-groups which are $E$ unitary.

Problem 2. Use Theorem 2 to derive the characterisations of $h$-groups in [**3**] or [**4**].

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