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ON M-STRUCTURE AND WEAKLY COMPACTLY GENERATED BANACH SPACES

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Abstract It is well known that every non-reflexive M-ideal is weakly compactly generated (in short, WCG). We present a family of Banach spaces $\{V_s : 0 < s < 1\}$ which are not WCG and such that every V_s satisfies the inequality

$$\|\varphi\| \geqslant \|\pi\varphi\| + s\|\varphi - \pi\varphi\| \quad \forall \varphi \in V_s^{***},$$

where π is the canonical projection from V_s^{***} onto V_s^{*}. In particular, no V_s can be renormed to be an M-ideal.

Keywords: M-ideal; $M(r,s)$-inequality; weakly compactly generated (WCG) Banach space

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1. Introduction

Let X be a Banach space. Recall that X is an *M-ideal* if

$$\|\varphi\| = \|\pi\varphi\| + \|\varphi - \pi\varphi\| \quad \forall \varphi \in X^{***},$$

where π denotes the canonical projection from X^{***} onto X^* (for complete information about M-ideals, see [**6**]). X is said to be *weakly compactly generated* (in short, WCG) if X is the closed linear span of a weakly compact subset of X. Fabian and Godefroy showed in [**4**] (cf. [**6**]) that every non-reflexive M-ideal is WCG. Indeed, it is proved in [**1**] that if X is an Asplund space with the property that for each $\varphi \in X^{***}$, $\|\varphi - \pi\varphi\| < \|\varphi\|$ whenever $\pi\varphi \neq 0$, then X is WCG.

Following [**1**] and [**2**], a Banach space X *satisfies the $M(r,s)$-inequality* if

$$\|\varphi\| \geqslant r\|\pi\varphi\| + s\|\varphi - \pi\varphi\| \quad \forall \varphi \in X^{***}$$

holds for given $r, s \in\ ]0,1]$. It is clear that the notion of $M(1,1)$-inequality coincides with that of M-ideal.

Following Johnson and Lindenstrauss [**7**], for every $s \in\ ]0,1[$, we present a Banach space V_s which is not WCG and which satisfies the $M(1,s)$-inequality, resolving the open question formulated in [**1**] of whether there are Banach spaces satisfying the $M(1,s)$-inequality without being WCG. As a consequence, V_s cannot be renormed to be an M-ideal (see Remarks 2.6 below). In Remarks 2.6, we collect some properties derived from the $M(1,s)$-inequality. For instance, we show that for every $s \in]0,1[$, V_s^* contains no proper norming subspaces, V_s is an Asplund space, V_s has the unique extension property, $V_s^* = \overline{\operatorname{lin}}(w^*\text{-sexp}\, B_{V_s^*})$ (where $w^*\text{-sexp}\, B_{V_s^*}$ denotes the set of the weak* strongly exposed points of $B_{V_s^*}$), and V_s has property (u) of Pełczyński.

In a Banach space X, the closed unit ball is denoted by B_X. The Banach space of all bounded linear operators on X will be denoted by $\mathcal{L}(X)$. For a set $A \subset X$, we denote by $\operatorname{lin} A$ its linear span.

In the next section, we exhibit some techniques for constructing Banach spaces satisfying the $M(r,s)$-inequality.

2. Example

First, we show some results which will be a key tool in the construction of examples of Banach spaces satisfying the $M(r,s)$-inequality. The next result generalizes a result in [**8**].

Proposition 2.1. *Let X be a Banach space and $r, s \in]0,1]$. If there exists a net (V_α) in $\mathcal{L}(X^{**})$ satisfying*

(1) *for each α, $\operatorname{Im} V_\alpha \subset X$;*

(2) *for each $x^* \in X^*$ and $x^{**} \in X^{**}$, $\lim_\alpha x^*(V_\alpha x^{**}) = x^{**}x^*$; and*

(3) *for each $x \in B_X$ and $x^{**} \in B_{X^{**}}$,*

$$\limsup_\alpha \|rV_\alpha x + s(x^{**} - V_\alpha x^{**})\| \leqslant 1;$$

then X satisfies the $M(r,s)$-inequality.

Proof. Let $\varepsilon > 0$ and $\varphi \in X^{***}$, and consider $x^{**} \in B_{X^{**}}$ and $x \in B_X$ such that

$$\pi\varphi(x) > \|\pi\varphi\| - \varepsilon, \qquad (\varphi - \pi\varphi)(x^{**}) > \|\varphi - \pi\varphi\| - \varepsilon.$$

By (2) and (3), there is α sufficiently large such that

$$\begin{aligned} \pi\varphi(V_\alpha x) &> \|\pi\varphi\| - \varepsilon, \\ \|rV_\alpha x + s(x^{**} - V_\alpha x^{**})\| &\leqslant 1 + \varepsilon, \\ |\pi\varphi(x^{**} - V_\alpha x^{**})| &< \varepsilon. \end{aligned}$$

Hence,

$$\begin{aligned} (1+\varepsilon)\|\varphi\| &\geqslant \varphi(rV_\alpha x + s(x^{**} - V_\alpha x^{**})) \\ &= r\pi\varphi(V_\alpha x) + s(\pi\varphi(x^{**} - V_\alpha x^{**}) + (\varphi - \pi\varphi)(x^{**} - V_\alpha x^{**})) \\ &> r(\|\pi\varphi\| - \varepsilon) - \varepsilon s + s(\|\varphi - \pi\varphi\| - \varepsilon). \end{aligned}$$

Letting $\varepsilon \to 0$, the result follows. $\square$

Corollary 2.2. *Let X be a Banach space and $r, s \in]0, 1]$. If there exists a net (T_α) in $\mathcal{L}(X)$ satisfying*

(1) *for each α, $\operatorname{Im} T_\alpha^{**} \subset X$;*

(2) *for each $x^* \in X^*$ and $x^{**} \in X^{**}$, $\lim_\alpha x^*(T_\alpha^{**} x^{**}) = x^{**} x^*$; and*

(3) *for each $x \in B_X$,*
$$\limsup_\alpha \sup_{y \in B_X} \|rT_\alpha x + s(y - T_\alpha y)\| \leqslant 1;$$

then X satisfies the $M(r, s)$-inequality.

Proof. This follows from the weak* density of B_X in $B_{X^{**}}$ and the above result. □

Proposition 2.3. *Let X be a Banach space, $r, s \in]0, 1]$ and a net (T_α) in $\mathcal{L}(X)$. The following assertions are equivalent.*

(1) $\limsup_\alpha \sup_{x,y \in B_X} \|rT_\alpha x + s(y - T_\alpha y)\| \leqslant 1.$

(2) $\limsup_\alpha \sup_{x^* \in B_{X^*}} (r\|T_\alpha^* x^*\| + s\|x^* - T_\alpha^* x^*\|) \leqslant 1.$

Proof. For each α, consider the operator $V_\alpha : X \oplus_\infty X \longrightarrow X$ defined by
$$V_\alpha(x, y) = rT_\alpha x + s(y - T_\alpha y) \quad \forall x, y \in X.$$
Then (1) means that $\limsup_\alpha \|V_\alpha\| \leqslant 1$, and (2) means that $\limsup_\alpha \|V_\alpha^*\| \leqslant 1$. □

Now we present a Banach space V similar to the Banach space U given in [**7**, Example 1]. Let $\{N_\gamma : \gamma \in \Gamma\}$ be a collection of infinite subsets of the integers such that $N_\gamma \cap N_{\gamma'}$ is finite for $\gamma \neq \gamma'$, and such that Γ has the cardinality of the continuum. Denote
$$V_0 = \left\{(y, z) \in \ell_\infty \bigoplus_\infty c_{00}(\Gamma) : \lim_n \left(y(n) - \sum_{\gamma \in \Gamma_n} z(\gamma)\right) = 0\right\},$$
where for each $n \in \mathbb{N}$, $\Gamma_n = \{\gamma \in \Gamma : n \in N_\gamma\}$, ℓ_∞ and $c_{00}(\Gamma)$ are equipped with the usual sup norm $\|\cdot\|_\infty$. V denotes the completion of V_0.

Observe that, given $(y, z) \in V_0$, we have that for each $\gamma \in \Gamma$, $z(\gamma) = \lim_{n \in N_\gamma} y(n)$. So, $\|(y, z)\| = \|y\|_\infty$. On the other hand, note that the functionals $(y, z) \mapsto y(n)$, $n \in \mathbb{N}$, form a total set in V_0^*. Hence, V^* is w^*-separable. Therefore, the proof of the next result follows from the one given in [**7**, Example 1].

Proposition 2.4. *V satisfies the following properties.*

(1) *V has a subspace X isometric to c_0 such that V/X is isometric to $c_0(\Gamma)$.*

(2) *V is not WCG.*

(3) *V^* is isomorphic to $\ell_1 \oplus \ell_1(\Gamma)$ and thus is not WCG.*

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Theorem 2.5. *For every $s \in \,]0,1[$, V admits an equivalent norm $\|\cdot\|_s$ for which $V_s = (V, \|\cdot\|_s)$ satisfies the $M(1,s)$-inequality.*

Proof. Let $s \in \,]0,1[$, and define

$$a_s = \begin{cases} (1-s)/(1+s) & \text{if } s \leqslant \frac{1}{2}, \\ (1-s)(1+2s^2) & \text{if } s > \frac{1}{2}, \end{cases} \qquad \text{and} \quad b_s = s a_s.$$

It is straightforward to show that

$$a_s + s(1 + \max\{2b_s, a_s\}) = 1. \tag{$*$}$$

Now consider in $\ell_1 \oplus \ell_1(\Gamma)$ the following norm:

$$\|(y^*, z^*)\|_s = \|y^*\|_{\ell_1} + \max\{b_s(\|y^*\|_{\ell_1} + \|z^*\|_{\ell_1(\Gamma)}), a_s\|z^*\|_{\ell_1(\Gamma)}\},$$

for all $y^* \in \ell_1$, $z^* \in \ell_1(\Gamma)$.

Following the correction given by Johnson and Lindenstrauss in [**7**], it is easy to show that $\|\cdot\|_s$ is an equivalent dual norm on V^*. In fact, from the form of the duality between V and V^*, it follows that the norm $\|\cdot\|_s$ is weak*-lower semicontinuous. As a consequence, V has an equivalent norm whose dual norm on V^* coincides with $\|\cdot\|_s$ (see [**3**, Theorem 3, p. 106]).

We will now prove that V_s satisfies the $M(1,s)$-inequality. In fact, consider the set

$$A = \{(n, F) : n \in \mathbb{N},\ F \subset \Gamma,\ F \text{ finite}\},$$

ordered as follows: $(n, F) \leqslant (m, G)$ if $n \leqslant m$ and $F \subset G$. For each $F \subset \Gamma$, we define

$$N_F = \bigcup_{\gamma \in F} N_\gamma,$$

and for each $(n, F) \in A$, we consider $P_{n,F} : V \longrightarrow V$ defined as follows: for $(y, z) \in V_0$, $P_{n,F}(y, z) = (y_0, z_0)$, where

$$y_0(m) = \begin{cases} y(m) & \text{if } m \leqslant n, \\ \displaystyle\sum_{\gamma \in \Gamma_m \cap F} z(\gamma) & \text{if } m > n \text{ and } m \in N_F, \\ 0 & \text{otherwise} \end{cases}$$

and

$$z_0(\gamma) = \begin{cases} z(\gamma) & \text{if } \gamma \in F, \\ 0 & \text{if } \gamma \notin F. \end{cases}$$

We extend $P_{n,F}$ to V by continuity.

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We prove that for each (n,F), $\operatorname{Im} P_{n,F}^{**} \subset V$. In fact, it is clear that if $(y,z) \in V^{**} = \ell_\infty \oplus \ell_\infty(\Gamma)$, then $P_{n,F}^{**}(y,z) = (y_0, z_0)$, where

$$y_0(m) = \begin{cases} y(m) & \text{if } m \leqslant n, \\ \displaystyle\sum_{\gamma \in \Gamma_m \cap F} z(\gamma) & \text{if } m > n \quad \text{and} \quad m \in N_F, \\ 0 & \text{otherwise} \end{cases}$$

and

$$z_0(\gamma) = \begin{cases} z(\gamma) & \text{if } \gamma \in F, \\ 0 & \text{if } \gamma \notin F. \end{cases}$$

So, $P_{n,F}^{**}(y,z) \in V$.

We now show that for each $x \in V$ and $(y^*, z^*) \in V^*$, $\|x - P_{n,F}(x)\| \to 0$ and $\|(y^*,z^*) - P_{n,F}^*(y^*,z^*)\|_s \to 0$. Indeed, let $\varepsilon > 0$, and suppose that $x = (y,z) \in V_0$. For (n,F) large enough, we have that $\{\gamma \in \Gamma : z(\gamma) \neq 0\} \subset F$ and

$$\Bigl| y(n) - \sum_{\gamma \in \Gamma_n \cap F} z(\gamma) \Bigr| < \varepsilon \quad \text{if } n \in N_F,$$
$$|y(n)| < \varepsilon \quad \text{if } n \notin N_F.$$

So, $\|(y,z) - P_{n,F}(y,z)\| < \varepsilon$.

On the other hand, it is easy to show that $P_{n,F}^*(y^*,z^*) = (y_0^*, z_0^*)$, where

$$y_0^*(m) = \begin{cases} y^*(m) & \text{if } m \leqslant n, \\ 0 & \text{if } m > n \end{cases}$$

and

$$z_0^*(\gamma) = \begin{cases} \displaystyle\sum_{m>n,\ m \in N_\gamma} y^*(m) + z^*(\gamma) & \text{if } \gamma \in F, \\ 0 & \text{if } \gamma \notin F. \end{cases}$$

Hence,

$$\begin{aligned} &\|(y^*,z^*) - P_{n,F}^*(y^*,z^*)\|_s \\ &\quad \leqslant \sum_{m=n+1}^{\infty} |y^*(m)| \\ &\qquad + \max\biggl\{ b_s\biggl(2\sum_{m=n+1}^{+\infty} |y^*(m)| + \sum_{\gamma \notin F} |z^*(\gamma)|\biggr), a_s\biggl(\sum_{m=n+1}^{+\infty} |y^*(m)| + \sum_{\gamma \notin F} |z^*(\gamma)|\biggr)\biggr\}. \end{aligned}$$

We obtain the desired conclusion by taking (n,F) sufficiently large that

$$\sum_{m=n+1}^{+\infty} |y^*(m)| < \varepsilon \quad \text{and} \quad \sum_{\gamma \notin F} |z^*(\gamma)| < \varepsilon.$$

Finally, we prove that $(P_{n,F})$ satisfies condition (2) of Proposition 2.3 for $r=1$ and s. In fact, by $(*)$, for each $(n,F)\in A$ and $(y^*,z^*)\in V^*$, we have that

$$\begin{aligned}
&\|P^*_{n,F}(y^*,z^*)\|_s + s\|(y^*,z^*) - P^*_{n,F}(y^*,z^*)\|_s \\
&\quad\leqslant \sum_{m=1}^{n} |y^*(m)| \\
&\qquad + \max\biggl\{b_s\biggl(\sum_{m=1}^{n}|y^*(m)| + \sum_{m=n+1}^{+\infty}|y^*(m)| + \sum_{\gamma\in F}|z^*(\gamma)|\biggr), \\
&\qquad\qquad a_s\biggl(\sum_{m=n+1}^{+\infty}|y^*(m)| + \sum_{\gamma\in F}|z^*(\gamma)|\biggr)\biggr\} \\
&\qquad + s\sum_{m=n+1}^{+\infty}|y^*(m)| \\
&\qquad + s\max\biggl\{b_s\biggl(2\sum_{m=n+1}^{+\infty}|y^*(m)| + \sum_{\gamma\notin F}|z^*(\gamma)|\biggr), a_s\biggl(\sum_{m=n+1}^{+\infty}|y^*(m)| + \sum_{\gamma\notin F}|z^*(\gamma)|\biggr)\biggr\} \\
&\quad\leqslant \sum_{m=1}^{n}|y^*(m)| + (a_s + s(1+\max\{2b_s,a_s\}))\sum_{m=n+1}^{+\infty}|y^*(m)| \\
&\qquad + \max\biggl\{b_s\biggl(\sum_{m=1}^{n}|y^*(m)| + \sum_{\gamma\in\varGamma}|z^*(\gamma)|\biggr), a_s\sum_{\gamma\in\varGamma}|z^*(\gamma)|\biggr\} \\
&\quad\leqslant \|y^*\|_{\ell_1} + \max\{b_s(\|y^*\|_{\ell_1} + \|z^*\|_{\ell_1(\varGamma)}), a_s\|z^*\|_{\ell_1(\varGamma)}\} \\
&\quad= \|(y^*,z^*)\|_s.
\end{aligned}$$

□

Remarks 2.6.

(1) Since every non-reflexive M-ideal is WCG [**4**, Theorem 3] (cf. [**6**, Theorem 4.6, p. 142]), by Proposition 2.4, V_s cannot be renormed to be an M-ideal.

(2) Note that $\|\cdot\|_s$ on V is not Fréchet differentiable. In fact, this follows from the equality
$$(1+b_s)\|x\|_s = \|x\|_\infty \quad \forall x\in c_0.$$

(3) Finally, by the $M(1,s)$-inequality, some properties of V_s are immediate. In fact, by [**1**, Proposition 2.5] and [**5**, Proposition 2.5], we have

  (i) V_s^* contains no proper norming subspaces and V_s is an Asplund space;

  (ii) V_s has the unique extension property;

  (iii) if X is a closed subspace of V_s such that there exists a Banach space Y with Banach–Mazur distance $d(X,Y^*)<1+s/2$, then X is reflexive.

By [**1**, Corollary 2.8], we obtain

(i) V_s does not contain an isomorphic copy of ℓ_1;

(ii) if X is a Banach space such that $V_s \subsetneq X \subset V_s^{**}$, then there are no norm-one projections from X onto V_s;

(iii) every subspace or quotient of V_s which is isometric to a dual space is reflexive.

By [**1**, Theorem 3.1], we have

(i) V_s has property U of Phelps;

(ii) V_s has property (u) of Pełczyński with constant $k_u(V_s) \leqslant 1/s$.

By [**1**, Corollary 3.4], the following results hold.

(i) Every subspace of V_s has property (V) of Pełczyński. In particular, V_s is not weakly sequentially complete and V_s fails the Radon–Nikodým property.

(ii) V_s^* is weakly sequentially complete. (In fact, since V_s^* is isomorphic to $\ell_1 \oplus \ell_1(\Gamma)$, V_s^* has the Schur property.)

(iii) V_s is not complemented in V_s^{**}.

(iv) V_s^{**}/V_s is not separable.

(v) Every operator from V_s to a space not containing c_0 (in particular, every operator from V_s to V_s^*) is weakly compact.

By [**1**, Theorem 3.6], we have that every slice of B_{V_s} has diameter greater than or equal to $2s$. In particular, B_{V_s} is not dentable.

By [**2**, Proposition 2.1], we have

$$V_s^* = \overline{\text{lin}}(w^*\text{-sexp}\, B_{V_s^*}).$$

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