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A NOTE ON SPACES WITH A STRONG RANK 1-DIAGONAL

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Abstract

We mainly prove that, assuming $\mathfrak{b} = \mathfrak{c}$, every regular star-compact space with a strong rank 1-diagonal is metrisable.

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1. Introduction

It is well known that the G_δ-diagonal property plays an important role in metrisation theorems. In 1945, Sneider [7] proved that every compact space with a G_δ-diagonal is metrisable. In 1976, Chaber [2] proved that every countably compact space with a G_δ-diagonal is compact and thus metrisable, which improved Sneider's result. The strong rank 1-diagonal property is stronger than the G_δ-diagonal property. The classical Mrowka space [6] demonstrates that a Tychonoff pseudocompact space with a strong rank 1-diagonal need not be metrisable. Notice that star-compactness is weaker than countable compactness and stronger than pseudocompactness. A natural question then arises.

Question 1.1. Is every Tychonoff (regular) star-compact space metrisable if it has a strong rank 1-diagonal?

In this paper we mainly prove that, assuming $\mathfrak{b} = \mathfrak{c}$, every regular star-compact space with a strong rank 1-diagonal is metrisable. This gives a consistent positive answer to Question 1.1.

2. Notation and terminology

All spaces are assumed to be Hausdorff unless otherwise stated.

A subset A of a space X is said to be bounded in X if every infinite family ξ of open sets of X such that $V \cap A \neq \emptyset$, for every $V \in \xi$, has an accumulation point in X. If X is

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bounded in itself, then we say that X is pseudocompact. It should be pointed out that the definition of pseudocompactness given here is equivalent to DFCC in [5]. It is easy to see that for Tychonoff spaces it is also equivalent to the usual one: every continuous real-valued function on X is bounded.

A space X is star-compact if whenever $\mathcal{U}$ is an open cover of X, there is a compact subset $A \subseteq X$ such that $\mathrm{St}(A, \mathcal{U}) = X$, where $\mathrm{St}(A, \mathcal{U}) = \bigcup\{U \in \mathcal{U} : U \cap A \neq \emptyset\}$.

A space X has a strong rank 1-diagonal [1] if there exists a sequence $\{\mathcal{U}_n : n \in \omega\}$ of open covers of X such that for each $x \in X$, $\{x\} = \bigcap\{\overline{\mathrm{St}(x, \mathcal{U}_n)} : n \in \omega\}$.

The Lindelöf number $l(X)$ of a topological space X is the smallest number κ such that every open cover of X has a subcover the cardinality of which is at most κ. The 'extent' $e(X)$ of X is the supremum of the cardinalities of closed discrete subsets of X.

A space is called pseudonormal if every countable closed subset has arbitrarily small closed neighbourhoods.

All notation and terminology not explained here is given in [3].

3. Results

Lemma 3.1. *Suppose that X is a regular pseudocompct space with a strong rank 1-diagonal. Then X is a Moore space.*

Proof. Since a pseudocompact space is bounded in itself, the conclusion is an easy corollary of [1, Theorem 3.7]. □

Lemma 3.2. *For a Moore space X, $l(X) = e(X) = nw(X) = w(X)$.*

Proof. Suppose that $l(X) = \kappa$. We prove that $e(X) \leq l(X)$. If not, let S be a closed discrete subspace of X with $|S| > \kappa$. For every $x \in S$ there exists an open set $U_x \subseteq X$ such that $S \cap U_x = \{x\}$. It is not difficult to see that the open cover $\{U_x\}_{x \in S} \cup \{X \setminus S\}$ of the space X has no subcover of cardinality at most κ. This is a contradiction. Next, we prove that $nw(X) \leq e(X)$. Since X is a Moore space, it is a σ-space [4, Theorem 4.5], that is, it has a σ-discrete network. Let $\mathcal{N} = \bigcup\{\mathcal{N}_n : n \in \omega\}$ be a σ-discrete network for X, where each $\mathcal{N}_n$ is discrete. Fix $n \in \omega$ and pick a point x_N from N for each $N \in \mathcal{N}_n$. Let $S_n = \{x_N : N \in \mathcal{N}_n\}$. Then S_n is closed discrete in X. Since $|S_n| \leq e(X)$, $|\mathcal{N}_n| \leq e(X)$. It follows immediately that $|\mathcal{N}| \leq e(X)$. Now we prove that $l(X) \leq nw(X)$. Let $\mathcal{U}$ be an open cover of X. For each $x \in X$ pick $N_x \in \mathcal{N}$ such that $x \in N_x \subseteq U$ for some $U \in \mathcal{U}$. Let $\mathcal{N}_0 = \{N_x : x \in X\}$; clearly $\mathcal{N}_0$ covers X and $|\mathcal{N}_0| \leq nw(X)$. For each $N \in \mathcal{N}_0$ we can pick $U_N \in \mathcal{U}$ such that $N \subseteq U_N$ since $\mathcal{N}_0$ refines $\mathcal{U}$. Let $\mathcal{U}_0 = \{U_N : N \in \mathcal{N}_0\}$. This is a subcover of $\mathcal{U}$ and $|\mathcal{U}_0| \leq nw(X)$. This shows that $l(X) \leq nw(X)$. Therefore we can conclude that $l(X) = e(X) = nw(X) = \kappa$. Finally, since a Moore space is a p-space, we have $nw(X) = w(X)$ [4, Theorem 4.2]. This completes the proof. □

Corollary 3.3. *If X is a Moore space with countable extent, then X is metrisable.*

Corollary 3.4. *If X is a regular pseudocompact space of countable extent and with a strong rank 1-diagonal, then X is metrisable.*

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Recall that $\mathfrak{b} = \min\{|B| : B$ is an unbounded subset of $\omega^\omega\}$. It is known that $\omega < \mathfrak{b} \le \mathfrak{c}$; see [4].

Lemma 3.5 [8, Lemma 2.2.9]. *Suppose that X is a regular first countable space with $l(X) < \mathfrak{b}$. Then X is pseudonormal.*

Theorem 3.6. *Suppose that X is a regular pseudocompact space with a strong rank 1-diagonal and $e(X) < \mathfrak{b}$. Then X is metrisable.*

Proof. It is sufficient to prove that X is countably compact, since a countably compact space with a G_δ-diagonal is metrisable [2]. Suppose that this is not so. Then there exists a countable infinite closed and discrete subset S of X. By Lemma 3.2, $l(X) = e(X) < \mathfrak{b}$; by Lemma 3.5, X is pseudonormal. So there exists a discrete family of open sets $\{U_x : x \in S\}$ of X such that $U_x \cap S = \{x\}$ for all $x \in S$ [4, Proposition 12.1]. Clearly, $\{U_x : x \in S\}$ is not finite. This contradicts the fact that X is pseudocompact. □

Recall that a space is said to be 1-star-compact if for every open cover $\mathcal{U}$ of X, there is some finite subset $\mathcal{V}$ of $\mathcal{U}$ such that $\mathrm{St}(\bigcup \mathcal{V}, \mathcal{U}) = X$.

Lemma 3.7 [8, Corollary 2.2.7]. *If X is a 1-star-compact Moore space, then $w(X)$ has countable cofinality.*

Theorem 3.8. *Assume $\mathfrak{b} = \mathfrak{c}$. Let X be a regular 1-star-compact space with a strong rank 1-diagonal. Then X is metrisable.*

Proof. Since a 1-star-compact space is pseudocompact [8, Theorem 2.1.8], X is a Moore space by Lemma 3.1. Apply [8, Lemma 2.2.1] to conclude that X is separable. Therefore it is not difficult to see that $w(X) \le \mathfrak{c}$. Since $\mathfrak{c}$ does not have countable cofinality, by Lemma 3.7, we can conclude that $w(X) < \mathfrak{c} = \mathfrak{b}$. By Lemma 3.2, $e(X) = w(X) < \mathfrak{b}$. It remains to apply Theorem 3.6. □

It is easy to see that a star-compact space is 1-star-compact. So the following result is an immediate consequence of Theorem 3.8.

Theorem 3.9. *Assume that $\mathfrak{b} = \mathfrak{c}$. Let X be a regular star-compact space with a strong rank 1-diagonal. Then X is metrisable.*

In Theorem 3.9, regularity cannot be weakened to the Hausdorff property.

Example 3.10. There exists a Hausdorff star-compact non-metrisable space with a strong rank 1-diagonal.

Proof. The space was constructed in [8, Example 2.2.4] as follows.

Let $Y = \bigcup\{[0, 1] \times \{n\} : n \in \omega\}$ and $X = Y \cup \{a\}$ where $a \notin Y$. Define a basis for a topology on X as follows. Basic open sets containing a take the form $\{a\} \cup \bigcup\{[0, 1) \times m : m \ge n\}$ where $n \in \omega$. Basic open sets about the other points of X are the usual induced metric open sets.

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It is easy to construct a sequence $\{\mathcal{V}_n : n \in \omega\}$ of open covers of Y which demonstrates that Y has a strong rank 1-diagonal. Let

$$\mathcal{U}_n = \mathcal{V}_n \cup \Big(\{a\} \cup \bigcup \{[0, 1) \times m : m \geq n\}\Big).$$

Then $\{\mathcal{U}_n : n \in \omega\}$ shows that X has a strong rank 1-diagonal. However, it is proved in [9] that X is a Hausdorff star-compact and non-metrisable space. □

However, the following question remains open.

QUESTION 3.11. Is a regular star-compact space metrisable if it has a G_δ-diagonal?

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