THE RADICAL OF THE ALGEBRA OF ANY FINITE SEMIGROUP OVER ANY FIELD

T. E. HALL

(Received 14 March 1969)

Communicated by G. B. Preston

1. Introduction and summary

For a finite semigroup S and a field Φ, denote by $\Phi[S]$ the semigroup algebra of S over Φ, and when S has a zero element, denote by $\Phi_0[S]$ the contracted semigroup algebra of S over Φ (see § 5.2 [1]). Then theorem 5.31 [1], due to E. Hewitt and H. S. Zuckerman, gives a determination of the radical of $\Phi[S]$ when S is commutative and the characteristic of Φ does not divide the order of any subgroup of S; and on page 168 [1] some results concerning the radical of $\Phi_0[S]$ when S is completely 0-simple are given, the determination of the radical in the general case remaining open.

In [2], G. Lallement and M. Petrich determine the radical of $\Phi_0[S]$ when S is any finite completely 0-simple semigroup, their result being as follows.

RESULT 1 (from [2]). *Let* $T = \mathscr{M}^0(G; m, n; P)$ be *any finite completely* 0-*simple semigroup and* Φ *any field. Then the radical of the Munn algebra* $\mathscr{B} = \mathscr{M}(\Phi[G]; m, n; P)$, *which we identify with* $\Phi_0[T]$ (*lemma* 5.17 [1]) *is determined by*

$$Rad\ \mathscr{B} = \{X \in \mathscr{B} : PXP \text{ is over } rad\ \Phi[G]\}.$$

In this paper we determine the radical of $\Phi_0[S]$, when S is any finite semigroup with a zero element (see theorem 2). As noted on page 160 [1], 'if $S \cup z$ is the semigroup resulting from the adjunction of a zero element z to a semigroup S (whether or not S has a zero to begin with) then $\Phi_0[S \cup z] \cong \Phi[S]$.' Thus the following also determines the radical of $\Phi[S]$, where S is any finite semigroup, with or without a zero element.

We use throughout the notations and conventions of A. H. Clifford and G. B. Preston [1].

2. Rad $\Phi_0[S]$

Let $S = S^0$ be a finite semigroup and let Φ be a field. Let J_i $(i = 1, \cdots, p)$ denote the $\mathscr{J}$-classes of S for which the corresponding principal factors, $Q_i = S^1 J_i S^1/I(J_i)$ $(i = 1, \cdots, p)$ say, are 0-simple (see page 72 [1]). Then by

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theorem 3.5 [1], each Q_i is isomorphic to a regular Rees matrix semigroup over a group with zero, i.e $Q_i \cong \mathscr{M}^0(G_i; m_i, n_i; P_i)$ say. We assume for convenience that $Q_i = \mathscr{M}^0(G_i; m_i, n_i; P_i)$, for $i = 1, \cdots, p$. Moreover, we shall also identify $\Phi_0[Q_i]$ with the Munn algebra $\mathscr{B}_i = \mathscr{M}(\Phi[G_i]; m_i, n_i; P_i)$ (lemma 5.17 [1]). Now $\Phi_0[Q_i]/\text{rad}\, \Phi_0[Q_i]$ is semisimple, and hence contains an identity, $E_i + \text{rad}\, \Phi_0[Q_i]$ say. Let ϕ_i denote the natural homomorphism of $\Phi_0[S^1 J_i S^1]$ onto $\Phi_0[Q_i]$ (see pages 170, 171 [1]).

Theorem 1. *The radical of* $\Phi_0[S]$ *is determined by*

$$\begin{aligned} Rad\, \Phi_0[S] &= \{X \in \Phi_0[S] : for\ i = 1, \cdots, p,\ (XE_i)\phi_i \in rad\, \Phi_0[Q_i]\} \\ &= \{X \in \Phi_0[S] : for\ i = 1, \cdots, p,\ P_i(XE_i)\phi_i P_i\ is\ over\ rad\, \Phi[G_i]\}. \end{aligned}$$

Proof. From lemma 5.6 [1] and the main representation theorem for semi-simple algebras (page 154 [1]) we see that the radical of $\Phi_0[S]$, as defined on page 149 [1], is simply the set

$\{X \in \Phi_0[S]$: for every proper irreducible representation Γ of S over Φ, $\Gamma(X) = 0\}$.

We now use extensively theorem 5.33 [1].

(i) Take any element X in $\Phi_0[S]$ such that $(XE_i)\phi_i \in \text{rad}\, \Phi_0[Q_i]$ for $i = 1, \cdots, p$, and let Γ be any proper non-null irreducible representation over Φ of S, and thus also of $\Phi_0[S]$. Since S is finite, Γ is principal, with apex J_k for some element $k \in \{1, \cdots, p\}$. Now Γ', the representation of Q_k induced by Γ (see page 171 [1]) is also irreducible and non-null, and by lemma 5.6 [1] is effectively a representation of $\Phi_0[Q_k]/\text{rad}\, \Phi_0[Q_k]$, whence $\Gamma'(E_k) = I_n$, where n is the degree of Γ'. Clearly then (from (2′) page 171 [1])

$$\Gamma(X) = \Gamma(X)I_n = \Gamma(X)\Gamma(E_k) = \Gamma(XE_k) = \Gamma'((XE_k)\phi_k) = 0,$$

since $(XE_k)\phi_k \in \text{rad}\, \Phi_0[Q_k]$. It follows that $X \in \text{rad}\, \Phi_0[S]$.

(ii) Take now any element X in rad $\Phi_0[S]$ and any element $i \in \{1, \cdots, p\}$. Let Γ' be any proper non-null irredicible representation of Q_i. Then as in (i), $\Gamma'(E_i) = I_n$, where n is the degree of Γ'. Define a representation Γ of $\Phi_0[S]$ by

$$\Gamma(Y) = \Gamma'((YE_i)\phi_i) \text{ for all } Y \text{ in } \Phi_0[S].$$

Then Γ is irreducible, whence

$$\Gamma'((XE_i)\phi_i) = \Gamma(X) = 0.$$

Hence $(XE_i)\phi_i \in \text{rad}\, \Phi_0[Q_i]$, and the result follows.

Remark 1. The determination of rad $\Phi_0[S]$ given by theorem 1 would be more complete if we could give a construction for the E_i, instead of merely proving their existence, i.e. if we could solve the following problem: given a finite completely 0-simple semigroup $T = \mathscr{M}^0(G; m, n; P)$ and a field Φ, construct, in terms of the variables Φ, G, P, an element E in $\Phi_0[T]$ such that $E + \text{rad}\, \Phi_0[T]$ is the identity

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element of $\Phi_0[T]/\text{rad } \Phi_0[T]$. However, in the next theorem, we are able to circumvent this problem and give a complete determination of rad $\Phi_0[S]$ by replacing each E_i by all the elements, one at a time, in J_i. It is important to note that J_i is finite, so that we have a finite procedure for determining, of any given element in $\Phi_0[S]$, whether it is in rad $\Phi_0[S]$ or not.

REMARK 2. A solution to the problem in remark 1 would make more complete the determination, given by theorem 5.33 [1], of all irreducible representations over Φ of a finite semigroup S (see the proof of theorem 1).

THEOREM 2. *The radical of $\Phi_0[S]$ is determined by*

$$\begin{aligned} \text{Rad } \Phi_0[S] &= \{X \in \Phi_0[S] : \text{for all } s \in J_i,\ (Xs)\phi_i \in \text{rad } \Phi_0[Q_i],\quad \text{for } i = 1, \cdots, p\} \\ &= \{X \in \Phi_0[S] : \text{for all } s \in J_i,\ P_i(Xs)\phi_i P_i \\ &\qquad\qquad \text{is over rad } \Phi[G_i], \text{ for } i = 1, \cdots, p\}. \end{aligned}$$

PROOF. Take any element $X \in \Phi_0[S]$ and any element $i \in \{1, \cdots, p\}$ such that $(XE_i)\phi_i \in \text{rad } \Phi_0[Q_i]$. Take any proper non-null irreducible representation Γ' of Q_i over Φ with degree n, and any element $s \in J_i$. Extend Γ' to Γ as above. Then

$$\begin{aligned} \Gamma'[(Xs)\phi_i] &= \Gamma(Xs) = \Gamma(X)\Gamma(s) = \Gamma(X)I_n\Gamma(s) \\ &= \Gamma(XE_i)\Gamma(s) = \Gamma'((XE_i)\phi_1)\Gamma(s) = 0. \end{aligned}$$

Hence $(Xs)\phi_i \in \text{rad } \Phi_0[Q_i]$ for each element $s \in J_i$.

Conversely, suppose $(Xs)\phi_i \in \text{rad } \Phi_0[Q_i]$ for each $s \in J_i$. Since $E_i \in \Phi[J_i]$, it follows that $(XE_i)\phi_i \in \text{rad } \Phi_0[Q_i]$. Hence, for any $X \in \Phi_0[S]$ and any $i \in \{1, \cdots, p\}$, we have $(XE_i)\phi_i \in \text{rad } \Phi_0[Q_i]$ if and only if $(Xs)\phi_i \in \text{rad } \Phi_0[Q_i]$ for each element $s \in J_i$. Theorem 2 now follows from theorem 1.

References

[1] A. H. Clifford and G. B. Preston, *The Algebraic Theory of Semigroups* (Math. Surveys, number 7, Amer. Math. Soc., Vol I, 1961).

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Monash University
Clayton

https://doi.org/10.1017/S1446788700006753 Published online by Cambridge University Press