ON THE PETTIS MEASURABILITY THEOREM

Dietrich Helmer

It is shown that, in Pettis’s criterion for Bochner measurability of a vector-valued function \( f : S \rightarrow X \), scalar measurability of \( f \) can be weakened to requiring that \( u \circ f \) be measurable for \( u \) in some subset of the dual \( X^* \) separating the points of \( X \). Even then, the separability hypotheses in Pettis’s Theorem can be weakened as well.

Throughout, \( (S, \Sigma, \mu) \) denotes a positive measure space. Then \( \Sigma_0 \) stands for the collection of all \( \mu \)-null sets and \( \Sigma_b \) for \( \{ E \in \Sigma \mid 0 < \mu(E) < \infty \} \). Moreover, \( X \) is a Fréchet space over \( K \), where \( K = \mathbb{R} \) or \( K = \mathbb{C} \). We fix a function \( f : S \rightarrow X \). Recall that \( f \) is \( \mu \)-measurable (in the sense of Bochner) if and only if \( \chi_E f \) is, for every \( E \in \Sigma_b \), the limit \( \mu \)-almost everywhere in \( S \) of a sequence of \( \mu \)-simple functions. The set of all \( \mu \)-measurable functions \( S \rightarrow X \) is denoted by \( \mathcal{M}(\mu, X) \).

One of the most frequently employed and useful criteria for \( \mu \)-measurability is the Pettis Measurability Theorem for Banach spaces \( X \) [12, p.278] (compare [2, p.149]): \( f \in \mathcal{M}(\mu, X) \) if (and only if) every \( E \in \Sigma_b \) admits an \( N \in \Sigma_0 \) such that \( f(E \setminus N) \) is separable and \( f \) is \( \mu \)-almost everywhere in \( S \) measurable.

A discussion of the importance of the theorem for Measure Theory and the theory of Banach spaces has been given by Uhl in [19]. It is known (compare [19] [1, p.43] for finite \( \mu \)) that scalar \( \mu \)-measurability of \( f \) in Pettis’s Theorem can be replaced by the weaker condition that \( U \circ f \subseteq \mathcal{M}(\mu) \) for some norming \( U \subseteq X^* \). We shall show in this note that a minimal requirement is already sufficient here: namely, that \( U \circ f \subseteq \mathcal{M}(\mu) \) for some \( U \subseteq X^* \) separating sufficiently many points of \( X \). (A key argument has already been utilised implicitly in [4] for Haar measures \( \mu \) and, for general \( \sigma \)-finite \( \mu \), in [7].) It is often easy to identify point-separating subsets \( U \) of \( X^* \); and, depending on \( X^* \) and \( f \), there may be flexibility in tailoring \( U \) so as to facilitate the verification of \( U \circ f \subseteq \mathcal{M}(\mu) \).

Let \( \mathbb{I} := [0, 1] \). If \( B \subseteq X \), then \( \text{span} \, B \) stands for the closed linear span of \( B \) in \( X \) and \( B_o \) denotes \( B \) with the subspace topology it inherits from the weak topology of \( X \).
Similarly, if $V \subseteq X^*$, then $V_\sigma$ is $V$ equipped with the restricted $w^*$-topology. $(S, \widehat{\Sigma}, \mu)$ denotes the Lebesgue extension of $(S, \Sigma, \mu)$. We call $\Sigma_c \subseteq \widehat{\Sigma}$ $\mu$-covering if and only if for every $E \in \Sigma_b$, there exists some $T \in \Sigma_c$ such that $E \cap T \notin \Sigma_0$, which is the case if and only if every $E \in \Sigma_b$ admits a sequence $T_n$ in $\Sigma_c$ with $E \setminus \bigcup_{n=1}^\infty T_n \in \Sigma_0$. And $\Gamma \subseteq \widehat{\Sigma}$ is said to be a $\mu$-regularity set if and only if $\mu(E) = \sup\{\mu(G) \mid G \in \Gamma, G \subseteq E\}$ for every $E \in \Sigma_b$. Any such $\Gamma$ is $\mu$-covering.

**Theorem 1.** $f \in \mathcal{M}(\mu, X)$ if and only if there exists a $\mu$-covering set $\Sigma_c$ such that every $T \in \Sigma_c$ admits some $N \in \Sigma_0$ with $f(T \setminus N)$ separable and some $U \subseteq X^*$ separating the points of $\text{span}(T \setminus N)$ with $U \circ f \subseteq \mathcal{M}(\mu)$.

**Proof:** If $f$ is $\mu$-measurable, then the conditions are satisfied with $\Sigma_c := \Sigma_\mu$ and $U := X^*$. Conversely, suppose now that there exists a $\mu$-covering collection $\Sigma_c$ as above. First, using routine arguments, one reduces the proof to the case where $\mu$ is finite, $X$ is separable, and where $U \circ f \subseteq \mathcal{M}(\mu)$ for some subset $U$ of $X^*$ separating the points of $X$. In this situation, it follows by means of Egoroff's Theorem that the linear subspace $D := \{u \in X^* \mid u \circ f \in \mathcal{M}(\mu)\}$ of $X^*$ is sequentially closed in $X^*_\sigma$. For every 0-neighbourhood $B$ in $X$, the polar $B^\circ := \{u \in X^* \mid |u(b)| \leq 1 \text{ for all } b \in B\}$ is compact and metrisable in $X^*_\sigma$; and, consequently, $D \cap B^\circ$ is closed in $(B^\circ)_{\sigma}$. According to the Krein-Smulian Theorem [9, 22.6], therefore, $D$ itself is closed in $X^*_\sigma$. On the other hand, $D$ is dense in $X^*_\sigma$ since it separates the points of $X$. Thus, $D = X^*$, that is $f$ is scalarly $\mu$-measurable. If $X$ is a Banach space, an appeal to Pettis's Theorem finishes the proof. In the general case, the usual arguments need adaptation: Fix a sequence $B_n$ of closed convex circled subsets of $X$ constituting a 0-neighbourhood base in $X$ such that $B_{n+1} \subseteq B_n$ for all $n$. Thereafter, pick, for every $n$, a sequence $u_{nk}$ that is dense in $(B_n^\circ)_{\sigma}$. Moreover, let $x_m$ be a dense sequence in $X$. Then

$$h_{nm} : s \mapsto \sup\{|u_{nk}(f(s) - x_m)| \mid k \in \mathbb{N}\} : S \to \mathbb{K}$$

is $\mu$-measurable for all $n, m$, and, consequently, $h_{nm}^\sigma(\mathbb{I}) \in \widehat{\Sigma}$. But $h_{nm}^\sigma(\mathbb{I}) = f^{-1}(x_m + B_n)$ by means of the Bipolar Theorem. Now the classical arguments carry over mutatis mutandis to show that $f$ is the uniform limit of a sequence $f_n$ of countably-valued functions with $\{f_n^{-1}(z) \mid n \in \mathbb{N}, z \in X\} \subseteq \widehat{\Sigma}$ and, finally, to show that $f$ is $\mu$-measurable.

**Example.** Suppose $S$ is also a topological space and $\Gamma$ is a $\mu$-regularity set consisting of closed subsets or of open subsets of $S$. Let $D$ be an open connected set in $\mathbb{C}$ and $h : S \times D \to \mathbb{C}$ a function that is analytic in the second argument such that $\{h(\cdot, z) \mid z \in A\} \subseteq \mathcal{M}(\mu)$ for some $A \subseteq D$ with an accumulation point in $D$. Then, given $\varepsilon > 0$, every $E \in \Sigma_b$ contains a $G \in \Gamma$ with $\mu(E \setminus G) < \varepsilon$ such that $h$ is continuous on $G \times D$.  

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PROOF: Consider \( \psi: s \mapsto h(s, -): S \to H(D) \), where \( H(D) \) denotes the separable Fréchet space consisting of all analytic functions \( D \to \mathbb{C} \) equipped with the topology of uniform convergence on compacta. The subset \( U \) of \( H(D)^* \), made up of the evaluation functionals at points of \( A \), separates the points of \( H(D) \) and satisfies \( U \circ \psi \subseteq \mathcal{M}(\mu) \). So \( \psi \) is \( \mu \)-measurable by Theorem 1; whence some sequence of countably-valued \( \mu \)-measurable functions \( S \to H(D) \) converges to \( \psi \) uniformly. Adapting standard arguments (compare [16, p.27]), given \( \epsilon \) and \( E \), we find some \( G \in \Gamma \) with \( G \subseteq E \) and \( \mu(E \setminus G) < \epsilon \) on which \( \psi \) is continuous. To complete the proof, it therefore suffices to utilise the continuity of the evaluation \( H(D) \times D \to \mathbb{C} \). \( \square \)

We now turn to the second ingredient of the Pettis Theorem, separability, and replace it by weaker conditions, while keeping our minimal hypotheses on scalar \( \mu \)-measurability.

We call a topology \( T \) on \( R \subseteq S \) a \( \mu \)-topology if and only if \( T \cap \Sigma_0 \subseteq \{ \emptyset, R \} \). If \( R \) is \( \sigma \)-finite in \( \widehat{\Sigma} \) and \( \theta \) is a lifting of \( \Delta := \{ D \in \widehat{\Sigma} \mid D \subseteq R \} \), or just any lower density of \( \Delta \), then the density topology

\[
T_\theta := \{ \theta(D) \mid M \in \Delta, M \in \Delta \cap \Sigma_0 \} = \{ F \in \Delta \mid F \subseteq \theta(F) \}
\]

(compare [8, p.54]) is a \( \mu \)-topology on \( R \). Generally: if \( R \in \widehat{\Sigma} \setminus \Sigma_0 \) and \( O \) is any topology on \( R \) with \( O \cap \Sigma_0 = \{ \emptyset \} \), then \( \{ Q \setminus M \mid Q \in O, M \in \Sigma_0 \} \) is a basis of a \( \mu \)-topology. If \( P \) is a topological space and \( V \subseteq C(P) \), then \( V \) is considered equipped with the topology of pointwise convergence and \( Alg V \) denotes the smallest closed \( K \)-subalgebra of \( C(P) \) containing \( V \). Recall that \( P \) satisfies the Souslin countable chain condition CCC if and only if every family of non-empty, pairwise disjoint, open subsets of \( P \) is countable. We say that a collection \( B \) of Borel subsets of \( K \) is a Borel subbase for \( K \) if and only if for every open \( W \subseteq K \) and every \( \alpha \in W \), there are \( B_1, \ldots, B_m \) in \( B \) with \( \alpha \in \bigcap_{k=1}^m B_k \subseteq W \).

**Theorem 2.** \( f \in \mathcal{M}(\mu, X) \) if there is a \( \mu \)-covering set \( \Sigma_c \) such that every \( T \in \Sigma_c \) admits pseudo-compact sets \( P_1, P_2, \ldots \) in \( X_\sigma \), \( \mu \)-null sets \( N_0, N_1, \ldots \) with

\[
f(T \setminus N_0) \subseteq \operatorname{span} \left( \bigcup_{i=1}^{\infty} f(T \setminus N_i) \cap P_i \right) =: Y,
\]

and a \( U \subseteq X^* \) separating points of \( Y \) with \( U \circ f \subseteq \mathcal{M}(\mu) \) satisfying, for every \( k \geq 1 \), one of these conditions:

1. There is a sequence \( V_{km} \) of pseudo-compact CCC-subspaces of \( C(P_{k\sigma}) \) with \( U \mid_{P_k} \subseteq \operatorname{Alg} \left( \bigcup_{m=1}^{\infty} V_{km} \right) \).
(2) There is a Borel subbase $B_k$ for $K$ so that every pairwise disjoint collection in

$$\left\{ T \cap \bigcap_{i=1}^{m} f^{-1}(P_k \cap u_i^{-1}(B_i)) \setminus N_k \mid m \in \mathbb{N}, u_i \in U, B_i \in B_k \right\}$$

is countable.

(3) $T \cap f^{-1}(P_k)$ is in $\tilde{\Sigma}$ and $\sigma$-finite; and there is a $\mu$-topology $T_k$ on $T \cap f^{-1}(P_k)$ such that $u \circ f$, restricted to $(T \cap f^{-1}(P_k)) \setminus N_k$, is $T_k$-continuous for $u \in U$.

Conversely, these conditions are necessary for $f \in M(\mu, X)$.

**Proof:** Suppose that there exists a $\mu$-covering collection $\Sigma_c$ with the properties listed above. Fix $T \in \Sigma_c$, and then let the $P_j$'s, the $N_j$'s and $U$ be as guaranteed by the hypotheses. In view of Theorem 1, it suffices to show that, for every fixed $k \geq 1$, the weak closure $C_k$ of $f(T \setminus N_k) \cap P_k$ is separable with respect to the weak topology. Note that $C_k \subseteq Y$, as $Y$ is closed in $X_{\sigma}$ by Mazur's Theorem [9, 17.1]. The class of Eberlein compacta (that is, those topological spaces that are homeomorphic with weakly compact subsets of Banach spaces) is well-known to be closed under the formation of countable products. And according to a result of Preiss and Simon [13], every weakly pseudo-compact subset of a Banach space is weakly compact. Thus, as $X$ admits a linear homeomorphic embedding into a product of countably many Banach spaces, $P_{k\alpha}$ is Eberlein compact.

First, suppose that (1) is satisfied. Fix $m$. Evaluation $g : V_{km} \times P_{k\alpha} \to K$ is separately continuous. Let $h \in C(K, I)$. Then an argument given by Pták [14, p.572] shows that

$$\lim_{i \to \infty} \lim_{j \to \infty} h(v_i(p_j)) = \lim_{j \to \infty} \lim_{i \to \infty} h(v_i(p_j)) \quad \text{in } I$$

for every sequence $(v_i, p_i)$ in $V_{km} \times P_k$ for which all limits involved exist. According to [5, 2.1.(2)], therefore, $h \circ g : V_{km} \times P_{k\alpha} \to I$ admits a separately continuous extension to $\beta V_{km} \times P_{k\alpha}$, where $\beta$ denotes the Stone-Čech compactification operator. Furthermore, if $A, B$ are any two disjoint closed subsets of $K$, then $h'(A) \subseteq \{0\}$ and $h'(B) \subseteq \{1\}$ for a suitable $h' \in C(K, I)$. Consequently, $g$ admits a separately continuous extension $\beta V_{km} \times P_{k\alpha} \to \beta K$ by [5, 2.1.(7)]. Moreover, all subspaces of $K$ of the form $V_{km}(p)$ with $p \in P_k$ and of the form $v(P_k)$ with $v \in V_{km}$ are compact. It therefore follows from [5, 2.5.(1), 2.8] that $V_{km}$ has Eberlein compact closure $\overline{V}_{km}$ in $C(P_{k\alpha})$. On the other hand, $\overline{V}_{km}$ satisfies CCC. Consequently, $\overline{V}_{km}$ is separable. This follows from the Rosenthal Separability Theorem [15, 4.6] and is also a rather direct consequence of Namioka's continuity Theorem 4.2 in [11]. Thus, $A := Alg\left(\bigcup_{m=1}^{\infty} V_{km}\right)$ is separable in...
\[ C(P_k, \sigma), \text{ whence } A|_{C_k} \text{ is separable in } C(C_k). \] But \( A|_{C_k} \) separates the points of \( C_k \) since it contains \( U|_{C_k} \). Consequently, \( C_k, \sigma \) is separable, indeed.

In cases (2), (3), it suffices to show that \( C_{k, \sigma} \) satisfies CCC because it is Eberlein compact. So let \( (Q_\lambda)_{\lambda \in \Lambda} \) be any family of non-empty, open, pairwise disjoint subsets of \( C_{k, \sigma} \). Fix \( \lambda \in \Lambda \), and choose some \( t_\lambda \in T \cap f^{-1}(P_k \cup Q_\lambda) \setminus N_k \). Since \( U|_{C_k} \) separates the points of \( C_k \), it generates the topology of \( C_{k, \sigma} \). So there exist \( u_1, \ldots, u_m \in U \) and open subsets \( B_1, \ldots, B_m \) in \( K \) such that

\[ f(t_\lambda) \in C_k \cap \bigcap_{i=1}^m u_i^{-1}(B_i) \subseteq Q_\lambda. \]

But then \( f(D_\lambda) \subseteq Q_\lambda \) for

\[ D_\lambda := T \cap f^{-1}(P_k) \cap \bigcap_{i=1}^m (u_i \circ f)^{-1}(B_i) \setminus N_k. \]

And \( D_\lambda \neq \emptyset \), as \( t_\lambda \in D_\lambda \).

Suppose now that (3) holds. Then \( D_\lambda \) is an open \( T_k \)-neighbourhood of \( t_\lambda \) in \( (T \cap f^{-1}(P_k)) \setminus N_k \). Since \( T_k \) is a \( \mu \)-topology on \( T \cap f^{-1}(P_k) \), it follows that \( D_\lambda \notin \Sigma_0 \), provided that \( D_\lambda \neq (T \cap f^{-1}(P_k)) \setminus N_k \). Moreover, in view of the hypothesis that \( U \circ f \subseteq \mathcal{M}(\mu) \) and that \( T \cap f^{-1}(P_k) \) is \( \sigma \)-finite in \( \hat{\Sigma} \), we obtain that \( D_\lambda \in \hat{\Sigma} \) (compare [2, p.148]). Consequently, the usual summability argument shows that \( \Lambda \) must be countable.

Finally, suppose that (2) is satisfied. Then we may assume that \( B_1, \ldots, B_m \) are, actually, members of \( B_k \) (though no longer open, perhaps). Consequently, (2) guarantees countability of \( \Lambda \) in this case as well.

Conversely, as for the necessity of the conditions, suppose that \( f \) is \( \mu \)-measurable.

Take \( \Sigma_c := \Sigma_k \) and \( U := X^* \). Fix \( T \in \Sigma_c \) and, thereafter, \( N_0 \in \Sigma_0 \) such that \( \bigcup_{i=1}^\infty P_i \) is dense in \( f(T \setminus N_0) \) for some sequence \( P_k \) of singletons. Let \( N_k := \emptyset \) for \( k \geq 1 \). Then (1) and (2) are, trivially, satisfied for every \( k \). As is (3) since \( T \cap f^{-1}(P_k) \subseteq \hat{\Sigma} \) (compare [2, p.148]).

REMARKS. 1. The proof of Theorem 1 essentially consisted in showing that, if \( U \circ f \subseteq \mathcal{M}(\mu) \) for some \( U \subseteq X^* \) separating the points of \( X \), then \( f \) is scalarly \( \mu \)-measurable, provided that \( X \) is separable. Scalar measurability has received considerable attention, in particular in the context of the Pettis integral (compare [18] and references there).

The Krein-Šmulian Theorem can be applied to the scalar measurability problem as well. Even if \( X \) is non-separable, the argument for scalar \( \mu \)-measurability of \( f \) in the proof of Theorem 1 goes through, provided that \( (B^*)_\sigma \) is sequential for every 0-neighbourhood \( B \) in \( X \). The latter condition is satisfied, for instance, if \( X \) is a closed

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linear subspace of a Fréchet space $Z$ such that $\bigcup_{n=1}^{\infty} P_n$ is total in $Z$ for some sequence $P_n$ of pseudo-compact subsets of $Z_\sigma$. But some additional assumption on $X$ is, generally, indispensable to arrive at the conclusion. Consider, for instance, the map $g: I \to \ell^\infty$, where $g(s)$ is the sequence making up the dyadic expansion of $s \in I$ such that $g(s)$ is not 1 eventually. Let $U \subseteq (\ell^\infty)^*$ consist of the evaluation functionals at points of $\mathbb{N}$. For every $u \in U$, then $u \circ g$ is measurable with respect to Lebesgue measure $\lambda$ (in fact, $u \circ g$ is of Baire class 1). It can be shown, however, that $g$ is not scalarly $\lambda$-measurable.

2. If $S$ is a Souslin space and $\mu$ a Borel measure, then the set $K(S)$ of all compact subsets of $S$ is a $\mu$-regularity set (compare [16, p.122]).

3. If $S$ is a completely regular space and $\mu$ a Baire measure, then it can be shown that the collection $\mathcal{Z}(S)$ of zero-sets of $S$ (that is, sets $g^{-1}(0)$ with $g \in C(S, I)$) is a $\mu$-regularity set. This, in turn, can be used to show that, if $S$ is a locally compact group and $\mu$ a Haar measure, then $\mathcal{Z}(S) \cap K(S)$ is a $\mu$-regularity set; if, moreover, the connected components of $S$ are metrisable, then those members of $\mathcal{Z}(S)$ that are homeomorphic with $\{0, 1\}^\omega$ form a $\mu$-regularity set, where $\omega$ is the local weight of $S$ [6, p.337].

4. In Theorems 1, 2, $U \circ f \subseteq M(\mu)$ can be replaced by $U \circ f|_T \subseteq M(\mu|_T)$, where $\mu|_T$ is the restriction of $\mu$ to $T$.

5. For certain measures $\mu$, a much stronger version of the $\sigma$-compactness condition for $f(T \setminus N_0)$ in Theorem 2 is a necessary by-product of $\mu$-measurability. Suppose $S$ is also a topological space and has a $\mu$-regularity set $\Gamma$ consisting of closed pseudo-compact sets of $S$. (For example, $\mu$ a Radon measure.) If $f \in M(\mu, X)$, then every $T \in \Sigma_b$ admits some $N_0 \in \Sigma_0$ such that $f(T \setminus N_0)$ is $\sigma$-compact, even in $X$. This follows from the fact (compare the Example) that $T$ contains, for every $\varepsilon > 0$, some $G \in \Gamma$ with $\mu(T \setminus G) < \varepsilon$ such that $f|_\sigma: G \to X$ is continuous.

6. Given the $P_k$'s in Theorem 2, condition (1) is satisfied if $U$ is pointwise bounded on $X$ and $U_\sigma$ satisfies CCC. For instance, if $Z$ is a Banach space for which $B := \{z \in Z \mid \|z\| \leq 1\}_\sigma$ satisfies CCC — such spaces have been studied and examples exhibited by Wheeler in [20] — and if $X$ is the strong dual of $Z$, then one may take the set $U \subseteq X^*$ corresponding to $B$ or any sufficiently large subset thereof.

7. Concerning the countable chain condition in (1), it would not be good enough to suppose that $\bigcup_{m=1}^{\infty} V_{km}$ be a CCC-space. Consider $X := \ell^2(\mathbb{N})$, $U := X^*$, $P_k := \{z \mid \|z\| \leq 1\}$, and $V_{km} := \{v \mid P_k \mid v \in X^*, \|v\| \leq m\}$. Since $X^*_\sigma$ is homeomorphic with a dense subset of some product $\mathbb{K}^A$, it satisfies CCC. But the scalarly Lebesgue measurable function $f: s \mapsto \chi_{\{s\}}: I \to X$ is not measurable (as Birkhoff observed long ago).

8. In Theorem 2: if $T \cap f^{-1}(P_k)$ is in $\widehat{\Sigma}$ and $\sigma$-finite and the set system in (2) has
only $\emptyset$ in common with $\Sigma_0$ for some $B_k$, then (2) is satisfied.

9. If (3) is satisfied for all $k$, if $T \setminus \bigcup_{k=1}^{\infty} f^{-1}(P_k) \in \Sigma_0$, and $\bigcup_{k=1}^{\infty} T_k \subseteq \widehat{\Sigma}$, then the hypothesis $U \circ f \in M(\mu)$ is redundant since $U \circ f \mid T \subseteq M(\mu_T)$ in this situation.

10. Let $F$ be the class of those Fréchet spaces in which all weakly compact, hence all weakly pseudo-compact, subsets are separable. (1), (2), (3) in Theorem 2 are redundant if $X \in F$. — Now let $X = L^\infty(\nu, Y)$, where $\nu$ is a $\sigma$-finite measure and $Y$ a Banach space. Clearly, $X \in F$, if $Y \in F$ and $L^1(\nu)$ is separable. It is shown in [7] that large subspaces of $X$ are in $F$ if $Y$ admits a continuous linear injection into some Banach space $Z$ such that $W_\sigma$ satisfies CCC for some bounded $W \subseteq Z^*$ separating the points of $Z$. More elementary is the fact that $X \in F$ if $Y^*_\sigma$ is separable. For a compact space $L$, the Banach space $C(L)$ may have a weak* separable dual, even when $L$ itself is not separable; compare [10] and [17]. In [3, 5.6], an example is given of a Banach space $X$ with $X^*_\sigma$ separable (equivalently: with a countable subset of $X^*$ separating the points of $X$) such that no countable subset of $X^*$ is norming.

**Corollary.** Let $\mu$ be $\sigma$-finite. If $f \in M(\mu, X)$, then there are $N \in \Sigma_0$, some $U \subseteq X^*$ separating points of $\overline{\text{span}} f(S \setminus N)$, and a topology $T$ on $S$ with $T \setminus \{0\} \subseteq \widehat{\Sigma} \setminus \Sigma_0$ rendering $U \circ f \mid S \setminus N$ $T$-continuous for all $u \in U$. The converse holds if $f(S \setminus M)$ is relatively compact in $X_\sigma$ for some $M \in \Sigma_0$.

**Proof:** Let $f \in M(\mu, X)$ and then $Z$ a separable closed linear subspace of $X$ such that $N_1 := f^{-1}(X \setminus Z) \in \Sigma_0$. Choose a countable subset $U$ of $X^*$ that separates the points of $Z$. Moreover, let $\theta$ be a lifting of $\widehat{\Sigma}$. (Such a lifting exists: if $(S, \Sigma, \mu)$ is not finite and complete, instead we may consider any measure $\nu: \widehat{\Sigma} \rightarrow \mathbb{I}$ producing the same null sets as $\mu$. Compare [8, p.46].) But then every $u \in U$ admits some $N_u \in \Sigma_0$ for which $u \circ f \mid S \setminus N_u$ is continuous with respect to the density topology $T_\theta$ on $S$ [8, p.59]. Consequently, $N := N_1 \cup \bigcup_{u \in U} N_u$ does the job. Conversely, suppose that, in addition to the conditions listed, there exist a weakly compact subset $P$ of $X$ and some $M \in \Sigma_0$ such that $f(S \setminus M) \subseteq P$. Then the $\mu$-measurability of $f$ follows by means of Theorem 2.(3) with $N_0 := N \cup M =: N_k$, $\Sigma_c := \{S \setminus N_0\}$, $P_0 := P$, and $T_k := \{Q \setminus N_0 \mid Q \in T\}$; compare Remark 9.

**References**


Department of Mathematics
University of Bahrain
PO Box 32038
Isa Town
Bahrain