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ON MAXIMALLY FROBENIUS DESTABILISED VECTOR BUNDLES

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Abstract

Let X be a smooth projective curve of genus $g \geq 2$ over an algebraically closed field k of characteristic $p > 0$. We show that for any integers r and d with $0 < r < p$, there exists a maximally Frobenius destabilised stable vector bundle of rank r and degree d on X if and only if $r \mid d$.

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1. Introduction

Let k be an algebraically closed field of characteristic $p > 0$ and X a smooth projective curve of genus $g \geq 2$ over k. The absolute Frobenius morphism $F_X : X \to X$ is induced by $\mathscr{O}_X \to \mathscr{O}_X$, $f \mapsto f^p$. A rank-r vector bundle $\mathscr{E}$ on X is called *maximally Frobenius destabilised* if the Harder–Narasimhan filtration of $F_X^*(\mathscr{E})$,

$$\mathrm{HN}(F_X^*(\mathscr{E})) : \quad 0 = \mathscr{E}_r \subset \mathscr{E}_{r-1} \subset \cdots \subset \mathscr{E}_1 \subset \mathscr{E}_0 = F_X^*(\mathscr{E}),$$

satisfies $\mathrm{rk}(\mathscr{E}_{i-1}/\mathscr{E}_i) = 1$ for $1 \leq i \leq r$ and $\mu(\mathscr{E}_{i-1}/\mathscr{E}_i) - \mu(\mathscr{E}_i/\mathscr{E}_{i+1}) = 2g - 2$ for any i with $1 \leq i \leq r - 1$.

If $r > p$, Zhao [11, Proposition 2.8] showed that any maximally Frobenius destabilised rank-r vector bundle over an arbitrary smooth projective curve of genus $g \geq 2$ in characteristic $p > 0$ is not semistable. If $r = p$, then $F_{X*}(\mathscr{L})$ is a maximally Frobenius destabilised rank-p stable vector bundle for any line bundle $\mathscr{L}$ on an arbitrary smooth projective curve X of genus $g \geq 2$ in characteristic $p > 0$ (see [5] and [11]). If $r < p$, Zhao [11, Proposition 2.14] showed that for any given natural numbers $p > 0$, $g \geq 2$ and $r > 0$ with $r < p$ and $p \nmid g - 1$, there exists some maximally Frobenius destabilised rank-r stable vector bundle over some smooth projective curve of genus $g \geq 2$ in characteristic p. Under the assumption $p > r(r-1)(r-2)(g-1)$, Joshi and Pauly [3] gave a correspondence between maximally Frobenius destabilised

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stable vector bundles of degree 0 and dormant operatic loci, and proved the existence of Frobenius destabilised stable vector bundles of rank r and degree 0. Further results about Frobenius destabilised stable vector bundles can be found in [4, 6, 7] and [8].

The main goal of this paper is to study the existence of maximally Frobenius destabilised stable vector bundles. We give a necessary and sufficient condition for the existence of a maximally Frobenius destabilised stable vector bundle in terms of its rank and degree. The main result of the paper is the following theorem.

Theorem 1.1. *Let k be an algebraically closed field of characteristic $p > 0$ and X a smooth projective curve of genus $g \geq 2$ over k. Then, for any integers r and d with $0 < r < p$, there exists a maximally Frobenius destabilised stable vector bundle of rank r and degree d on X if and only if $r \mid d$.*

Moreover, we show that any maximally Frobenius destabilised stable vector bundle can be realised as a subsheaf of the Frobenius direct image of some line bundle (Proposition 2.5).

2. Maximally Frobenius destabilised vector bundles

Let k be an algebraically closed field of characteristic $p > 0$ and X a smooth projective curve of genus g over k. For any coherent sheaf $\mathscr{F}$ on X, there exists a *canonical connection* $(F_X^*(\mathscr{F}), \nabla_{\text{can}})$ on the coherent sheaf $F_X^*(\mathscr{F})$,

$$\nabla_{\text{can}} : F_X^*(\mathscr{F}) \to F_X^*(\mathscr{F}) \otimes_{\mathscr{O}_X} \Omega_X^1,$$

which is locally defined by $f \otimes m \mapsto m \otimes d(f)$, where $m \in \mathscr{F}$, $f \in \mathscr{O}_X$ and $d : \mathscr{O}_X \to \Omega_X^1$ is the canonical exterior differentiation.

Definition 2.1 (Joshi *et al.* [4]). Let k be an algebraically closed field of characteristic $p > 0$ and X a smooth projective curve over k. For any coherent sheaf $\mathscr{F}$ on X, let

$$\nabla_{\text{can}} : F_X^* F_{X*}(\mathscr{F}) \to F_X^* F_{X*}(\mathscr{F}) \otimes_{\mathscr{O}_X} \Omega_X^1$$

be the canonical connection on $F_X^* F_{X*}(\mathscr{F})$. Set

$$\begin{aligned} V_1 &:= \ker(F_X^* F_{X*}(\mathscr{F}) \twoheadrightarrow \mathscr{F}), \\ V_{l+1} &:= \ker\{V_l \xrightarrow{\nabla_{\text{can}}} F_X^* F_{X*}(\mathscr{F}) \otimes_{\mathscr{O}_X} \Omega_X^1 \to (F_X^* F_{X*}(\mathscr{F})/V_l) \otimes_{\mathscr{O}_X} \Omega_X^1\}. \end{aligned}$$

The filtration

$$\mathbb{F}^{\text{can}}_{\mathscr{F}\bullet} : F_X^* F_{X*}(\mathscr{F}) = V_0 \supset V_1 \supset V_2 \supset \cdots \supset V_{p-1} \supset V_p = 0$$

is called the *canonical filtration* of $F_X^* F_{X*}(\mathscr{F})$.

Theorem 2.2 (Joshi *et al.* [4] and Sun [9]). *Let k be an algebraically closed field of characteristic $p > 0$, X a smooth projective curve of genus g over k and $\mathscr{E}$ a vector bundle on X. Then the canonical filtration of $F_X^* F_{X*}(\mathscr{E})$,*

$$0 = V_p \subset V_{p-1} \subset \cdots \subset V_{l+1} \subset V_l \subset \cdots \subset V_1 \subset V_0 = F_X^* F_{X*}(\mathscr{E}),$$

has the following properties.

(1) $\nabla_{\mathrm{can}}(V_{i+1}) \subset V_i \otimes_{\mathscr{O}_X} \Omega^1_X$ *for* $0 \leq i \leq p-1$.
(2) $V_l/V_{l+1} \xrightarrow{\nabla_{\mathrm{can}}} \mathscr{E} \otimes_{\mathscr{O}_X} \Omega_X^{\otimes l}$ *is an isomorphism for* $0 \leq l \leq p-1$.
(3) *If* $g \geq 1$, *then* $F_{X*}(\mathscr{E})$ *is semistable whenever* $\mathscr{E}$ *is semistable. If* $g \geq 2$, *then* $F_{X*}(\mathscr{E})$ *is stable whenever* $\mathscr{E}$ *is stable.*
(4) *If* $g \geq 2$ *and* $\mathscr{E}$ *is semistable, then the canonical filtration of* $F_X^*F_{X*}(\mathscr{E})$ *is just the Harder–Narasimhan filtration of* $F_X^*F_{X*}(\mathscr{E})$.

Theorem 2.3 (Sun [10, Corollary 2.4]). *Let* k *be an algebraically closed field of characteristic* $p > 0$, X *a smooth projective curve of genus* $g \geq 2$ *over* k *and* $\mathscr{L} \in \mathrm{Pic}(X)$. *Then, for any coherent subsheaf* $\mathscr{E} \subseteq F_{X*}(\mathscr{L})$,

$$\mu(\mathscr{E}) - \mu(F_{X*}(\mathscr{L})) \leq -\frac{p - \mathrm{rk}(\mathscr{E})}{p}(g-1).$$

Proposition 2.4. *Let* k *be an algebraically closed field of characteristic* $p > 0$, X *a smooth projective curve of genus* $g \geq 2$ *over* k *and* $\mathscr{E}$ *a maximally Frobenius destabilised vector bundle of rank* r *and degree* d *on* X. *Then* $r \mid pd$.

Proof. Suppose that $\mathrm{rk}(\mathscr{E}) = r$ and $\deg(\mathscr{E}) = d$. Let

$$\mathrm{HN}(F_X^*(\mathscr{E})): \quad 0 = \mathscr{E}_r \subset \mathscr{E}_{r-1} \subset \cdots \subset \mathscr{E}_1 \subset \mathscr{E}_0 = F_X^*(\mathscr{E})$$

be the Harder–Narasimhan filtration of $F_X^*(\mathscr{E})$ and $F_X^*(\mathscr{E}) \twoheadrightarrow \mathscr{L} \cong \mathscr{E}_0/\mathscr{E}_1$ the quotient line bundle with minimal degree in the Harder–Narasimhan filtration of $F_X^*(\mathscr{E})$. Since $\mathscr{E}$ is a maximally Frobenius destabilised vector bundle on X,

$$\deg(F_X^*(\mathscr{E})) = \sum_{i=0}^{r-1} \deg(\mathscr{E}_i/\mathscr{E}_{i+1}) = r\deg(\mathscr{L}) + r(r-1)(g-1) = pd.$$

It follows that $\deg(\mathscr{L}) = (pd/r) - (r-1)(g-1) \in \mathbb{Z}$. Hence, $r \mid pd$. □

Proposition 2.5. *Let* k *be an algebraically closed field of characteristic* $p > 0$ *and* X *a smooth projective curve of genus* $g \geq 2$ *over* k. *Let* $\mathscr{E}$ *be a maximally Frobenius destabilised semistable vector bundle of rank* r *and degree* d *on* X *and* $F_X^*(\mathscr{E}) \twoheadrightarrow \mathscr{L}$ *the quotient line bundle with minimal degree in the Harder–Narasimhan filtration of* $F_X^*(\mathscr{E})$. *Then:*

(1) $\mu(F_{X*}(\mathscr{L})) - \mu(\mathscr{E}) = ((p-r)/p)(g-1)$; *and*
(2) *the adjoint homomorphism* $\mathscr{E} \hookrightarrow F_{X*}(\mathscr{L})$ *is an injection, that is,* $\mathrm{rk}(\mathscr{E}) \leq p$.

Proof. Since $\mathscr{E}$ is a maximally Frobenius destabilised vector bundle on X, we have $\deg(\mathscr{L}) = (pd/r) - (r-1)(g-1)$ from the proof of Proposition 2.4. Moreover, from [4, Section 2.9],

$$\deg(F_{X*}(\mathscr{L})) = \frac{pd}{r} - (r-1)(g-1) + (p-1)(g-1) = \frac{pd}{r} + (p-r)(g-1).$$

It follows that

$$\mu(F_{X*}(\mathscr{L})) - \mu(\mathscr{E}) = \frac{p-r}{p}(g-1).$$

By adjunction, there is a nontrivial homomorphism $\mathscr{E} \to F_{X*}(\mathscr{L})$. Denote the image by $\mathscr{G}$. If $\mathrm{rk}(\mathscr{G}) < \mathrm{rk}(\mathscr{E})$, then, by [10, Corollary 2.4] and the stability of $F_{X*}(\mathscr{L})$,

$$\mu(\mathscr{G}) - \mu(F_{X*}(\mathscr{L})) \leq -\frac{p - \mathrm{rk}(\mathscr{G})}{p}(g-1),$$

$$\mu(\mathscr{G}) \leq \mu(F_{X*}(\mathscr{L})) - \frac{p - \mathrm{rk}(\mathscr{G})}{p}(g-1) = \frac{d}{r} - \frac{r - \mathrm{rk}(\mathscr{G})}{p}(g-1).$$

Thus, $\mu(\mathscr{G}) < \mu(\mathscr{E})$. This contradicts the semistability of $\mathscr{E}$, so $\mathrm{rk}(\mathscr{G}) = \mathrm{rk}(\mathscr{E})$. Therefore, $\mathscr{E} \cong \mathscr{G}$, that is, the adjoint homomorphism $\mathscr{E} \hookrightarrow F_{X*}(\mathscr{L})$ is an injection. □

It is easy to deduce [11, Proposition 2.8] from Proposition 2.5.

Corollary 2.6 (Zhao [11, Proposition 2.8]). *Let k be an algebraically closed field of characteristic $p > 0$, X a smooth projective curve of genus $g \geq 2$ over k and $\mathscr{E}$ a maximally Frobenius destabilised vector bundle on X with $\mathrm{rk}(k) > p$. Then $\mathscr{E}$ is not semistable.*

Proposition 2.7. *Let k be an algebraically closed field of characteristic $p > 0$, X a smooth projective curve of genus $g \geq 2$ over k and $\mathscr{L} \in \mathrm{Pic}(X)$. Let $\mathscr{E}$ be a coherent subsheaf of $F_{X*}(\mathscr{L})$ such that $\mathrm{rk}(\mathscr{E}) = r < p$ and*

$$\mu(F_{X*}(\mathscr{L})) - \mu(\mathscr{E}) = (1 - r/p)(g-1).$$

Then:

(1) *$F_{X*}(\mathscr{L})$ is a maximally Frobenius destabilised stable vector bundle;*
(2) *$F_{X*}(\mathscr{L})/\mathscr{E}$ is a maximally Frobenius destabilised stable vector bundle; and*
(3) *the adjoint homomorphism $F_X^*(\mathscr{E}) \to \mathscr{L}$ is the quotient line bundle with the minimal slope in the Harder–Narasimhan filtration of $F_X^*(\mathscr{E})$.*

Proof. The canonical filtration of $F_X^* F_{X*}(\mathscr{L})$,

$$\mathbb{F}^{\mathrm{can}}_{\mathscr{F}\bullet}: \quad 0 = V_p \subset V_{p-1} \subset \cdots \subset V_1 \subset V_0 = F_X^* F_{X*}(\mathscr{L}),$$

induces the filtration

$$0 \subset V_m \cap F_X^*(\mathscr{E}) \subset V_{m-1} \cap F_X^*(\mathscr{E}) \subset \cdots \subset V_1 \cap F_X^*(\mathscr{E}) \subset V_0 \cap F_X^*(\mathscr{E}) = F_X^*(\mathscr{E}),$$

where $m = \max\{l \mid V_l \cap F_X^*(\mathscr{E}) \neq 0, 0 \leq l \leq p-1\}$. Let

$$W_l := \frac{V_l \cap F_X^*(\mathscr{E})}{V_{l+1} \cap F_X^*(\mathscr{E})} \subset \frac{V_l}{V_{l+1}}, \quad r_l := \mathrm{rk}(W_l) \quad \text{for } 0 \leq l \leq m.$$

Then the injections $W_l \hookrightarrow W_{l-1} \otimes_{\mathscr{O}_X} \Omega^1_X$, $1 \leq l \leq m$, imply that $r_0 \geq r_1 \geq r_2 \geq \cdots \geq r_m$. Since $r_0 = 1$, it follows that $m = r - 1$ and $r_0 = r_1 = r_2 = \cdots = r_{r-1} = 1$.

Let $\mathscr{G} \subset \mathscr{E}$ be a subsheaf of $\mathscr{E}$ with $\mathrm{rk}(\mathscr{G}) < \mathrm{rk}(\mathscr{E})$. By [10, Corollary 2.4] and the stability of $F_{X*}(\mathscr{L})$,

$$\mu(\mathscr{G}) - \mu(F_{X*}(\mathscr{L})) \leq -\frac{p - \mathrm{rk}(\mathscr{G})}{p}(g-1).$$

It follows that

$$\mu(\mathscr{G}) \le \mu(F_{X*}(\mathscr{L})) - \frac{p-\mathrm{rk}(\mathscr{G})}{p}(g-1) = \mu(\mathscr{E}) - \frac{r-\mathrm{rk}(\mathscr{G})}{p}(g-1).$$

Thus, $\mathscr{E}$ is a stable vector bundle. Summing over i,

$$\deg(F_X^*(\mathscr{E})) = \sum_{i=0}^{r-1}\deg(W_i) \le \sum_{i=0}^{r-1}\deg(\mathscr{L}\otimes_{\mathscr{O}_X}\Omega_X^{\otimes i}) = pd.$$

Hence, the previous inequality is an equality. It follows that

$$W_l \cong \frac{V_l}{V_{l+1}} \cong \mathscr{L}\otimes_{\mathscr{O}_X}\Omega_X^{\otimes l} \quad \text{for } 0 \le l \le r-1.$$

Thus, $\mu(W_i) - \mu(W_{i-1}) = 2g-2$ for $1 \le i \le r-1$ and the Harder–Narasimhan filtration of $F_X^*(\mathscr{E})$ is

$$0 \subset V_{r-1}\cap F_X^*(\mathscr{E}) \subset V_{r-2}\cap F_X^*(\mathscr{E}) \subset \cdots \subset V_1\cap F_X^*(\mathscr{E}) \subset V_0\cap F_X^*(\mathscr{E}) = F_X^*(\mathscr{E}).$$

Hence, $\mathscr{E}$ is a maximally Frobenius destabilised stable vector bundle and the adjoint homomorphism $F_X^*(\mathscr{E}) \twoheadrightarrow \mathscr{L}$ is the quotient line bundle with the minimal slope in the Harder–Narasimhan filtration of $F_X^*(\mathscr{E})$. This completes the proof of (1) and (3) of the proposition.

Now we will prove the stability of $F_{X*}(\mathscr{L})/\mathscr{E}$. Let $\mathscr{F}$ be the subsheaf of $F_{X*}(\mathscr{L})/\mathscr{E}$ with $\mathrm{rk}(\mathscr{F}) = t < p-r = \mathrm{rk}(F_{X*}(\mathscr{L})/\mathscr{E})$ and $\widetilde{\mathscr{F}}$ the preimage of $\mathscr{F}$ under the projection $F_{X*}(\mathscr{L}) \twoheadrightarrow F_{X*}(\mathscr{L})/\mathscr{E}$. By [10, Corollary 2.4] and the stability of $F_{X*}(\mathscr{L})$,

$$\mu(\widetilde{\mathscr{F}}) \le \mu(F_{X*}(\mathscr{L})) - \frac{p-\mathrm{rk}(\widetilde{\mathscr{F}})}{p}(g-1) = \mu(\mathscr{E}) + \frac{t}{p}(g-1).$$

Hence,

$$\deg(\mathscr{F}) = \deg(\widetilde{\mathscr{F}}) - \deg(\mathscr{E}) \le t\cdot\mu(\mathscr{E}) + \frac{t(r+t)}{p}(g-1),$$
$$\mu(\mathscr{F}) \le \mu(\mathscr{E}) + \frac{r+t}{p}(g-1).$$

On the other hand, $\mu(F_{X*}(\mathscr{L})) - \mu(\mathscr{E}) = (1-r/p)(g-1)$ implies that

$$\deg(F_{X*}(\mathscr{L})/\mathscr{E}) = \deg(F_{X*}(\mathscr{L})) - \deg(\mathscr{E}) = (p-r)\mu(\mathscr{E}) + (p-r)(g-1),$$
$$\mu(F_{X*}(\mathscr{L})/\mathscr{E}) = \mu(\mathscr{E}) + (g-1).$$

Thus,

$$\mu(\mathscr{F}) \le \mu(\mathscr{E}) + \frac{r+t}{p}(g-1) < \mu(\mathscr{E}) + (g-1) = \mu(F_{X*}(\mathscr{L})/\mathscr{E}).$$

Hence, $F_{X*}(\mathscr{L})/\mathscr{E}$ is a stable vector bundle.

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Projecting the canonical filtration $\mathbb{F}^{\mathrm{can}}_{\mathscr{F}\bullet}$ of $F_X^* F_{X*}(\mathscr{L})$ to the quotient sheaf $F_X^*(F_{X*}(\mathscr{L})/\mathscr{E})$ gives the filtration

$$0 = \widetilde{V}_p \subset \widetilde{V}_{p-1} \subset \cdots \subset \widetilde{V}_1 \subset \widetilde{V}_0 = F_X^*(F_{X*}(\mathscr{L})/\mathscr{E}),$$

where $\widetilde{V}_i := V_i/(V_i \cap F_X^*(\mathscr{E}))$ for $0 \leq i \leq p-1$.

Consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V_{i+1} \cap F_X^*(\mathscr{E}) & \longrightarrow & V_{i+1} & \longrightarrow & V_{i+1}/(V_{i+1} \cap F_X^*(\mathscr{E})) & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\ 0 & \longrightarrow & V_i \cap F_X^*(\mathscr{E}) & \longrightarrow & V_i & \longrightarrow & V_i/(V_i \cap F_X^*(\mathscr{E})) & \longrightarrow & 0 \end{array}$$

for $0 \leq i \leq r-2$. Since $(V_i \cap F_X^*(\mathscr{E}))/(V_{i+1} \cap F_X^*(\mathscr{E})) \cong V_i/V_{i+1} \cong \mathscr{L} \otimes_{\mathscr{O}_X} \Omega_X^{\otimes i}$,

$$V_i/(V_i \cap F_X^*(\mathscr{E})) \cong F_X^*(F_{X*}(\mathscr{L})/\mathscr{E})$$

for $0 \leq i \leq r-1$ by the snake lemma. This yields the filtration on $F_X^*(F_{X*}(\mathscr{L})/\mathscr{E})$:

$$0 = \widetilde{V}_p \subset \widetilde{V}_{p-1} \subset \cdots \subset \widetilde{V}_r \subset \widetilde{V}_{r-1} = F_X^*(F_{X*}(\mathscr{L})/\mathscr{E}),$$

where $\widetilde{V}_i/\widetilde{V}_{i+1} \cong \mathscr{L} \otimes_{\mathscr{O}_X} \Omega_X^{\otimes i}$ for $r \leq i \leq p-1$. This is just the Harder–Narasimhan filtration of $F_X^*(F_{X*}(\mathscr{L})/\mathscr{E})$. Hence, $F_{X*}(\mathscr{L})/\mathscr{E}$ is also a maximally Frobenius destabilised stable vector bundle. □

3. Existence of maximally Frobenius destabilised vector bundles

Let k be an algebraically closed field of arbitrary characteristic and X a smooth projective curve of genus g over k. Let $\mathscr{E}$ be a vector bundle of rank n and degree d over X. For any integer r with $0 < r < n$, define

$$s_r(\mathscr{E}) := r \cdot d - n \cdot \max\{\deg(\mathscr{E}') | \mathscr{E}' \subsetneq \mathscr{E}, \mathrm{rk}(\mathscr{E}') = r\}.$$

Then $s_r(\mathscr{E}) \equiv r \cdot d \pmod{n}$ and $\mathscr{E}$ is semistable (respectively stable) if and only if $s_r \geq 0$ (respectively $s_r > 0$) for any integer $0 < r < n$.

Lemma 3.1 (Hirschowitz [1, Theorem 4.4]). *Let k be an algebraically closed field of arbitrary characteristic, X a smooth projective curve of genus $g \geq 2$ over k and $\mathscr{E}$ a vector bundle over X. Then, for any integer r with $0 < r < n$, there exists a subbundle $\mathscr{E}' \subset \mathscr{E}$ of rank r with*

$$r \cdot d - n \cdot \deg(\mathscr{E}') \leq r(n-r)(g-1) + \varepsilon,$$

where ε is the unique integer with $r(n-r)(g-1) + \varepsilon \equiv r \cdot d \pmod{n}$ and $0 \leq \varepsilon < n$, that is, $s_r(\mathscr{E}) \leq r(n-r)(g-1) + \varepsilon$.

Proposition 3.2. *Let k be an algebraically closed field of characteristic $p > 0$, X a smooth projective curve of genus $g \geq 2$ over k and $\mathscr{L} \in \mathrm{Pic}(X)$. Then, for any integer r with $0 < r < p$,*

$$s_r(F_{X*}(\mathscr{L})) = r(p-r)(g-1) + \varepsilon,$$

where ε is the unique integer satisfying $r(p-r)(g-1) + \varepsilon \equiv r \cdot d \pmod p$ and $0 \leq \varepsilon < p$.

Proof. Let $\mathscr{E}$ be a rank-r subsheaf of $F_{X*}(\mathscr{L})$. By [10, Corollary 2.4],

$$p \cdot \deg(\mathscr{E}) - r \cdot \deg(F_{X*}(\mathscr{L})) \leq -r(p-r)(g-1).$$

This implies that $s_r(F_{X*}(\mathscr{L})) \geq r(p-r)(g-1)$. Then, by Lemma 3.1,

$$s_r(F_{X*}(\mathscr{L})) = r(p-r)(g-1) + \varepsilon,$$

where ε is the unique integer satisfying $r(p-r)(g-1) + \varepsilon \equiv r \cdot d \pmod p$ and $0 \leq \varepsilon < p$. □

Theorem 3.3. *Let k be an algebraically closed field of characteristic $p > 0$ and X a smooth projective curve of genus $g \geq 2$ over k. Then, for any integers r and d with $0 < r < p$, there exists a maximally Frobenius destabilised stable vector bundle of rank r and degree d on X if and only if $r \mid d$.*

Proof. For any maximally Frobenius destabilised stable vector bundle of rank r and degree d on X, we have $r \mid d$ by Proposition 2.4.

Conversely, suppose that $0 < r < p$ and $r \mid d$. Let $\mathscr{L}$ be a line bundle of degree $(pd/r) - (r-1)(g-1)$ on X. From [4, Section 2.9],

$$\deg(F_{X*}(\mathscr{L})) = \frac{pd}{r} + (p-r)(g-1).$$

Let $\mathscr{E}$ be a rank-r subsheaf of $F_{X*}(\mathscr{L})$ with maximal degree. Then

$$s_r(F_{X*}(\mathscr{L})) = r\Big(\frac{pd}{r} + (p-r)(g-1)\Big) - p \cdot \deg(\mathscr{E}) = r(p-r)(g-1) + \varepsilon,$$

where ε is the unique integer satisfying $r(p-r)(g-1) + \varepsilon \equiv r \cdot d \pmod p$ and $0 \leq \varepsilon < p$. It follows that $\varepsilon = 0$ and $\deg(\mathscr{E}) = d$. Thus, $\mathscr{E}$ is a maximally Frobenius destabilised stable vector bundle of rank r and degree d on X by Proposition 2.7. □

Remark 3.4. Joshi provided a conjectural formula [2, Conjecture 8.1] for the degree of the dormant operatic locus under the assumption $p > r(r-1)(r-2)(g-1)$. By Theorem 3.3, the left-hand side of the conjectural formula still makes sense after removing the assumption $p > r(r-1)(r-2)(g-1)$. So, we can propose the conjectural formula for any integer r under the condition $0 < r < p$.

Corollary 3.5. *Let k be an algebraically closed field of characteristic $p > 0$ and X a smooth projective curve of genus $g \geq 2$ over k. Then there exists a maximally Frobenius destabilised rank-p nonsemistable vector bundle on X.*

Proof. Fix two integers r and d with $0 < r < p$ and $r \mid d$. Choosing any line bundle $\mathscr{L}$ of degree $(pd/r) - (r-1)(g-1)$ on X, by the proof of Theorem 3.3, there exists a rank-r and degree-d subsheaf $\mathscr{E}$ of $F_{X*}(\mathscr{L})$. By Proposition 2.7, $\mathscr{E}$ and $F_{X*}(\mathscr{L})/\mathscr{E}$ are maximally Frobenius destabilised stable vector bundles on X. It is easy to check that $\mathscr{E} \oplus F_{X*}(\mathscr{L})/\mathscr{E}$ is a maximally Frobenius destabilised rank-p nonsemistable vector bundle on X. □

By Corollary 3.5, we see that the condition $r < p$ is necessary in [11, Proposition 2.6].

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