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RESEARCH ARTICLE

On $A_n \times C_m$-unramified extensions over imaginary quadratic fields

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Abstract

Let n be an integer congruent to 0 or 3 modulo 4. Under the assumption of the ABC conjecture, we prove that, given any integer m fulfilling only a certain coprimeness condition, there exist infinitely many imaginary quadratic fields having an everywhere unramified Galois extension of group $A_n \times C_m$. The same result is obtained unconditionally in special cases.

1. Introduction

Understanding unramified extensions of number fields is of crucial importance in algebraic number theory, much like understanding the geometric fundamental groups of manifolds is in geometry.

In particular, a well-known open problem in number theory is the question whether, given any finite group G, there exist infinitely many quadratic number fields possessing an everywhere unramified Galois extension with Galois group G. It is furthermore of interest to determine explicit families of number fields having such an unramified extension. For the case of cyclic groups C_m, these questions are part of class field theory, and it has long been known that for any given integer $n > 1$, there exist infinitely many quadratic number fields such that the ideal class group has a cyclic subgroup of order m (see [1, 18, 23] for existence results as well as [6, 7, 9, 14, 16, 17, 20, 24] for stronger quantitative as well as explicit computational results). See also the survey paper [2]. On the other end of the scale is the case of non-abelian simple groups. The symmetric and alternating groups are among the relatively few groups for which the problem has been solved, and several previous results such as those of Uchida [22], Yamamoto [23], Elstrodt–Grunewald–Mennicke [5], Kondo [13], and Kedlaya [8] are related to the existence of unramified extensions over quadratic fields whose Galois group is isomorphic to a symmetric or alternating group. Similar existence results for some other (small) nonabelian simple groups are contained in [12]. Going one step further, one may wish to investigate the problem for nonsolvable groups which are not generated by involutions, a condition that in practice tends to add to the difficulty, as explained in [10]. Some first results were obtained in [10] and [11], which solved the problem for certain nonsplit central extensions of simple groups such as $\mathrm{SL}_2(\mathbb{F}_7)$ and $\mathrm{SL}_2(\mathbb{F}_5)$.

In the same spirit, it is also natural to ask whether it is possible to find quadratic number fields having an A_n-unramified extension and a C_m-unramified extension simultaneously. This is particularly interesting, since the solution requires to combine ideas from the realization of cyclic groups and simple groups, whose methods are otherwise often quite different. We give a partial answer to this question. Here is the main theorem.

Main Theorem. *Let n be an integer congruent to* 0 *(resp* 3*) modulo* 4 *and m be an integer such that* $\gcd(n, m) = 1$ *(resp.* $\gcd(n-1, m) = 1$*). If we assume that the ABC conjecture is true, then there exist infinitely many imaginary quadratic fields having an $A_n \times C_m$-unramified extension.*

The theorem will be proven in Section 3. The usefulness of the ABC conjecture to problems related to class number divisibility has been previously noticed by Murty, see [16]. Furthermore, the relevance of the ABC conjecture to counting squarefree discriminants of trinomials, and consequently to counting quadratic fields with unramified A_n-extensions, was pointed out in [15]. It seems, however, that previously the two directions of "class number divisibility" and "unramified non-solvable (namely, A_n)-extensions" have not been put together. The proof of our main theorem uses careful specialization of trinomial families to achieve these two goals simultaneously.

In Section 4, we point out a special case in which the assertion follows unconditionally, without relying on the ABC conjecture.

2. Preliminaries

2.1. Trinomials and construction of A_n-unramified extensions over quadratic fields

There are several ways to construct A_n-unramified extensions over quadratic fields. A well-known approach works with trinomials, that is, polynomials of the form $X^n + aX^k + b$. The following discriminant formula for trinomial extensions, due to Swan, will be very useful.

Theorem 2.1. (Theorem 2 of [21]): *The trinomial $f(x) = X^n - aX^k + b$ with $0 < k < n$ has discriminant:*

$$D(f) = (-1)^{\frac{n(n-1)}{2}} b^{k-1} \left[n^N b^{N-K} - (n-k)^{N-K} k^K a^N \right]^d,$$

where $d = (n, k), N = n/d, K = k/d$.

Theorem 2.2. (Theorem 1 of [19]) *Let $f(X) = X^n + aX^k + b$ be a polynomial of rational integral coefficients, that is, $f(X) \in \mathbb{Z}[X]$. Let $a = a_0 c^n$ and $b = b_0^k c^n$. Then the Galois group of f is isomorphic to the symmetric group S_n of degree n if the following conditions are satisfied:*

1. *$f(X)$ is irreducible ouer $\mathbb{Q}$,*
2. *$a_0 c(n-k)k$ and nb_0 are relatively prime, that is, $(a_0 c(n-k)k, nb_0) = 1$.*

Theorem 2.3. (Theorem 1 of [22]) *Let k be an algebraic number field of finite degree. Let a and b be integers of k. Let*

$$f(X) = X^n - aX + b,$$

and let K denote the splitting field of f over k, that is, $K = k(\alpha_1, \alpha_2, \cdots, \alpha_n)$ where $\alpha_1, \alpha_2, \cdots, \alpha_n)$ are the roots of $f(X) = 0$. Let D be the discriminant of $f(X)$. If $(n-1)a$ and nb are relatively prime, K is unramified over $k(\sqrt{D})$.

2.2. Divisibility of class numbers of imaginary quadratic fields

We collect some useful results on class number divisibility.

Proposition 2.4. (Proposition 1 of [20]) *Let $g \geq 3$ be an integer. Suppose that $d \geq 63$ is a squarefree integer such that $t^2 d = m^g - n^2$. Here, t, m, and n are positive integers with $(m, 2n) = 1$ and with $m^g < (d+1)^2$. Then,* $\mathrm{CL}(-d)$ *contains an element of order g where* $\mathrm{CL}(-d)$ *is the class group of* $\mathbb{Q}(\sqrt{-d})$.

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Notice the condition $m^g < (d+1)^2$ in the above proposition, which is exactly the reason why the ABC conjecture will be needed in our proof. There are also numerous results on class number divisibility of certain families of quadratic fields which do not require such extra conditions. Among those, the following is particularly well suited to our problem.

Theorem 2.5. (Main Theorem of [24]) *Let k be an odd integer with $k > 1$ and $2^{2m} < k^n$ where m is a positive integer and n is an odd positive integer with $n > 3$. Suppose that $2^{2m} - k^n \not\equiv 5 \pmod 8$. Then the class group of the imaginary quadratic field $\mathbb{Q}(\sqrt{2^{2m} - k^n})$ contains an element of order n.*

Remark 2.6. *Note that in [24], the assumptions of the theorem are weaker; we only extracted the case relevant to our application. On the other hand, in [24], the conclusion is worded only in terms of order of the class group, not in terms of order of elements. The latter is a priori stronger, but the proof in [24] does indeed yield this stronger version. We briefly summarize the points relevant to this stronger conclusion. Let $-dy^2 = 2^{2m} - k^n$ where d is a squarefree integer. Then, we know that $2^{2m} + y^2 d = k^n$ and $\left(2^m + y\sqrt{-d}\right) = \mathfrak{k}^n$ for some ideal $\mathfrak{k}$. We want to show that the ideal class of $\mathfrak{k}$ has order n. To do this, one has to show that $2^m + y\sqrt{-d} \neq \left(x_1 + y_1\sqrt{-d}\right)^t$ for all divisors $t \neq 1$ of n. $\left(\text{If } 2^m + y\sqrt{-d} = \left(x_1 + y_1\sqrt{-d}\right)^t \text{ for some } t, \text{ then we know that the order of } \mathfrak{k} \text{ is a divisor of } n/t.\right)$*

Suppose that $2^m + y\sqrt{-d} = \left(x_1 + y_1\sqrt{-d}\right)^t$ for some t. Then, [24] deduces a contradiction unless $t = 3$. If $t = 3$, there is a contradiction when $2^{2m} - k^n \not\equiv 5 \pmod 8$. (See the bottom of page 153 and the top of page 154 in [24].) Thus, $2^m + y\sqrt{-d} \neq \left(x_1 + y_1\sqrt{-d}\right)^t$ for all $t|n, t \neq 1$ when $2^{2m} - k^n \not\equiv 5 \pmod 8$. In other words, the order of $\mathfrak{k}$ is n and the class group of the ring of integer of $\mathbb{Q}\left(\sqrt{2^{2m} - k^n}\right)$ has an element of order n.

2.3. The ABC conjecture

The ABC conjecture is stated in terms of three positive integers a, b, and c that are relatively prime and satisfy $a + b = c$. The conjecture essentially states that in this situation the radical of abc, that is, the product of the distinct prime factors of abc, cannot be too small. Here is the concrete statement.

The ABC Conjecture. *For each $\epsilon > 0$, there are at most finitely many coprime triples a, b, c of positive integers with $a + b = c$ and $\mathrm{rad}(abc) < c^{1-\epsilon}$. Here, $\mathrm{rad}(n)$ denotes the radical of a positive integer n.*

3. Proof of the main theorem

We will now prove our main result.

Proof. Case 1: $n \equiv 3 \pmod 4$. Let us consider the following polynomial:

$$f(x) = X^n - X - n^k, \tag{3.1}$$

where k is an arbitrary integer. By Theorem 2.1, we know that the discriminant of $f(x)$ is

$$(n-1)^{n-1} - n^{n+k(n-1)}.$$

Assume for the moment that f is irreducible. Then the Galois group of the splitting field K of f is isomorphic to S_n by Theorem 2.2. On the other hand, due to [4, Theorem 1.2], if the polynomial f is reducible for infinitely many choices of k, then the two-variable polynomial $F(T, X) = X^n - X - n^u T^e$ must be reducible for some integer u and positive integer e. However, that polynomial is an Eisenstein polynomial in T (for the prime X of $\mathbb{Q}[X]$) and therefore clearly irreducible. We have thus obtained that f has Galois group S_n for all but finitely many choices of k. We also know that K is unramified over $\mathbb{Q}\left(\sqrt{(n-1)^{n-1} - n^{n+k(n-1)}}\right)$, that is, $\mathbb{Q}\left(\sqrt{(n-1)^{n-1} - n^{n+k(n-1)}}\right)$ has an A_n-unramified

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extension K. The remaining task is to show that, for infinitely many choices of k, the class group of $\mathbb{Q}\left(\sqrt{(n-1)^{n-1}-n^{n+k(n-1)}}\right)$ contains an element of m. The discriminant of $f(x)$ can be written as follows:

$$(n-1)^{n-1}-n^{n+k(n-1)}=-b^2D$$

Here, D is a squarefree integer. Set $A:=(n-1)^{n-1}$, $B=b^2D$ and $C:=n^{n+k(n-1)}$ so that $A+B=C$. By Proposition 2.4, it suffices to show that $(D+1)^2>C$. If we assume the ABC conjecture, for a given $\epsilon>0$, for all but finitely many (A,B,C) of this kind, one has $\mathrm{rad}(ABC)>C^{1-\epsilon}$. From now on, we will write $a:=n+k(n-1)$. We easily see that

$$(n-1)\mathrm{rad}(B)n\geq \mathrm{rad}(ABC)>C^{1-\epsilon}=n^{a(1-\epsilon)}.$$

In other words,

$$\mathrm{rad}(B)>n^{a(1-\epsilon)-2}\frac{n}{n-1}>n^{a(1-\epsilon)-2}=(n^a)^{1-\epsilon-2/a}>B^{1-\epsilon-2/a}.$$

As soon as $a>8$, one has $\mathrm{rad}(B)>B^{3/4}$. This means that

$$B/\mathrm{rad}(B)=\prod_{p|B}p^{e_p-1}<B^{1/4}$$

where e_p is the multiplicity of p in the prime factorization of B. Since all the primes contributing to b^2 must divide B more than once, it follows that the term b^2 in the expression $B=b^2D$ must be smaller than $(B^{1/4})^2=B^{1/2}$. Therefore, we know that $B=b^2D<B^{1/2}D$ and $B<D^2$. Since A is a fixed value (not depending on k), if a is sufficiently large, we obtain the condition $A<B^{1/2}$. In conclusion,

$$n^a=C=A+B<D^2+D<(D+1)^2.$$

Since $n-1$ is relatively prime to m, we can find positive integers k such that $m|\big(n+k(n-1)\big)$. Therefore, if we choose k satisfying $m|(n+k(n-1))$ and $n+k(n-1)=a>8$, the class group of $\mathbb{Q}\big(\sqrt{(n-1)^{n-1}-n^{n+k(n-1)}}\big)$ contains an element of order m under the assumption of the ABC conjecture.

Case 2: $n\equiv 0 \pmod 4$. Let us consider the polynomial:

$$f(x)=X^n-(n-1)^kX+1, \tag{3.2}$$

where k is an arbitrary integer. We know that the discriminant of $f(X)$ is

$$n^n-(n-1)^{n-1+nk}.$$

By an argument analogous to the above, we know that f has Galois group S_n and the splitting field K of f is unramified over $\mathbb{Q}\left(\sqrt{n^n-(n-1)^{n-1+nk}}\right)$ for all but finitely many choices of k. We also want to show that $\mathbb{Q}\left(\sqrt{n^n-(n-1)^{n-1+nk}}\right)$ has a C_m-unramified extension by the similar way. The discriminant of $f(x)$ can be written as follows:

$$n^n-(n-1)^{n-1+nk}=-b^2D$$

where D is a squarefree integer. Set $A:=n^n$, $B:=b^2D$, and $C:=(n-1)^{n-1+nk}$ so that $A+B=C$. By the similar argument in the above, under the assumption of the ABC conjecture, for a given $\epsilon>0$, for all but finitely many (A,B,C), we have $\mathrm{rad}(ABC)>C^{1-\epsilon}$. Set $a=n(k+1)-1$. We know that

$$n\cdot\mathrm{rad}(B)\cdot(n-1)\geq\mathrm{rad}(ABC)>C^{1-\epsilon}=(n-1)^{a(1-\epsilon)}.$$

It means that,

$$\mathrm{rad}(B)>(n-1)^{a(1-\epsilon)-3}\frac{(n-1)^2}{n}>(n-1)^{a(1-\epsilon)-3}=((n-1)^a)^{1-\epsilon-3/a}>B^{1-\epsilon-3/a}.$$

In particular, for $a>12$, we have $\mathrm{rad}(B)>B^{3/4}$. This implies that

$$B/\mathrm{rad}(B)=\prod_{p|B}p^{e_p-1}<B^{1/4}$$

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Table 1. *Numerical examples for* $n = 7, 8, 11, 12, 15, 16$

n	m	k	$f(x)$	$\mathrm{Gal}(K/\mathbb{Q}(\sqrt{-d}))$	$\mathrm{CL}(-d)$
7	13	1	$x^7 - x - 7^1$	A_7	$C_{13\times 17352}$
7	19	2	$x^7 - x - 7^2$	A_7	$C_{19\times 2265609}$
7	25	3	$x^7 - x - 7^3$	A_7	$C_2 \times C_{25\times 462766508}$
7	31	4	$x^7 - x - 7^4$	A_7	$C_2 \times C_2 \times C_2 \times C_2 \times C_{31\times 11748043524}$
7	37	5	$x^7 - x - 7^5$	A_7	$C_2 \times C_2 \times C_{37\times 17184049472416}$
7	43	6	$x^7 - x - 7^6$	A_7	$C_4 \times C_{43\times 3529527988166436}$
7	49	7	$x^7 - x - 7^7$	A_7	$C_2 \times C_4 \times C_{49\times 996646374538593224}$
8	15	1	$x^8 - 7x + 1$	A_8	$C_{15\times 9444}$
8	23	2	$x^8 - 7^2x + 1$	A_8	$C_2 \times C_{23\times 85502314}$
8	31	3	$x^8 - 7^3x + 1$	A_8	$C_2 \times C_2 \times C_2 \times C_2 \times C_{31\times 18978581872}$
8	39	4	$x^8 - 7^4x + 1$	A_8	$C_4 \times C_{39\times 30473144843060}$
8	47	5	$x^8 - 7^5x + 1$	A_8	$C_2 \times C_2 \times C_2 \times C_{47\times 122165495694239114}$
11	21	1	$x^{11} - x - 11$	A_{11}	$C_{21\times 850416880}$
11	31	2	$x^{11} - x - 11^2$	A_{11}	$C_2 \times C_2 \times C_6 \times C_{31\times 14559528313494}$
11	41	3	$x^{11} - x - 11^3$	A_{11}	$C_2 \times C_6 \times C_{41\times 3368337361585347756}$
12	23	1	$x^{12} - 11x + 1$	A_{12}	$C_2 \times C_{23\times 4534229550}$
12	35	2	$x^{12} - 11^2x + 1$	A_{12}	$C_2 \times C_2 \times C_{35\times 9499707828478530}$
15	29	1	$x^{15} - x - 15$	A_{15}	$C_2 \times C_{29\times 2247764426365470}$
16	31	1	$x^{16} - 15x + 1$	A_{16}	$C_2 \times C_2 \times C_{31\times 22784561683366030}$

With the same development as in Case 1, we can deduce

$$(n-1)^a = C = A + B < D^2 + D < (D+1)^2.$$

for all sufficiently large a. With the same idea as in Case 1, if we put k satisfying $m|(n-1+nk)$ (which is possible since m and n are coprime) and $n-1+kn = a > 12$, the class group of $\mathbb{Q}\left(\sqrt{n^n - (n-1)^{n-1+kn}}\right)$ contains an element of order m under the assumption of the ABC conjecture. □

4. An unconditional result

Although we have proven our Main Theorem under the assumption of the ABC conjecture, in special cases, it is possible to prove it without this assumption. We will explain such a case in this section.

Theorem 4.1. *Let* $n = 2^\ell$ *be a* 2*-power. Then there exist infinitely many imaginary quadratic fields having an* $A_n \times C_m$*-unramified extension where* m *is any odd integer.*

Proof. Let us again consider the following polynomial:

$$f(X) = X^n - (n-1)^k X + 1. \tag{4.1}$$

We know that the discriminant of f is of the form:

$$2^{n\ell} - (n-1)^{n-1+nk}.$$

As we have already seen in the proof of the previous theorem, $f(x)$ is irreducible and the Galois group of the splitting field K of f is isomorphic to S_n for all but finitely many choices of k. By Theorem 2.3, we also know that K is unramified over $\mathbb{Q}\left(\sqrt{2^{n\ell} - (n-1)^{n-1+nk}}\right)$. Since m is an odd integer, we can find (infinitely many) k such that $m|\left(n-1+nk\right)$. Since $n = 2^\ell$ and $n-1+nk$ is an odd integer, we know

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that $2^{n\ell} - (n-1)^{n-1+nk} \equiv 1 \pmod 8$. By Theorem 2.5, the class group of the imaginary quadratic field $\mathbb{Q}\left(\sqrt{2^{n\ell} - (n-1)^{n-1+nk}}\right)$ contains an element of order $n - 1 + nk$, and hence one of order m. □

5. Numerical examples

In this section, we will give several explicit examples of quadratic fields with $A_n \times C_m$-unramified extensions illustrating the proof of the Main Theorem. All computations in this section were carried out with Magma (see [3]). The contents of Table 1 are described as follows (with the two cases depending on the mod-4 residue of n):

n: an integer congruent to 3 (resp. 0) modulo 4,
k: an auxiliary integer,
m: the integer $m = n + k(n-1)$ (resp. $m = n - 1 + nk$),
$f(x)$: the polynomial $f(x) = x^n - x - n^k$ (resp. $f(x) = x^n - (n-1)^k x + 1$),
K: the splitting field of $f(x)$,
$-d$: the squarefree part of the discriminant of $f(x)$.

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