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PURE N-HIGH SUBGROUPS, P-ADIC TOPOLOGY AND DIRECT SUMS OF CYCLIC GROUPS

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This paper is divided into two sections. In the first, we characterize the subgroups N of a reduced abelian primary group for which all pure N-high subgroups are bounded. This condition on pure N-high subgroups occurs in several instances, for instance, all pure N-high subgroups of a primary group G are bounded if G is the smallest pure subgroup of G containing N; all N-high subgroups are bounded if $N \neq 0$ and all N-high subgroups are closed in the p-adic topology.

In the second section, we give a new characterization of direct sums of cyclic groups. This characterization allows us to construct automatically examples that show that the class of direct sum of cyclic groups is not closed under extensions, even in the case where there is no elements of infinite height.

All groups considered here are abelian and primary for a fixed prime p. The notation is that of [**6**]. For the definitions of N-high subgroups, p-adic topology, etc., we refer the reader to [**1**] or [**6**]. We use the symbol $\oplus_C$ to denote direct sum of cyclic groups. G^1 denotes the subgroup of elements of infinite height of G. Z^+ denotes the set of nonnegative integers.

1. Bounded pure S-high subgroups. Let G be a reduced p-group and let N be a subgroup of G. If one N-high subgroup H is not bounded (i.e., $p^nH \neq 0$ for all $n \in Z^+$) then H contains a proper basic subgroup B and

$$G/B = (H/B) \oplus (K/B)$$

where K can be chosen to contain N and by [**2**, Lemma 2.2], K is a pure proper subgroup of G. Thus if G is the smallest pure subgroup of G containing N, we see that all N-high subgroups must be bounded. This motivates our first result. For any p-groups G, let $G_n = (p^nG)[p]$, $n \in Z^+$. Further, since a subgroup is N-high if and only if it is $N[p]$-high, we work with subgroups of $G_0 = G[p]$.

In [**3**] we have shown that for any subgroup N of a p-group there exists pure N-high subgroups which can be chosen so as to contain any pure subgroup disjoint from N. In what follows, we consider only the family of pure N-high subgroups.

THEOREM 1.1. *Let S be a subgroup of $G[p]$ where G is a reduced primary group.*

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Then all pure N-high subgroups are bounded if and only if

$$(G_n + S)/S$$

is finite for some $n \in Z^+$.

Proof. It is a simple exercise to show that if $(G_n + S)/S$ is finite for some $n \in Z^+$, then K_n is finite for every pure N-high subgroup K, and since G is reduced, K is bounded.

Suppose now that $(G_n + S)/S$ is infinite for every $n \in Z^+$. We construct inductively a sequence of elements of $G[p]$ whose heights are strictly increasing.

Since G is reduced there exists $x_1 \in G[p]\backslash S$ such that $h(x_1) = r(1) < \infty$, otherwise $G[p] \subset G^1$ and by [**9**, Lemma 8], G is divisible. Suppose that we have found n elements $x_1, x_2, \ldots, x_n$ in $G[p]$ such that

$$h(x_i) = r(i) \quad \text{and} \quad r(i) < r(i+1) \quad i = 1, 2, \ldots, n-1,$$

and $K^n \cap S = 0$ where $K^n = \langle x_1, x_2, \ldots, x_n \rangle$.

It is easy to verify that the finiteness of K^n implies

$$(K^n + S) \cap G_{r(n)+1} \subsetneq G_{r(n)+1}.$$

thus there exists $x_{n+1} \in G_{r(n)+1}$ such that $x_{n+1} \notin K^n + S$ and

$$h(x_{n+1}) = r(n+1) < \infty$$

otherwise $G_{r(n)+1} \subset G^1$ and G would not be reduced. Now,

$$r(n+1) \geqq r(n) + 1 > r(n), \quad \text{and} \quad K^{n+1} = K^n + \langle x_{n+1} \rangle,$$

is disjoint from S. The induction step is finished.

Choose $y_i \in G$ such that $x_i = p^{r(i)} y_i$ and let $K = \langle \{y_i\}_{i=1}^{\infty} \rangle$. Then K is a pure unbounded subgroup of G (see [**7**, Lemma 4.4]) disjoint from S which can be extended to a pure unbounded S-high subgroup of G. This completes the proof.

The following corollary is straightforward.

COROLLARY. *Let S be a subgroup of $G[p]$ where G is a primary group. Then all pure S-high subgroups of G are bounded if and only if $(G_n + S)/S$ is finite for some $n \in Z^+$ and S contains the socle of the divisible subgroup of G.*

The next result is another motivation for Theorem 1.1. We need first this lemma.

LEMMA 1.2. *The intersection of the family of pure S-high subgroups of a primary group G is trivial whenever S is a nontrivial subgroup of G.*

Proof. We show that, for every $y \in K[p]$, where K is a pure S-high subgroup of G, there exists another pure S-high subgroup which does not contain y. Two cases may occur; either there exists $s \in S$ such that $h(s + y) < \infty$ or $h(s + y) = \infty$ for all $s \in S$. In the first case $s + y$ can be embedded in a pure subgroup (see the proof of Theorem 9 in [**9**]) disjoint from S which can be extended to a pure S-high subgroup H which clearly does not contain y. In the second case we have

$$h(s) = h(y), \text{ for all } s \in S$$

so that if $h(y) = \infty$, $S \subset G^1$ and all S-high subgroups of G are pure (see [**8**]) and if $h(y) < \infty$, S supports a pure subgroup M (i.e., $M[p] = S$) and $G = M \oplus R$ since M is bounded. Now then $R \supset G^1$ and thus $s + y \in R$ for all $s \in S$. Clearly R is pure S-high and does not contain y.

THEOREM 1.3. *Let S be a nontrivial subgroup of $G[p]$ where G is a primary group. Then all pure S-high subgroups of G are closed in the p-adic topology of G if and only if they are bounded and $G^1 = 0$.*

Proof. Suppose all pure S-high subgroups of G are closed. Then G^1, the closure of 0, is contained in their intersection and by Lemma 1.2., $G^1 = 0$. Suppose that some pure S-high subgroup H of G is not bounded. Then H contains a proper basic subgroup B and we have

$$(G/B)^1 \neq 0$$

and by the first observation there exists a pure $(S \oplus B)/B$-high subgroup K/B of G/B which is not closed in the p-adic topology of G/B. (i.e.,

$$((G/B)/(K/B)^1 \neq 0)$$

and thus K is not closed in the p-adic topology of G, $((G/K)^1 \neq 0)$. Now K is pure, since B is pure, and it is S-high, which is a contradiction. Therefore all pure S-high subgroups are bounded. The converse is trivial.

As a direct consequence of Theorem 1.3. we have the following

COROLLARY. *Let S be a nontrivial subgroup of $G[p]$, where G is a primary group, and A a pure subgroup of G disjoint from S. Then all pure S-high subgroups H of G containing A are closed in the p-adic topology of G if and only if $(G/A)^1 = 0$ and H/A is bounded for all H.*

Proof. This follows from the fact H is a pure S-high subgroup of G containing A if and only if H/A is a pure $(S \oplus A)/A$-high subgroup of G/A and H is closed in G if and only if H/A is closed in G/A.

We apply these results to obtain the following results, of which we prove only the first.

THEOREM 1.4. *Let G be a primary group. If there exists a nontrivial subgroup S of $G[p]$ such that $G/H = \oplus_C$ for all pure S-high subgroups H of G then $G = \oplus_C$.*

Proof. Since $G/H = \oplus_C$ for all pure S-high subgroups of G we see that all pure S-high subgroups of G are closed in the p-adic topology. Thus all such subgroups are bounded and $G = H \oplus K$ where H is a bounded direct sum of cyclic groups and $G = \oplus_C$.

THEOREM 1.5. *If G/H is bounded for all pure S-high subgroups H for some nontrivial subgroup S of $G[p]$ where G is a primary group then G is itself bounded.*

THEOREM 1.6. *Let G be a primary group. If there exists a nontrivial subgroup S of $G[p]$ such that G/H is torsion complete for every pure S-high subgroup of G then G is torsion complete.*

2. Subsocles and direct sum of cyclic groups. In this section, we establish a characterization of direct sum of cyclic groups. Our techniques give us as a by-product, a simple way to construct examples of extensions of direct sums of cyclic groups by direct sums of cyclic groups which are not themselves direct sums of cyclic groups although they have no elements of infinite height. Such an example was first given in [**4**].

We will need the following lemma and its corollary.

LEMMA 2.1. *Let G be a primary group and K a pure subgroup of G. Then*
(a) *$(G/K)[p] = (T \oplus K)/K$, where T is a complementary summand of $K[p]$ in $G[p]$;*
(b) *for every subgroup H of G such that $H[p] = K[p]$, $h_{G/H}(t + H) \geqq h_{G/K}(t + K)$ for every $t \in T$.*

Proof. (a) This is well-known.
(b) Suppose $t + K = p^n g + K$, for some $g \in G$, and $n \in Z^+$. Then

$$t - p^n g \in K, \quad \text{and} \quad p^{n+1} g \in K.$$

Since K is a pure subgroup of G, there exists $k \in K$, such that $p^{n+1}g = p^{n+1}k$. This shows that $t - p^n(g - k) \in K[p] = H[p]$. Therefore $h(t + H)$ in G/H is greater or equal to $h(t + K)$ in G/K.

COROLLARY. *In the previous lemma, $h_{G/H}(t + H) = h_{G/K}(t + K)$ in the following cases:*
(a) *H and K are both pure subgroups of G,*
(b) *H is a subgroup of K.*

Proof. (a) This follows by symmetry.
(b) If $H \subset K$, then there exists $f: G/H \rightarrow G/K$ such that $f(t + H) = t + K$ for all $t \in K$, and $h_{G/H}(t + H) \leqq h_{G/K}(t + K)$ since homomorphisms do not decrease heights. Therefore $h_{G/H}(t + H) = h_{G/K}(t + K)$.

Using the previous result, we obtain the following interesting criterion for pure subgroups of primary groups to be direct summands.

THEOREM 2.2. *Let K be a pure subgroup of a primary group G. Suppose that for some subgroup H of K containing $K[p]$, and some complementary summand T*

of $H[p]$ in $G[p]$, $(T \oplus H)/H$ supports a pure subgroup of G/H which is a direct sum of cyclic groups. Then K is a summand of G. In fact, $G/K = \oplus_C$.

Proof. Suppose $(T \oplus H)/H$ supports a pure subgroup of G/H which is a direct sum of cyclic groups. Then by [**9**, Theorem 12], we know that $(T \oplus H)/H$ is an ascending chain of subgroups of bounded heights in G/H, thus $(T \oplus K)/K$ is an ascending chain of subgroups of bounded height in G/K. But $(T \oplus K)/K = (G/K)[p]$. Therefore using [**9**, Theorem 12] we conclude that $G/K = \oplus_C$. Since K is a pure subgroup of G, [**9**, Theorem 5] implies that K is a summand of G.

Note that the hypothesis of Theorem 2.2. is satisfied in the case where $G/H = \oplus_C$ for some subgroup H of G such that $K[p] \subset H \subset K$. A special case of this gives the following result.

THEOREM 2.3. *Let G be a primary group and suppose that there exists a pure subgroup K of G such that $G/K[p^n] = \oplus_C$ for some $n \in Z^+$. Then $G = \oplus_C$.*

Proof. By Theorem 2.2., we see that $G = K \oplus A$ where $A = \oplus_C$. Now $K/K[p^n] \subset G/K[p^n] = \oplus_C$ implies that $K/K[p^n] = \oplus_C$, but this is isomorphic to $p^n K$ and by [**5**, Theorem 12.4], $K = \oplus_C$. Therefore $G = \oplus_C$.

In order to obtain the examples we have mentioned in the introduction to this section, we need the following theorem.

THEOREM 2.4. *Let F be a direct sum of cyclic groups and S a subgroup of $F[p]$. Then the following are equivalent:*

(a) *$F/S = \oplus_C$.*
(b) *F/S is pure-complete,*
(c) *S supports a summand of F.*

Proof. (a) $\Rightarrow$ (b). It is a known fact that in a direct sum of cyclic groups every subsocle supports a pure subgroup, i.e., is pure-complete.

(b) $\Rightarrow$ (a). If F/S is pure complete, then $F[p]/S$ supports a pure subgroup of F/S. But $(F/S)/(F[p]/S) \simeq F/F(p) = \oplus_C$ and by Theorem 2.3., $F/S = \oplus_C$.

(a) $\Rightarrow$ (c). Since F is pure-complete, there exists a pure subgroup K of F such that $K[p] = S$ and by Theorem 2.2., K is a summand of F.

(c) $\Rightarrow$ (a). If K is a summand of F then $F = K \oplus A$ and

$$F/K[p] = (K/K[p]) \oplus ((A \oplus K[p])/K[p]) = \oplus_C.$$

We now apply the preceeding results to obtain quite a general method of constructing extensions of direct sums of cyclic groups by direct sums of cyclic groups which are not themselves direct sums of cyclic groups and which have no elements of infinite height.

It is known that any primary group G is isomorphic to F/K where F is a direct sum of cyclic primary groups and K is a pure subgroup of F. (i.e., the direct sums of cyclic p-groups are the pure projectives in the category of

p-groups). Choose G a group which is not a direct sum of cyclic groups, for example

$$G = t\left(\prod_{n=1}^{\infty} C(p^n)\right).$$

Then $F/K[p]$ is not a direct sum of cyclic groups, as follows from Theorem 2.4, but in the following exact sequence,

$$0 \rightarrow F[p]/K[p] \rightarrow F/K[p] \rightarrow F/F[p] \rightarrow 0,$$

the first group and the last are easily seen to be direct sums of cyclic groups.

Note also that $F/K[p]$ is an example of a group without elements of infinite height which is not pure complete because $F[p]/K[p]$ does not support a pure subgroup.

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