as is often the case in an economic context, the methods of calculus, analysis and even topology are appropriate [6]. Returning to "pure" Arrow's theorem, there are alternative approaches and proofs: Roberts' treatment in [7] is graph theoretic, and in [8] he points to connections with graph theory, semi-orders and indifference relations. In [9] the individual rankings are points in a metric space and the election is decided by minimising a certain distance in this space. [2] contains a version of Arrow's theorem in terms of Boolean matrices. Finally, a totally elementary and entertaining account of the shortcomings of several "reasonable" ways of conducting elections is given in [10], using nothing more sophisticated than arithmetic. Most of these books contain references to more detailed papers.

References


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A-level mathematics in an information age

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Mathematics and the information society

Since the time of Gutenberg the volume of printed information has been doubling roughly every fifteen years. Only relatively recently, however, has the growth of knowledge reached a point at which it challenges human intellectual capacity. Our society has responded by a progressive specialisation of intellectual roles. Very broadly, the nineteenth century saw the development of specialism in distinct fields; the twentieth, the growth of...
specialism within those fields. To take just mathematics, it has been estimated that, after allowing for redundancy, our present knowledge fills the equivalent of something of the order of 60 thousand average-sized volumes. While at the start of this century, according to Ostrowski, it was expected that a mathematics graduate would be prepared to deal with any question in any branch of mathematics, by mid century Von Neumann was estimating that a skilled mathematician might be familiar with some 10% of the whole field.

Over the course of this century we have been moving towards what has been called an information society. This shift is signalled by the increasing importance in our society of technologies and practices which are based on theorised knowledge, and the emergence of a major sector of the economy directly concerned with the generation and exploitation of information and knowledge. The ideas and techniques of mathematics play a central role in many of these knowledge-based activities, bringing order, structure and precision to complex situations. As a result, not only have mathematics specialists become involved in a wider range of activities, but others involved in these activities require a greater appreciation of mathematics.

For those involved in knowledge-based activities there is a growing need for a recurrent, if not continuous updating of personal expertise primarily through contact with others in the field, the reading of appropriate publications, and continuing education and training. Such work increasingly takes place in teams to which each member brings a different expertise, and in which the members must collaborate effectively to produce results. Making effective use of the growing store of published information requires more sophisticated techniques for locating and exploiting relevant information, notably those based on new technologies. This style of working heightens the importance of learning, communication, collaboration, and information skills.

Recent years have seen the development of a new information technology, built around the computer, which is set to play a central role in our society. Although computer technology has only been widely available for less than one working lifetime, it has already had a significant influence on mathematics. It has given rise to new areas of mathematics, directly concerned with automata and algorithms; their scope, reliability, efficiency, and accuracy. The computational power of the computer has greatly increased the accessibility of certain "real" problems to mathematical analysis, stimulating the growth of new areas of applied mathematics, notably statistics and operational research. Moreover, the power of the computer has encouraged the development of algorithmic and computational approaches within traditional areas of mathematics.

The computer also promises to emancipate mathematicians from many routine tasks. We can look forward, in the relatively near future, to hand-held machines capable of a range of symbolic manipulation and graphical display, as well as calculation. Their impact is likely to be even
greater than that of the calculator. In the longer term, the influence of the computer may be even more profound. An experimental mode of research, where the computer is used to generate and examine possibilities, is becoming increasingly common among mathematicians. Moreover, it has been suggested that familiarity with the computer is encouraging a shift in the interests of mathematicians away from non constructive results.

The curriculum of the academic sixth form

The central issue for the curriculum is no longer whether specialisation is desirable; it is when it should begin. Within the present English educational system, it is in the sixth form that the academic curriculum narrows in terms of the number and range of subjects studied at examination level. It would be mistaken, however, to imagine that specialisation is simply a matter of the number, or the range of subjects that a student studies. Long before the sixth form, even within a broader curriculum framework, individual subject curricula tend to become specialist in their aims and outlook. This is a point which is often missed by those who argue for an increase in the number of examination subjects in the academic sixth form. Such a change would simply broaden the range of specialist studies. Perhaps a more fundamental problem is that the lack of breadth within specialist studies in the sixth form.

According to a recent HMI report, sixth form mathematics courses often amount to little more than training in a sequence of techniques, each illustrated by a limited range of stereotyped exercises. The majority of sixth form mathematics lessons were “mainly instructional in character, made only limited provision for the interchange of ideas with students and offered insufficient opportunity for students to gain more than restricted view of the subject. Teaching in these circumstances was highly predictable and tailored to the needs of what had become a passive and uncritical audience.” The fundamental problem is the narrow instrumentalism of the present curriculum. Not only is A-level mathematics seen as a pre-university course, intended largely as a preparation for further study of mathematics, engineering or the physical sciences. That preparation is seen primarily in terms of mastery of a body of mathematical technique.

To view A-level in this instrumental way is no longer consistent with the reality of the academic sixth form. It ignores the many A-level mathematics students who do not continue into higher education, and fails to recognise the diversity of courses entered by those who do. Nor is the present rationale consistent with the idea of a broad curriculum. The breadth of the sixth form curriculum is as much related to the specialist’s view of his own subject, as to his knowledge of others. Particularly important in this respect is the ability to take an overview of mathematics; to understand the nature and purpose of mathematical activity, and the structure of beliefs and values which lies behind it; to be aware of the evolution of mathematical knowledge and of its wider intellectual and social context.
The central weakness of the present rationale, however, is its preoccupation with the transmission of an established body of knowledge. The time has passed when any mathematician could expect to be familiar with the whole range of mathematics. Yet the desire to cover an ever growing field remains a powerful influence on the design of curricula. Twenty five years ago, the influential Royaumont Seminar argued that “the only solution (to the ‘squeeze’ on the mathematics curriculum) is for the secondary school to take on some of the burden now resting on the university”. “Secondary school mathematics must be intensified.” In retrospect we can see that the “new mathematics” was based on a fallacious synoptic principle; the belief that the concepts developed to produce a unifying logical structure for mathematics could also provide a more powerful psychological structure for the learning of mathematics. In practice, while the curriculum was indeed intensified, this was at the cost of a superficial and often excessively abstract treatment.

The development of intellectual capability

Not only have we failed to find a suitable synoptic principle for the intensification of school mathematics. A curriculum primarily concerned with the transmission of knowledge appears less than adequate in an information society with its new patterns of intellectual activity. Enquiry, problem solving, communication, information and study skills are sometimes seen as general and widely transferable. But this is only superficially so. In formulating and solving problems in mathematics, for example, we make use of schemata and strategies which are deeply embedded in our knowledge of the subject area. In fact, most of our heuristic knowledge of mathematical enquiry is tacit; built on our experience and our unconscious systematisation of that experience.

In recent years a number of curricula have attempted to give sixth form students a more authentic experience of mathematical enquiry and problem solving. But in most cases there is something rather artificial about the way in which the mathematical process is subsidiary to the exposition of particular mathematical topics. The investigations and models used often seem “whimsical”; chosen to illustrate the topic rather than to provide an authentic experience of the process of enquiry. At the same time, many students have difficulty in working in a non-routine way with concepts which they are still in the process of acquiring. The implication of these observations is that experience designed to allow students to engage in authentic mathematical activity must be accessible in terms of the mathematical knowledge with which students are already confident.

Like enquiry and problem solving, effective communication is often tied to contextual knowledge and experience. In the case of mathematics it includes facility in employing a precise formal language and symbolism, and appropriate informal and visual languages. Even some of the information
and study skills required in mathematics are distinctive. As the Cockcroft Report points out, reading a typical piece of mathematics calls for close attention to detail, and the use of pen and paper to follow and flesh out the bare statements of the text.

Encouraging the development of such skills within the mathematics curriculum would entail the use of a much wider range of learning and assessment tasks. Students might be asked to solve a "real" problem, to carry out an investigation, to evaluate a number of solutions to a problem, or to find out about a particular area of mathematics; to produce a written report of their work, to draw up a brief summary of it, or to give a talk on it; to work as a member of a group or as an individual.

The corollary of the adoption of a more varied style of working and assessment at A-level is some diminution in the coverage of specified mathematical content. The Cockcroft Report has already endorsed such a principle for courses at an earlier stage of secondary education. It is equally valid in the sixth form. In the past, institutions of higher education have shown themselves quite capable of accommodating the diversity of mathematics syllabuses at A-level. There is no reason why they should not do even better with students who have covered a reduced but more standard content at A-level, and demonstrated greater intellectual independence and versatility.

The impact and role of the new technology

The development of computer-based technologies of graphing and symbolic manipulation will make a number of complex and previously highly valued mathematical skills redundant. In particular, the importance of facility in a wide range of symbolic manipulations (such as the factorisation, expansion and simplification of algebraic and trigonometric expressions, and symbolic differentiation and integration) will diminish. No doubt the process of adaptation will be dogged by the same kind of controversy which accompanied the introduction of the calculator.

These new technologies free the user from the need to carry out complex and time-consuming routines, and allow him to focus his attention on the structure of a problem. Such systems may also help to promote a more sophisticated level of conceptual understanding. My own experience is that calculators and graphing packages both facilitate exploratory play and serve as sources of imagery. Their design also reflects, and hence transmits, certain aspects of an established mathematical culture. The graphics commands of BBC BASIC, for example, are based on the coordinate and vector geometry of the plane: if only they extended to matrix transformations!

This raises the vexed question of whether experience of programming should form part of the mathematics curriculum. It is sometimes claimed that the experience of programming in certain languages helps students to
develop capability in problem solving. Given the difficulty of defining the nature of problem solving, let alone the processes by which it is accomplished, my claim would be more modest. It is that some programming languages employ structures which might usefully be assimilated to the student's repertoire of problem solving concepts. But to argue a deeper significance for structured programming and modular design is to take a highly idealised view of the problem solving process. Programming languages offer a set of useful conventions. By and large, the structure of a (well written) program bears little resemblance to the problem solving process which generated it, as does the formal statement of a proof: it is no less a matter of expressing ideas within a conventional structure.

My own argument for the value of programming would be quite different. The computer, like the calculator, enables more realistic problems to be tackled and offers new approaches to solving them. Moreover, in designing and implementing a program the student is constructing in the first instance a purely symbolic representation of a mathematical idea (as program code), but one in which the symbolism creates and corresponds directly to an active, and sometimes visual representation (in execution). In carrying out the task, the student moves between these modes, using information gained in the active mode to guide work in the symbolic mode. In interactive computer systems this movement is particularly easy, allowing the student control of his access to feedback. In short, programming exploits to the full the particular advantages of the cognitive environment provided by the computer. Present computer languages are relatively rich in algorithmic, numerical, and probabilistic structures and it is these areas that currently offer the most promising fields for a programming approach. For example, the use of the computer as a tool for simulating random processes offers the possibility of a less combinatorial approach to probability, linking empirical and theoretical perspectives.

There is one other way in which computers may enrich the curriculum. There is evidence that work with computers, particularly group work, can stimulate the kind of discussion which is so lacking in present sixth form mathematics classrooms. It offers a context for the development of capabilities in communication and collaboration. In addition, the computer appears to facilitate the adoption of new roles and new styles of working by both students and teachers. These may of course be transitional effects. But for the present they suggest that reform of the sixth form curriculum may be very closely related to exploitation of the new technology.

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