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DIRECT PRODUCTS AND PROPERLY 3-REALISABLE GROUPS

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In this paper, we show that the direct product of infinite finitely presented groups is always properly 3-realisable. We also show that classical hyperbolic groups are properly 3-realisable. We recall that a finitely presented group G is said to be properly 3-realisable if there exists a compact 2-polyhedron K with $\pi_1(K) \cong G$ and whose universal cover $\widetilde{K}$ has the proper homotopy type of a (p.l.) 3-manifold with boundary. The question whether or not every finitely presented is properly 3-realisable remains open.

1. Introduction

The following question was formulated in [**9**] for an arbitrary finitely presented group G: *does there exist a compact 2-polyhedron K with $\pi_1(K) \cong G$ and whose universal cover $\widetilde{K}$ is proper homotopy equivalent to a 3-manifold*? If so, the group G is said to be *properly 3-realisable*. It is known that the proper homotopy type of any locally finite 2-dimensional CW-complex can be represented by a subpolyhedron in $\mathbf{R}^4$ (see [**4**]), thus $\widetilde{K}$ would always be proper homotopy equivalent to a 4-manifold. The question of whether or not every finitely presented G group is properly 3-realisable still remains open. In case of a positive answer, this property would allow us to use duality arguments in the study of certain low-dimensional ((co)homological) proper invariants of the group G, see [**9**]. There are several results in the literature regarding the proper 3-realisability question for finitely presented groups (see [**1, 5, 9, 10**]). See also [**6**] for a survey on this question. In this paper, we prove the following.

Theorem 1.1. *If G and H are infinite finitely presented groups, then the direct product $G \times H$ is properly 3-realisable.*

Observe that if G is properly 3-realisable and H is finite, then the direct product $G \times H$ has a copy of G as a subgroup of finite index and hence it is properly 3-realisable, by ([**1**, Theorem 1.1]).

Corollary 1.2. *Every finitely generated Abelian group is properly 3-realisable.*

The techniques used in the Proof of Theorem 1.1 also yield the following.

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Proposition 1.3. *If G is the fundamental group of a manifold which can be covered by an Euclidean space, then G is properly 3-realisable.*

As an example, we have that all "classical" hyperbolic groups (that is, the fundamental group of a closed Riemannian manifold with negative sectional curvature) are properly 3-realisable.

2. Preliminaries

In order to prove Theorem 1.1, we first need some preliminaries from proper homotopy theory. In what follows, we shall be working within the category tow – Gr of *towers of groups* whose objects are inverse sequences of groups

$$\underline{A} = \{A_0 \xleftarrow{\phi_1} A_1 \xleftarrow{\phi_2} A_2 \longleftarrow \cdots\}$$

A morphism in this category will be called a *pro-morphism*. See [**2, 11**] for a general reference.

A tower $\underline{L}$ is a *free tower* if it is of the form

$$\underline{L} = \{L_0 \xleftarrow{i_1} L_1 \xleftarrow{i_2} L_2 \longleftarrow \cdots\}$$

where $L_i = \langle B_i \rangle$ are free groups of basis B_i such that $B_{i+1} \subset B_i$, the differences $B_i - B_{i+1}$ are finite and $\bigcap_{i=0}^{\infty} B_i = \emptyset$, and the bonding homomorphisms i_k are given by the corresponding basis inclusions. On the other hand, a tower $\underline{P}$ is a *telescopic tower* if it is of the form

$$\underline{P} = \{P_0 \xleftarrow{p_1} P_1 \xleftarrow{p_2} P_2 \longleftarrow \cdots\}$$

where $P_i = \langle D_i \rangle$ are free groups of basis D_i such that $D_{i-1} \subset D_i$, the differences We shall also use the full subcategory (Gr, tow – Gr) of Mor(tow – Gr) whose objects are arrows $\underline{A} \longrightarrow G$, where $\underline{A}$ is an object in tow – Gr and G is a group regarded as a constant tower whose bonding maps are the identity. Morphisms in (Gr, tow – Gr) will also be called pro-morphisms.

From now on, X will be a (strongly) locally finite CW-complex. A proper map $\omega : [0,\infty) \longrightarrow X$ is called a *proper ray* in X. We say that two proper rays ω, ω' *define the same end* if their restrictions $\omega|_{\mathbf{N}}, \omega'|_{\mathbf{N}}$ are properly homotopic. Moreover, we say that they *define the same strong end* if ω and ω' are in fact properly homotopic.

Given a base ray ω in X and a collection of compact subsets $C_1 \subset C_2 \subset \cdots \subset X$ so that $X = \bigcup_{n=1}^{\infty} C_n$, the following tower

$$\text{pro}-\pi_1(X,\omega) = \Big\{\pi_1\big(X,\omega(0)\big) \leftarrow \pi_1\big(X - C_1, \omega(t_1)\big) \leftarrow \pi_1\big(X - C_2, \omega(t_2)\big) \leftarrow \cdots\Big\}$$

can be regarded as an object in (Gr, tow – Gr) and it is called the *fundamental pro-group of* (X,ω), where $\omega\big([t_i,\infty)\big) \subset X - C_i$ and the bonding homomorphisms are induced by the inclusions. This tower does not depend (up to pro-isomorphism) on the sequence of subsets $\{C_i\}_i$. It is worth mentioning that if ω and ω' define the same strong end, then pro $-\pi_1(X,\omega)$ and pro $-\pi_1(X,\omega')$ are pro-isomorphic. In particular, we may always assume that ω is a cellular map. Moreover, if X is strongly connected at each end (that is, any two proper rays defining the same end define the same strong end), then $\pi_1^e(X,\omega) = \varprojlim$ pro $-\pi_1(X,\omega)$ is a well-defined useful invariant which only depends (up to isomorphism) on the end determined by ω (see [**8**]). In a similar way, one can define objects in (Gr, tow − Gr) corresponding to the higher homotopy pro-groups of (X,ω).

DEFINITION 2.1: Given $n \geqslant 1$, a tree T and a proper ray $\omega : [0,\infty) \longrightarrow T$, a *spherical object* S^n_ω *under* T is a space obtained from T by attaching finitely n-spheres S^n at each vertex of $\omega([0,\infty))$. Observe that any two of such spherical objects (along ω) are proper homotopy equivalent (under T), by ([**2**, Propisition 4.5(b)]).

The following result, which characterises those one-ended 2-dimensional proper co-H-spaces, will be crucial for the Proof of Theorem 1.1.

THEOREM 2.2. [**7**, Corollary 6.4]. *If X is a one-ended 2-dimensional locally finite CW-complex, then the following are equivalent*

(a) pro $-\pi_1(X,\omega)$ *is pro-isomorphic to a (coproduct) tower of the form* $\underline{L} \vee \underline{P}$.

(b) *There exist spherical objects S^2_ω and $S^2_{\omega'}$ and a proper homotopy equivalence (under $[0,\infty)$) $X \vee S^2_\omega \simeq B(\underline{L} \vee \underline{P}) \vee S^2_{\omega'}$.*

Here, $(B(\underline{L} \vee \underline{P}),\omega')$ is the properly based 2-polyhedron defined as the proper wedge (that is, along a base ray) of a one-ended spherical object S^1_ε, with pro $-\pi_1(S^1_\varepsilon,\omega') \cong \underline{L}$ *($\omega' : [0,\infty) \hookrightarrow S^1_\varepsilon$ the canonical inclusion), and a proper wedge C of a decreasing sequence (possibly infinite) of cylinders $C_n = S^1 \times [n,\infty)$ and/or Euclidean planes $\mathbf{R}^2_m = S^1 \times [m,\infty)/S^1 \times \{m\}$ attached along the half line $[0,\infty)$ for which* pro $-\pi_1(C,\omega') \cong \underline{P}$, *with $\omega' : [0,\infty) \hookrightarrow C$ the canonical inclusion. Thus, $B(\underline{L} \vee \underline{P})$ can be seen as a "proper Eilenberg-MacLane space" $K(\underline{L} \vee \underline{P}, 1)$.*

3. DIRECT PRODUCTS

The purpose of this section is to prove Theorem 1.1 and Proposition 1.3. First, we shall roughly outline the generalised van Kampen theorem in a naive way (see [**3**, **12**] for a proof using groupoids). For simplicity, we shall not take care of base points in what follows (see [**14**] for details).

Let X_0, X_1, X_2 be subcomplexes of a CW-complex X so that X_1, X_2 are connected and satisfy $X = X_1 \cup X_2$, $X_1 \cap X_2 = X_0$. Suppose X_0 is not connected, say it has two connected components Y and Z. Let $\widetilde{Z}$ denote the CW-complex obtained by identifying a copy of X_1 with a copy of X_2 along Z, and let $\widetilde{X}$ denote the CW-complex (homotopy equivalent to X) obtained from $Y \times I$ and $\widetilde{Z}$ by identifying $Y \times \{i\}$ to the copy Y_{i+1} of Y in X_{i+1}, $i = 0,1$. Then, one can check that we have the following push-out diagrams

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in the category of groups:

$$\begin{array}{ccc} \pi_1(Z) & \longrightarrow & \pi_1(X_1) \\ \downarrow & & \downarrow \\ \pi_1(X_2) & \longrightarrow & \pi_1(\widetilde{Z}) \end{array} \qquad \begin{array}{ccc} \pi_1(Y) * \pi_1(Y) & \xrightarrow{\varphi} & \langle t\rangle * \pi_1(\widetilde{Z}) \\ \downarrow & & \downarrow \\ \pi_1(Y) & \longrightarrow & \pi_1(\widetilde{X}) \end{array}$$

where t is represented by a loop $\big(\{y_0\} \times I\big) \cup \gamma$, with $y_0 \in Y$ and γ a path in $\widetilde{Z}$ from $(y_0, 0)$ to $(y_0, 1)$; and φ is given by $\theta \mapsto \theta$ on the first factor, and by $\theta \mapsto t\theta t^{-1}$ on the second factor. From now on, we shall denote $\pi_1(X_1)\widehat{*}_{\pi_1(Z)}\pi_1(X_2) \equiv \pi_1(\widetilde{Z})$ and $\pi_1(\widetilde{Z})\widehat{*}_{\pi_1(Y)} \equiv \pi_1(\widetilde{X}) \cong \pi_1(X)$ the corresponding fundamental groups obtained by the process described above.

Proposition 3.1. *Let X and Y be locally finite, simply connected non-compact CW-complexes. Then,* pro $-\pi_1(X \times Y)$ *is pro-isomorphic to a telescopic tower* $\underline{P}$.

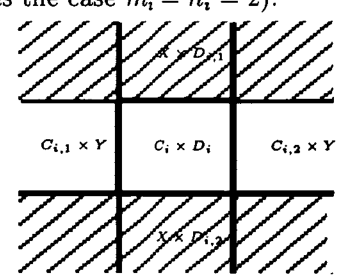
Corollary 3.2. *With X and Y as above, we have* $\varprojlim^i$ pro $-\pi_1(X \times Y) = \{1\}$.

Note that Mihalik [**13**] already showed that the product $X \times Y$ of locally finite, connected non-compact CW-complexes is semistable at ∞.

Proof of Proposition 3.1: Let X and Y be locally finite, simply connected non-compact CW-complexes. In [**8**], the computation of pro $-\pi_1(X \times Y)$ is done in detail for $Y = \mathbf{R}$ (and X not necessarily simply connected). The computations in the general case we are concerned with are similar to those in [**8**], so we shall not take care of base rays or base points in what follows, for simplicity. Notice that $X \times Y$ is strongly connected at infinity, that is, it only has one strong end (see [**13**]).

Let $C_1 \subset C_2 \subset \cdots \subset X$ and $D_1 \subset D_2 \subset \cdots \subset Y$ be sequences of compact subsets with $X = \bigcup_{i=1}^{\infty} C_i$ and $Y = \bigcup_{i=1}^{\infty} D_i$, and so that $X - C_i = C_{i,1} \cup \cdots \cup C_{i,m_i}$ and $Y - D_i = D_{i,1} \cup \cdots \cup D_{i,n_i}$ are the corresponding collections of connected components satisfying $C_{i+1,1} \subset C_{i,1}$, for all i, and $C_{i+1,2} \subset C_{i,2}$ if $m_i \geqslant 2$.

Consider $U_i = (X \times Y) - (C_i \times D_i) = \Big(\bigcup_{j=1}^{n_i} X \times D_{i,j}\Big) \cup \Big(\bigcup_{j=1}^{m_i} C_{i,j} \times Y\Big)$, $i \geqslant 1$. We wish to compute $\pi_1(U_i)$ as well as the bonding homomorphism $\pi_1(U_{i+1}) \longrightarrow \pi_1(U_i)$ induced by inclusion. By the generalised van Kampen theorem, $\pi_1(U_i)$ can be expressed as follows (the picture roughly describes the case $m_i = n_i = 2$):



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$$\Big(\Big\{\big(\pi_1(X \times D_{i,1})\widehat{*}_{\pi_1(C_{i,1}\times D_{i,1})}\pi_1(C_{i,1} \times Y)\big)\cdots\widehat{*}_{\pi_1(C_{i,m_i}\times D_{i,1})}\pi_1(C_{i,m_i} \times Y)$$

$$\widehat{*}_{\pi_1(C_{i,1}\times D_{i,2})}\pi_1(X \times D_{i,2})\cdots\widehat{*}_{\pi_1(C_{i,1}\times D_{i,n_i})}\pi_1(X \times D_{i,n_i})\Big\}\widehat{*}_{\pi_1(C_{i,2}\times D_{i,2})}\Big)$$

$$\cdots\widehat{*}_{\pi_1(C_{i,2}\times D_{i,n_i})} \ldots \widehat{*}_{\pi_1(C_{i,m_i}\times D_{i,2})}\cdots\widehat{*}_{\pi_1(C_{i,m_i}\times D_{i,n_i})}$$

Moreover, if we take $P_i = F\big(\{t_{i,j,k}, 2 \leqslant j \leqslant m_i, 2 \leqslant k \leqslant n_i\}\big)$ (here, $F(A)$ stands for the free group on the set A), then there are homomorphisms $\alpha_i : P_i \longrightarrow \pi_1(U_i)$ and commutative diagrams

$$\begin{array}{ccc} P_i & \xleftarrow{\ \beta_{i+1}\ } & P_{i+1} \\ \downarrow{\scriptstyle \alpha_i} & & \downarrow{\scriptstyle \alpha_{i+1}} \\ \pi_1(U_i) & \longleftarrow & \pi_1(U_{i+1}) \end{array}$$

where:

(i) α_i maps each $t_{i,j,k}$ to the new generator added when considering the corresponding push-out $(\ldots)\widehat{*}_{\pi_1(C_{i,j}\times D_{i,k})}$ in the expression of $\pi_1(U_i)$ given above.

(ii) $\beta_{i+1}(t_{i+1,j,k}) = t_{i,j',k'}$ whenever $C_{i+1,j} \subset C_{i,j'}$ $(j' \geqslant 2)$ and $D_{i+1,k} \subset D_{i,k'}$ $(k' \geqslant 2)$.

(iii) $\beta_{i+1}(t_{i+1,j,k}) = 1$ if $C_{i+1,j} \subset C_{i,1}$.

Observe that the group inside the curly brackets in the expression of $\pi_1(U_i)$ given above is the trivial group, since X and Y are both simply connected and hence the group homomorphisms involved in the corresponding push-out diagrams can be regarded as induced by the corresponding projections $C_{i,j} \times D_{i,k} \longrightarrow C_{i,j}$ and $C_{i,j} \times D_{i,k} \longrightarrow D_{i,k}$. Moreover, it is not hard to check that each homomorphism $\alpha_i : P_i \longrightarrow \pi_1(U_i)$ is in fact an isomorphism, with the above considerations.

Finally, after an appropriate change of basis for the free groups P_i (in terms of the original generators $t_{i,j,k}$), one can see that the tower

$$\underline{P} = \big\{\{1\} \longleftarrow P_1 \xleftarrow{\beta_2} P_2 \xleftarrow{\beta_3} P_3 \longleftarrow \cdots\big\}$$

can be regarded as a telescopic tower, and the conclusion of the proposition follows. ∎

Proof of Theorem 1.1: Let G and H be infinite finitely presented groups, and let X and Y be compact 2-polyhedra with $\pi_1(X) \cong G$ and $\pi_1(Y) \cong H$. Let $\widetilde{X}$ and $\widetilde{Y}$ denote the universal covers of X and Y respectively. Observe that $\widetilde{X}$ and $\widetilde{Y}$ are non-compact polyhedra, since G and H are infinite. It is clear that $\pi_1(X \times Y) \cong G \times H$ and $\widetilde{X} \times \widetilde{Y}$ is the universal cover of $X \times Y$. Let $p : \widetilde{X} \times \widetilde{Y} \longrightarrow X \times Y$ be the universal covering projection and let W denote the 2-skeleton of $X \times Y$. Then, $\pi_1(W) \cong G \times H$ and $\widetilde{W} = p^{-1}(W) \subset \widetilde{X} \times \widetilde{Y}$ is the universal cover of W, with pro $-\pi_1(\widetilde{W}) \equiv$ pro $-\pi_1(\widetilde{X} \times \widetilde{Y})$ which is pro-isomorphic to a telescopic tower $\underline{P}$, by Proposition 3.1. Pick a base ray ω in

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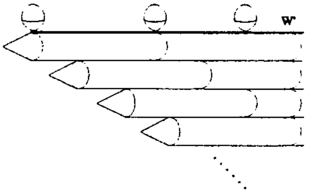
$\widetilde{W}$. Since $\widetilde{W}$ is 2-dimensional and (strongly) one-ended, there exist spherical objects S^2_ω and $S^2_{\omega'}$ and a proper homotopy equivalence $\widetilde{W} \vee S^2_\omega \simeq B(\underline{P}) \vee S^2_{\omega'}$, by Theorem 2.2. Let $V \subset \widetilde{W}$ be the set of vertices in $\omega([0,\infty))$, with $p(V) = \{v_1, \dots, v_r\} \subset W$, and denote by $\widehat{W}$ the polyhedron obtained from $\widetilde{W} \vee S^2_\omega$ by attaching one sphere S^2 through every vertex in $p^{-1}(p(V)) - \omega([0,\infty))$. Thus, $\widehat{W}$ is the universal cover of the compact 2-polyhedron obtained from W by attaching one sphere S^2 at each of the vertices $v_1, \dots, v_r$ (which is homotopy equivalent to a wedge $W \vee \left(\bigvee_{i=1}^{r} S^2\right)$). On the other hand, $\widehat{W}$ is proper homotopy equivalent to a polyhedron Q obtained from $B(\underline{P}) \vee S^2_{\omega'}$ by attaching infinitely many spheres S^2 in a proper way (that is, via the corresponding proper homotopy equivalence given by Theorem 2.2). Finally, the proper homotopy type of the proper wedge $B(\underline{P}) \vee S^2_{\omega'}$ can be represented by the closed subpolyhedron in $\mathbf{R}^3$ shown in the figure below. It is then easy to check that the proper homotopy type of Q can also be represented by a closed subpolyhedron $\widehat{Q}$ in $\mathbf{R}^3$.



Therefore, the universal cover of the compact 2-polyhedron $W \vee \left(\bigvee_{i=1}^{r} S^2\right)$ (with $\pi_1(W \vee \left(\bigvee_{i=1}^{r} S^2\right)) \cong G \times H$) turns out to be proper homotopy equivalent to the 3-manifold obtained by taking a regular neighbourhood of $\widehat{Q}$ in $\mathbf{R}^3$, and the conclusion of the theorem follows. ∎

Proof of Proposition 1.3: Let G be the fundamental group of an n-manifold M whose universal cover $\widetilde{M}$ can be identified with the Euclidean space $\mathbf{R}^n$. Thus, pro $-\pi_1(\widetilde{M})$ is clearly pro-isomorphic to a telescopic tower. Therefore, using an argument similar to that of Theorem 1.1, there is a finite wedge $\bigvee_{i \in I} S^2$ so that the universal cover of $W \vee \left(\bigvee_{i \in I} S^2\right)$ is proper homotopy equivalent to a 3-manifold, where W is the 2-skeleton of M. ∎

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