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# SCREENING PROPERTIES OF THE SUBBASE OF ALL CLOSED CONNECTED SUBSETS OF A CONNECTEDLY GENERATED SPACE

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**1. Introduction.** In [**1**] de Groot has introduced the notation "connectedly generated" (or cg) for those spaces in which the closed connected sets form a subbase for the topology. He pointed out that these are the semi-locally connected spaces of Whyburn. See [**5**; **6**].

If $X$ is cg, then, since $X$ is closed, $X$ is the union of a finite number of closed connected sets and, thus, has only a finite number of components. If $p$ is any point in a cg space, and $N_p$ is any neighbourhood of $p$, then the complement of $N_p$ may be covered by a finite number of closed connected sets, none of which contain $p$.

In [**1**] and in [**2**] the concept of "screening" is introduced and shown to be usefully related to local connectedness and construction of compactifications for completely regular spaces. We review this concept in § 2.

In this paper we confine our attention to the subbase $\mathscr{C}$ of closed connected sets, and the base $\mathscr{B}$ of finite unions of members of $\mathscr{C}$ in a cg space. Using the definitions in [**2**], we show, in § 3, that subbase-regularity, with respect to $\mathscr{C}$, is equivalent to local connectedness for regular, cg spaces. In § 4, we show that, for a cg space $X$, subbase-normality with respect to $\mathscr{C}$ is equivalent to "locally connected and normal with respect to closed, connected sets". Also we give two other equivalent screening properties, one basic and the other subbasic, and an example which shows that (sub)base-regularity is not equivalent to (sub)base-normality for $(\mathscr{C})\mathscr{B}$.

The closure of a set $A$ is denoted by $\bar{A}$, and its boundary by $B(A)$. The notation $A\backslash B$ stands for $\{x: x \in A \text{ and } x \notin B\}$.

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**2. Screening.** If $\mathscr{A}$ is any collection of subsets of a set $X$, and if $B$ and $C$ are any two subsets of $X$ which are disjoint, then we say "$\mathscr{A}$ screens $B$ and $C$" if there exists a finite collection of sets in $\mathscr{A}$ such that their union is $X$, and no member of the collection intersects both $B$ and $C$. We let $\mathscr{A}(B, C)$ stand for the smallest integer $n$ such that there exist $A_1, \ldots, A_n \in \mathscr{A}$ and

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$\{A_1, \ldots, A_n\}$ screens $B$ and $C$. In this case we call $\{A_1, \ldots, A_n\}$ a "clean screen" of $B$ and $C$. (We often drop the brackets at the ends of a list of members from $\mathscr{A}$ screening $B$ and $C$.) If $\mathscr{A}$ fails to screen $B$ and $C$, then we let $\mathscr{A}(B, C) = 1$. We can extend this notation as follows. If $\mathscr{F}$ and $\mathscr{G}$ are two collections of subsets of $X$, we let $\mathscr{A}(\mathscr{F}, \mathscr{G})$ stand for

$$\min\{\mathscr{A}(F, G) \mid F \in \mathscr{F}, G \in \mathscr{G}, F \cap G = \emptyset\}.$$

Then, if $X$ is allowed to stand for the collection of one point subsets of itself, we have:

(1) Base-regularity with respect to $\mathscr{B}$ is equivalent to $\mathscr{B}(\mathscr{B}, X) = 2$,
(2) Base-normality with respect to $\mathscr{B}$ is equivalent to $\mathscr{B}(\mathscr{B}, \mathscr{B}) = 2$,
(3) Subbase-regularity with respect to $\mathscr{C}$ is equivalent to $\mathscr{C}(\mathscr{C}, X) \geqq 2$,
(4) Subbase-normality with respect to $\mathscr{C}$ is equivalent to $\mathscr{C}(\mathscr{C}, \mathscr{C}) \geqq 2$.

It is easy to see that (1) implies (3) and (2) implies (4). If $X$ is a Hausdorff space, then (2) implies (1) and (4) implies (3).

**3. Equivalence of regularity.** Subbase- and base-regularity and normality will always refer to $\mathscr{C}$ and $\mathscr{B}$ as given in the introduction.

3.1. THEOREM. *If $X$ is Hausdorff and cg, then the following conditions on $X$ are equivalent:*

(i) *$X$ is regular and locally connected,*
(ii) *$X$ is base-regular with respect to $\mathscr{B}$,*
(iii) *$X$ is subbase-regular with respect to $\mathscr{C}$.*

In [**1**], (i) and (ii) were shown to be equivalent. Since (ii) implies (iii) and (iii) clearly implies that $X$ is regular, it need only be shown that (iii) implies $X$ is locally connected. This involves three lemmas.

The following lemma is an obvious generalization of [**5**, 6.1].

3.2. LEMMA. *In a Hausdorff cg space $X$, the complement of every point $p$ is a topological union of connected sets each of which is the union of all closed connected sets containing some fixed point but not $p$.*

*Proof.* If $p$ is a point of $X$ and $q \neq p$, let $K_q$ be the union of all those closed connected sets in $X$ which contain $q$ but not $p$. Clearly the $K_q$s partition $X \backslash \{p\}$. It need only be shown that $K_q$ is open. If $x \in K_q$ let O be a neighbourhood of $p$ such that $x \notin \bar{\mathrm{O}}$.

Since $X$ is cg, $X \backslash \mathrm{O}$ can be covered by a finite number of closed connected sets none of which contain $p$. Since any two of these sets which intersect can be replaced by their union, there exists a disjoint cover of $X \backslash \mathrm{O}$ by closed connected sets $C_1, \ldots, C_n$ none of which contain $p$. Assume that $x \in C_1$. Then $X \backslash (\bar{\mathrm{O}} \cup C_2 \cup C_3 \cup \ldots \cup C_n)$ is an open set containing $x$ and contained in $K_q$.

3.3. LEMMA. *If $X$ is Hausdorff and cg, $p \in X$, O is an open neighbourhood of $p$, $C$ is a closed connected set containing $p$, and $c \in C \cap \mathrm{O}$ is such that $c$ and $p$*

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*are not in the same component of* O $\cap$ $C$, *then there exists a closed connected set* $K$ *such that* $c \in K$, $p \notin K$, *and* $K$ *meets* $B(\mathrm{O}) \cap C$.

*Proof*. Applying Lemma 3.2, if $c$ is in a component of $X\backslash\{p\}$, which meets $B(\mathrm{O}) \cap C$, then we can find such a $K$. If not, we apply [**7**, p. 21, Theorem **9.11**] to remove the components of $X\backslash\{p\}$, which contain $B(\mathrm{O})$, from $C$ (notice that since $X$ is cg there is only a finite number of them) and thus show that $c$ and $p$ are in the same component of $C \cap \mathrm{O}$, a contradiction.

A space is said to be weakly locally connected at a point $p$ if, whenever $N$ is a neighbourhood of $p$, the component of $p$ in $N$ is a neighbourhood of $p$. It is well known that, if $X$ is weakly locally connected at every point, then $X$ is locally connected. For more details see [**3**].

3.4. Lemma. *Suppose that* $X$ *is Hausdorff, cg, and subbase regular with respect to* $\mathscr{C}$. *If* $p \in X$, *and* $p$ *has arbitrarily small neighbourhoods whose boundary may be covered by a closed connected set not containing* $p$, *then* $X$ *is weakly locally connected at* $p$.

*Proof*. Given an arbitrary neighbourhood $N$ of $p$, we take a neighbourhood O of $p$ and a closed connected set $K$ such that $\bar{\mathrm{O}} \subset N$, $p \notin K$, and $B(\mathrm{O}) \subset K$. Then we clean screen $K$ and $p$ with a subset of $\mathscr{C}$. The member of the screen which contains $p$ is a connected neighbourhood of $p$ contained in $N$.

*Proof of Theorem* 3.1. If $X$ is not locally connected, then it is not weakly locally connected. Thus we can apply Lemma 3.4 to find a point $p$ with a neighbourhood $N$ such that $N$ does not contain any neighbourhood whose boundary is covered by a closed connected set not containing $p$. Since $X$ is cg, we can pick $F_1, \ldots, F_n$ closed and connected such that

$$p \notin \bigcup_{i=1}^{n} F_i = F \qquad \text{and} \qquad F \cup N = X.$$

Assume that $F_1, \ldots, F_r$, $2 \leqq r \leqq n$, are all the $F_i$s meeting the boundary of $F$, and assume that $r$ is minimal among all such collections. For each $i = 1, \ldots, n$, let $C_1{}^i, \ldots, C_{s_i}{}^i$ be a clean screen of $p$ and $F_i$ such that $p \in C_1{}^i$. Clearly $C_1{}^i$ is a closed connected neighbourhood of $p$, and $C = \bigcap_{i=1}^{n} C_1{}^i$ is a neighbourhood of $p$ contained in $X\backslash F = \mathrm{O}$.

Since $N$ does not contain any connected neighbourhood of $p$, we can pick $c \in C$ such that $c$ and $p$ are not in the same component of O. Thus, by Lemma 3.3, for each $i = 1, \ldots, r$ we can find a closed connected set $K_i$ such that $c \in K_i$, $p \notin K_i$, and $K_i$ meets the intersection of the boundary of O and $C_1{}^i$. Let $K = \bigcup_{i=1}^{r} K_i$. Then $K$ is a closed connected set not containing $p$ and meeting at least two of the $F_1, \ldots, F_r$ since, if $K_1$ meets $F_{i_0}$, then $K_{i_0}$ must meet some other $F_j$ such that $j \neq i_0$. Suppose that $K$ meets $F_m, \ldots, F_r$. Then we may replace $F_m, \ldots, F_r$ in the list by $K \cup \bigcup_{i=m}^{r} F_i$ which contradicts the minimality of $r$.

**4. Equivalence of normality.** If $F_1, \ldots, F_n$ are pairwise disjoint members of $\mathscr{C}$, we say that "$\mathscr{C}$ pairwise screens $F_1, \ldots, F_n$" if there exists a collection $C_1, \ldots, C_m$ of members of $\mathscr{C}$ such that $X = \bigcup_{i=1}^m C_i$ and no $C_i$ meets more than one $F_j$. If $\mathscr{C}$ pairwise screens every finite pairwise disjoint collection of its members, we say that "$\mathscr{C}$ is pairwise screening".

If every pair of disjoint closed connected sets are contained in disjoint open sets, we say that "$X$ is normal with respect to closed connected sets" (or $X$ is nwc).

4.1. THEOREM. *If $X$ is cg and Hausdorff, then the following properties of $X$ are equivalent:*

(i) *$\mathscr{C}$ is pairwise screening,*

(ii) *$X$ is base-normal with respect to $\mathscr{B}$,*

(iii) *$X$ is subbase-normal with respect to $\mathscr{C}$,*

(iv) *$X$ is locally connected and nwc.*

*Proof.* Clearly (i) implies (ii), (ii) implies (iii), and (iii) implies that $X$ is nwc. Now, (iii) implies that $X$ is subbase-regular with respect to $\mathscr{C}$, and an application of Theorem 3.1 shows that (iii) implies (iv).

It remains to show that (iv) implies (i): If $F_1, \ldots, F_n$ is a pairwise disjoint collection of members of $\mathscr{C}$, then, since $X$ is nwc, we may find pairwise disjoint open sets $O_1, \ldots, O_n$ such that $F_i \subset O_i$, $i = 1, \ldots, n$. Since $X$ is locally connected, find an open connected set $N_x$ for each $x \in F_i$ such that $N_x \subset O_i$. Let

$$P_i = \overline{\bigcup_{x \in F_i} N_x},$$

so that $P_i$ is closed and connected, and $P_i \cap O_j = \emptyset$ if $i \neq j$. Let $P = \bigcup_{i=1}^n P_i$. Let $Q_i$ be the union of $P_i$ and all those components of $X \backslash P$ whose closure meets $P_i$. Let $C_i = \overline{Q_i}$. Clearly, $C_i$ does not meet the interior of $P_j$ for $j \neq i$, and so $C_i$ does not meet $F_j$ for $j \neq i$. Now, if $K$ is a component of $X \backslash P$ whose closure does not meet any $P_i$, then, since $X$ is locally connected, $K$ is both open and closed in $X$. Therefore, $K$ is a component of $X$, and, since $X$ is cg, there can be only a finite number of such $K$s. Let them be denoted by $C_{n+1}, \ldots, C_m$. Then, the list $C_1, \ldots, C_m$ pairwise screens $F_1, \ldots, F_n$. Thus (iv) implies (i).

4.2. *Remark.* In a Hausdorff cg space the subbase of all closed connected sets may be subbase-regular without being subbase-normal. The following example, whose conceptual antecessors are the long line and the Tychonoff plank, has this property.

4.3. *Example.* Let $\theta$ be the first ordinal with cardinality greater than $c$. Let $\theta' = (\theta \times [0, 1)) \cup \{\theta\}$ be the topological space given by the order topology where $\theta \times [0, 1)$ has the lexicographic order and $\theta$ is taken to be the last element. $\theta'$ is then longer than the long line but has similar properties. See, for example, [**4**, problem 16H]. Let $P = [0, 1] \times \theta'$ with the product topology. $P$, unlike the Tychonoff plank, is compact, cg, connected, and locally connected. If $p = (1, \theta)$, then $P\backslash\{p\}$ is Hausdorff, cg, regular, and locally connected; thus the subbase of all closed connected sets is subbase-regular. However, as in the case of the analogous subset of the Tychonoff plank, the two subsets $F_1 = [0, 1) \times \theta$ and $F_2 = 1 \times (\theta \times [0, 1))$ are disjoint and closed but are not contained in disjoint open sets. Here, however, $F_1$ and $F_2$ are connected, and so $P\backslash\{p\}$ is not nwc. Thus, the subbase of all closed connected sets is not subbase-normal.

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